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Embedding Complete Bipartite Graphs into Necklace Graphs

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Abstract

Graph embedding is an important technique used in studying the problem of efficiently implementing parallel algorithms on parallel computers. Wirelength is an embedding parameter widely studied in data structures and data representations, electrical networks, VLSI network and chemical graphs. This parameter had been studied for embedding complete bipartite graphs into complete necklace, star necklace and windmill graphs.

Keywords: Embedding; wirelength; Complete bipartite graph; Necklace graph; maximum subgraph

1. Introduction

The processors in the parallel computer systems are connected in a specific way on the basis of interconnection networks. Mathematically, the interconnection network's structure can be shown as a graph, in which the vertices represent processors and the edges represent the communication links between the processors in the network. The topology of the interconnection network decides how the processors are connected to one another, in other words, how the edges connect the vertices of the graph. The capacity in efficiently simulating the programs written for other architectures is also an important feature of an interconnection network. For simulating different interconnection networks, Graph Embedding is used. Thus embedding of graphs from one network into another network is mostly prevalent in the field of interconnection parallel architectures. In general, these problems are NP-complete [1]. This makes the study of wirelength problems in graph embeddings a challenging problem. Inspite of

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the fact that there are many discussions and results on the wirelength problem, many deal with only approximate results. In this paper, we find the wirelength of embedding complete bipartite graphs into necklace graphs.

2. Basic Concepts

The basic definitions and preliminaries that are required for the subsequent study are provided in this section.

Definition 2.1 [2] *Let G and H be graphs of finite order. An embedding $f: G \to H$ is defined as follows:*

1. f is a injective map from $V(G) \to V(H)$
2. f is an injective map from $E(G)$ to $\{P_f(u,v): P_f(u,v)$ is a path in H between $f(u)$ and $f(v)$ for $(u,v) \in E(G)\}$.

Definition 2.2 [2] The *edge congestion* of an embedding $f: G \to H$ is the maximum number of edges of the graph G that are embedded on any single edge of H. Let $C_f(G, H(e))$ denote the number of edges (u,v) of G such that e is in the path $P_f(u,v)$between $f(u)$ and $f(v)$ in H. In other words,

$$C_f\big(G, H(e)\big) = \big|\{(u,v) \in E(G): e \in P_f(u,v)\}\big|$$

where $P_f(u,v)$ denotes the path between $f(u)$ and $f(v)$ in H with respect to f.

Definition 2.3 [3] *The wirelength of an embedding f of G into H is given by*

$$WL_f(G,H) = \sum_{(u,v) \in E(G)} d_H\big(f(u), f(v)\big) = \sum_{e \in E(H)} C_f(G, H(e))$$

where $d_H(f(u), f(v))$ denotes the length of the path $P_f(u,v)$ in H. Then, the *wirelength* of G into H is defined as

$$WL(G,H) = \min WL_f(G,H)$$

where the minimum is taken over all embeddings f of G into H.

The *wirelength problem* [2, 3] of a graph G into H is to find an embedding of G into H that induces the minimum wirelength $WL(G,H)$.

Definition 2.4 [3] Let G be a graph and $A \subseteq V(G)$. Denote

$$I_G(A) = \{(u,v) \in E(G) / u, v \in A\},\ I_G(m) = \max_{A \subseteq V(G), |A| = m} |I_G(A)|.$$

For a given m, where $m = 1, 2, \dots, n$, we consider the problem of finding a subset A of vertices of G such that $|A|=m$. Such subset is called optimal [4, 5]. The problem of finding $I_G(A)$ is called *maximum subgraph problem* [6].

Lemma 2.5 (Congestion Lemma) [3] *Let G be an r-regular graph and $f: G \to H$ be an embedding. Let S be an edge cut of H such that the removal of edges of S leaves H into 2 components H_1 and H_2 and let $G_1 = f^{-1}(H_1)$ and $G_2 = f^{-1}(H_2)$. Also S satisfies the following conditions:*

1. *For every edge $(a,b) \in G_i, i = 1,2, P_f(a,b)$has no edges in S.*
2. *For every edge (a,b) in G with $a \in G_1$ and $b \in G_2$, $P_f(a,b)$ has exactly one edge in S.*
3. *G_1 is an optimal set.*

Then $EC_f(S)$ is minimum and $EC_f(S) = r|V(G_1)| - 2|E(G_1)|$.

Lemma 2.6 (k-Partition Lemma) [3] *Let $f: G \to H$ be an embedding. Let $|kE(H)|$ denote the collection of edges of H repeated exactly k times, $k \geq 1$. Let $\{S_1, S_2, \ldots, S_p\}$ be a partition of $|kE(H)|$ such that each S_i is an edge cut of H. Then*

$$WL_f(G,H) = \frac{1}{k}\sum_{i=1}^{p} C_f(S_i).$$

Definition 2.7 [7] A *complete bipartite graph is a graph whose vertex-set is partitioned into two subsets A and B, so that each vertex of A is joined to all vertices of B and vice-versa. Moreover, each edge must have one end vertex in A and another in B. It is denoted by $K_{t,s}$, where $|A| = t$ and $|B| = s$.*

Lemma 2.8. [8] *The maximum subgraph of $K_{t,s}$ on r vertices is given by $K_{\lceil \frac{r}{2} \rceil, \lfloor \frac{r}{2} \rfloor}$.*

3. Complete bipartite graph into complete necklace graph

The exact wirelength of embedding complete bipartite graph into necklace graph is evaluated in this section.

Definition 3.1. *Let K_m be a complete graphs on m vertices $v_1, v_2, \ldots, v_m$ and K_{t_i} denote the complete graphs on t_i vertices, t_i even, $1 \leq i \leq m$ such that $t_1 + t_2 + \ldots + t_m = 2^n$. Let $K_m * K_{t_i}$ denote the graph obtained from Km and K_{t_i} by identifying any one vertex of K_{t_i} with v_i, $1 \leq i \leq m$. The resultant graph $K_m * \left(\bigcup_{i=1}^{m} K_{t_i}\right)$ is a complete necklace denoted by $CN(K_m; K_{t_1}, K_{t_2}, \ldots K_{t_m})$. See Fig 1. For brevity, the complete necklace $CN(K_m; K_{t_1}, K_{t_2}, \ldots K_{t_m})$ will be represented by $CN(K_m, K)$.*

Embedding Algorithm

Input : The complete bipartite graph, $K_{2^{n-1},2^{n-1}}$ and a complete necklace $H = CN(K_m, K)$.

Algorithm : The hamiltonian cycle HC_{2^n} in $K_{2^{n-1},2^{n-1}}$ are labeled consecutively in the clockwise sense from 1 to 2^n. Let $\sum_{j=0}^{p} t_j = k_p$, $0 \leq p \leq m$, where $t_0 = 0$. The vertices of K_{t_i} in H are labeled as $k_{l-1} + r$, $r = 0, 1, 2, \ldots, t_i - 1$ such that k_{l-1} is the label of v_i, $1 \leq i \leq m$.

Output : An embedding $f : K_{2^{n-1},2^{n-1}} \to H$ yields a minimum wirelength for $f(x) = x$-1.

Proof of correctness : The labels refer to the vertices which have been assigned. Assume $S_i = \{(k_i - 1, k_r - 1)/1 \leq r \leq m, i \neq r\}$, $S_i^j = \{(k_{i-1} + j, k_{r-1} + p)/1 \leq p \leq t_i - 1, j \neq p\}$ and $S_i' = \{(k_i - 1, k_r - 1 - r)/1 \leq r \leq t_i - 1\}$ for $1 \leq i \leq m$. See Fig.1. Now $\{S_i, S_i'\} \cup \{S_i^j : 0 \leq j \leq t_i - 2\}, 1 \leq i \leq m$ partitions $|2H|$. Clearly for every i, H_{i1} and H_{i2} are the two components of $E(H)\backslash S_i$, $1 \leq i \leq m$, where $V(H_{i1}) = \{s_{i-1}, s_{i-1} + 1, \ldots, s_{i-1}\}$. Let $G_{i1} = f^{-1}(H_{i1})$ and $G_{i2} = f^{-1}(H_{i2})$. It is clear that G_{i1} is an optimal set by Lemma 2.8. Moreover, the conditions $(i), (ii)$ *and* (iii) of the Lemma 2.5 are satisfied by each $S_i, 1 \leq i \leq m$. Hence $C_f(S_i)$ is minimum.

For every i, H_{i1}' and H_{i2}' are the two components of $E(H)\backslash S'_i$, $1 \leq i \leq m$, where $V(H_{i1}') = \{k_{i-1}, k_{i-1} + 1, \ldots, k_{i-1} + t_i - 2\}$. Let $G_{i1}' = f^{-1}(H_{i1}')$ and $G_{i2}' = f^{-1}(H_{i2}')$. It is clear that G_{i1}' is an optimal set by Lemma 2.8. Moreover, the conditions $(i), (ii)$ *and* (iii) of the Lemma 2.5 are satisfied by each $S'_i, 1 \leq i \leq m$. Hence $C_f(S_i')$ is minimum.

For every i, j, H_{i1}^{j} and H_{i2}^{j} are the two components of $E(H)\setminus S_i^j$, $1 \le i \le m$ and $0 \le j \le t_{i-2}$ where $V(H_{i1}^{j}) = \{s_{i-1} + j\}$. Let $G_{i1}^{j} = f^{-1}(H_{i1}^{j})$ and $G_{i2}^{j} = f^{-1}(H_{i2}^{j})$. It is clear that G_{i1}^{j} is an optimal set by Lemma 2.8. Moreover, the conditions $(i), (ii)$ *and* (iii) of the Lemma 2.5 are satisfied by each S_i^j, $1 \le i \le m$ and $0 \le j \le t_{i-2}$. Hence $C_f(S_i^j)$ is minimum. Therefore the wirelength is minimum by the Lemma 2.6.

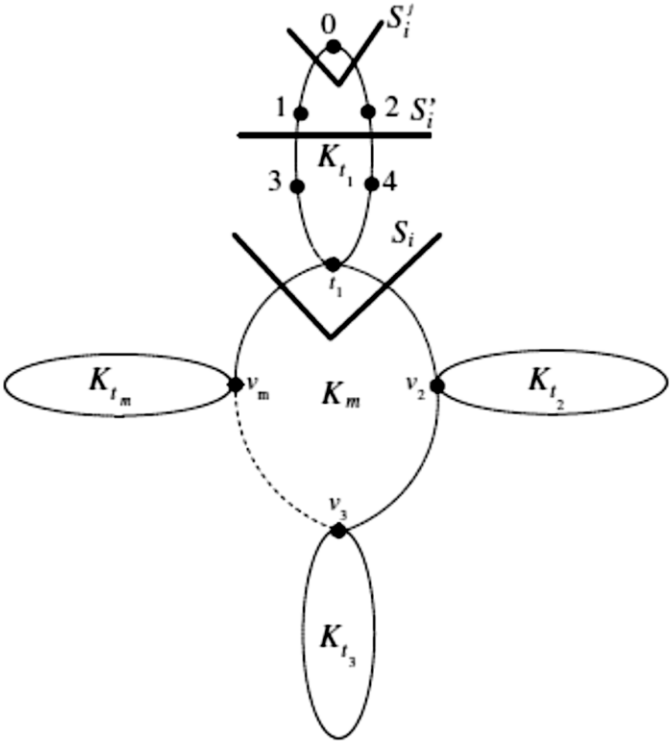

Fig. 1. Edge cut of $CN(K_m, K)$

Theorem 3.2. The wirelength of embedding complete bipartite graph $K_{2^{n-1},2^{n-1}}$ into complete necklace $CN(K_m, K)$ is given by

$$WL(K_{2^{n-1},2^{n-1}}, CN(K_m, K)) = \frac{1}{2}\left(\sum_{i=1}^{m}(2^{n-1}t_i - 2I_G(t_i)\right) + \frac{1}{2}\left(\sum_{i=1}^{m}(2^{n-1}(t_i - 1)) - 2I_G(t_i -)\right) + 2^{n-2}(S_m - m).$$

Proof: By congestion lemma, $C_f(S_i) = 2^{n-1}t_i - 2I_G(t_i)$, for $i = 1, 2, ..., m$.
$C_f(S_i') = 2^{n-1}(t_i - 1) - 2I_G(t_i - 1)$ for $i = 1, 2, \ldots, m$ and $C_f(S_i^j) = 2^{n-1}(S_m - m)$, for $i = 1, 2, \ldots, m$ and $j = 0, 1, 2, \ldots, t_i - 2$.

$$\begin{aligned} WL(K_{2^{n-1},2^{n-1}}, CN(K_m, K)) &= \frac{1}{2}\left(\sum_{i=1}^{m} C_f(S_i) + \sum_{i=1}^{m} C_f(S_i')\right) + \frac{1}{2}\left(\sum_{i=1}^{m}\sum_{j=0}^{t_i-2} C_f(S_i^j)\right) \\ &= \frac{1}{2}\left(\sum_{i=1}^{m}(2^{n-1}t_i - 2I_G(t_i)\right) + \frac{1}{2}\left(\sum_{i=1}^{m}(2^{n-1}(t_i - 1)) - 2I_G(t_i - 1)\right) + 2^{n-2}(S_m - m). \end{aligned}$$

4. More Results

Definition 4.1. *Let $K_{1,m}$be a star graphs on m+1 vertices v_0 v_1, v_2, . . . , v_m and K_{t_i} denote the complete graphs on t_i vertices, t_i even, i=1,2,...m-1. Let $K_{1,m} * K_{t_i}$ denote the graph obtained from $K_{1,m}$ and K_{t_i} by identifying any one vertex of K_{t_i} with i=1,2,...m-1 The resultant graph $K_{1,m} * (\bigcup_{i=1}^{m} K_{t_i})$ is a star necklace denoted by $SN(K_{1,m}: K_{t_1}, K_{t_2}, \ldots . K_{t_m})$.*

Theorem 4.2. The wirelength of embedding complete bipartite graph $K_{2^{n-1},2^{n-1}}$ into star necklace $SN(K_{1,m}: K_{t_1}, K_{t_2}, \ldots . K_{t_m})$ is given by

$$WL(K_{2^{n-1},2^{n-1}}, SN(K_{1,m}: K_{t_1}, K_{t_2}, \ldots . K_{t_m})) = \frac{1}{2}(\sum_{i=1}^{m}(2^{n-1}t_i - 2I_G(t_i)) + \sum_{i=1}^{m}(2^{n-1}(t_i-1) - 2I_G(t_i-1))) + 2^{n-2}(S_m - m - 1).$$

Definition 4.3. *Let K_{t_i}be a complete graphs on t_i vertices and t_i,even for i=2,3,...,m-1. If one vertex from each K_{t_i} is merged to a single vertex v_1 then the resultant graph $\bigcup_{i=1}^{m} K_{t_i}$ is a windmill graph incident with a common vertex v_1 denoted by $WM(K_{t_1}, K_{t_2}, \ldots, K_{t_m})$.*

Theorem 4.4. The wirelength of embedding complete bipartite graph $K_{2^{n-1},2^{n-1}}$ into wind mill graph $WM(K_{t_1}, K_{t_2}, \ldots, K_{t_m})$ is given by

$$WL(K_{2^{n-1},2^{n-1}}, WM(K_{t_1}, K_{t_2}, \ldots, K_{t_m})) = \frac{1}{2}(\sum_{i=2}^{m}(2^{n-1}t_i - 2I_G(t_i))) + 2^{n-2}(S_m - 1).$$

5. Concluding Remarks

In this paper, we have embed complete bipartite graphs into graphs like complete necklace, star necklace and windmill graphs to obtain exact wirelength.

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