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Embedding of Complete graphs and Cycles into Grids with holes

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Abstract

An important feature of an interconnection network is its ability to efficiently simulate one architecture by another. Such a simulation problem can be mathematically formulated as a graph embedding problem. Although the definition of an embedding is an into mapping from Guest Graph to Host Graph, so far in the literature, the embedding has been considered as a mapping from G onto H. In other words, the number of processors in G and H are considered to be the same. In this paper, we increase the number of processors in H by 1. The question is to find the processor in H which does not have the pre-image under the embedding mapping, so that the wirelength of the embedding is minimum. We relate this problem to finding transmission of vertices in the host graph.

Keywords: Embedding; Congestion Lemma; Wirelength; Transmission; Wiener Index

1. Introduction

Embedding of a graph G into a graph H is a function from the vertex set $V(G)$ of G into the vertex set $V(H)$of H such that f maps every edge (u,v) of G to $\big(f(u), f(v)\big)$-path in H. G is called the guest graph and H, a host graph [2]. Embedding plays a key role in Computer Science wherein an algorithm designed for network (graph) G can be efficiently modified for network H. Dilation, congestion, wirelength are some of the parameters associated with the embedding of graphs.

Interconnection networks are graphs with vertices representing processors and edges representing communication links between processors. Embedding one network onto another helps to efficiently simulate one architecture by another [2]. The circulant networks have been used for decades in creating telecommunication networks [3, 4, 5, 6, 7]. They are also used in designing binary codes [8]. The complete graph K_n and the cycle C_n on n vertices are special cases of circulant graphs [9].

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So far, in the literature, the embedding has been considered as mapping one network G onto another network H. In this paper, we consider it as an 'into' function implying that a few processors in the host graph H do not have preimages in the guest graph G. We call the position of these vertices in H as 'holes' in H. In the subsequent sections, we take the complete graph K_n and the cycle C_n, n odd as the guest graphs and a $2 \times \left\lceil \frac{n}{2} \right\rceil$ mesh, also called a ladder, as the host graph.

2. Basic Concepts

In this section, we give the definitions related to embedding and transmission problems.

Definition 2.1: Let $\boldsymbol{G}$ and $\boldsymbol{H}$ be finite graphs with $\boldsymbol{n}$ vertices. $\boldsymbol{H}$ is called a Host Graph and $\boldsymbol{G}$ a Guest Graph. $\boldsymbol{V(G)}$ and $\boldsymbol{V(H)}$ denote the vertex set of $\boldsymbol{G}$ and $\boldsymbol{H}$ respectively. $\boldsymbol{E(G)}$ and $\boldsymbol{E(H)}$ denote the edge set of $\boldsymbol{G}$ and $\boldsymbol{H}$ respectively. An *embedding* [2] $\boldsymbol{f}$ of $\boldsymbol{G}$ into $\boldsymbol{H}$ is defined as

1. $\boldsymbol{f}$ is a one-to-one map from $\boldsymbol{V(G)} \to \boldsymbol{V(H)}$.
2. $\boldsymbol{P_f}$ is a one-to-one map from $\boldsymbol{E(G)}$ to $\boldsymbol{P_f(f(u), f(v))}$: $\boldsymbol{P_f(f(u), f(v))}$ is a path in $\boldsymbol{H}$ between $\boldsymbol{f(u)}$ and $\boldsymbol{f(v)}$ for $\boldsymbol{(u, v)} \in \boldsymbol{E(G)}$}.

Definition 2.2: The *edge congestion* of an embedding f of G into H is the maximum number of edges of the graph G that are embedded on any single edge of H. Let $EC_f\ (G, H(e))$ denote the number of edges (u, v) of G such that e is in the path $P_f(f(u), f(v))$ between $f(u)$ and $f(v)$ in H. In other words,

$$EC_f\ (e) = \left|(u, v) \in E(G) :\ e \in P_f(f(u), f(v))\right|$$

where $P_f(u, v)$ denotes the path between $f(u)$ and $f(v)$ in H with respect to f.

Lemma 2.1: (Congestion Lemma [1]). Let G be an r-regular graph and f be an embedding of G into H. Let S be an edge cut of H such that the removal of edges of S leaves H into two components H_1 and H_2 and let $G_1 = f^{-1}(H_1)$ and $G_2 = f^{-1}(H_2)$. Also S satisfies the following conditions:

1. For every edge $(a, b) \in G_i, i = 1, 2, P_f(f(a), f(b))$ has no edges in S.
2. For every edge $(a, b) \in G$ with $a \in G_1$ and $b \in G_2$; $P_f(f(a), f(b))$ has exactly one edge in S.
3. G_1 is a maximum subgraph on k vertices, where $k = |V(G_1)|$.

Then $EC_f(S)$ is minimum, that is, $EC_f(S) \leq EC_g(S)$ for any other embedding g of G into H.

Definition 2.3: The *wirelength of an embedding* f [1, 2] of G into H is given by

$$WL_f(G, H) = \sum_{(u,v)\in E(G)} \left|P_f(f(u), f(v))\right|$$

where $\left|P_f(f(u), f(v))\right|$ denotes the length of the path $P_f(f(u), f(v))$ in H.

For an embedding f of G into H, the wirelength of f is

$$WL_f(G, H) = \sum_{(u,v)\in E(G)} \left|P_f(f(u), f(v))\right| = \sum_{e \in E(H)} EC_f(G, H(e))$$

The *wirelength* of G into H is defined as

$$WL(G,H) = min\ WL_f(G,H)$$

where the minimum is taken all over all embeddings f of G into H.

Definition 2.4: (Transmission of a vertex). [10, 11, 12] The Transmission $T(u)$ of a vertex u in a graph G is defined as $T(u) = \sum_{v \in V} d(u,v)$ and the Wiener Index is defined as $WI(G) = \sum \frac{T(u)}{2}$.

Definition 2.5: [13] Let $u \in V(G)$ and let $S = \{S_1, S_2, \ldots, S_k\}$ be a partition of $E(G)$ where each S_i, $1 \leq i \leq k$ is an edge cut of G such that removal of edges in S_i leaves two components G_i and G_i^* of G, with u belongs to one of the components, say G_i. For$1 \leq i \leq k$, suppose

1. For any two vertices v and w in G_i^*, every shortest path between v and w lies in G_i^*.
2. For a vertex v in G_i^*, every shortest path between u and v passes through exactly one edge in S_i. Then

$$T(u) = \sum_{i=1}^{k} |V(G_i^*)|$$

Definition 2.6: (Grid Network [14]). Let P_n denote a path on n vertices. Then the cartesian product $P_n \times P_m$is called a mesh or grid network denoted by $M\ (n \times m)$. In particular, when $n = 2$, the network is referred to as a ladder.

3. Main Results

Let $\boldsymbol{Wl_v}$ denote the wirelength of an embedding $\boldsymbol{G}$ into $\boldsymbol{H}$ with hole in the location of vertex $\boldsymbol{v}$ in $\boldsymbol{H}$ implying that $\boldsymbol{v}$ has no preimage under the embedding. As there are $|\boldsymbol{V(H)}|$ possibilities of selecting the location of $\boldsymbol{v}$ in $\boldsymbol{H}$, the following becomes a significant problem: Find the location of the hole in the host graph $\boldsymbol{H}$, so that the wirelength of embedding $\boldsymbol{G}$ into $\boldsymbol{H}$ is minimum. In other words, the problem is to find the location of $\boldsymbol{v}$ such that $\boldsymbol{WL_v} = \mathbf{min}_{\boldsymbol{u \in V(H)}} \boldsymbol{WL_u}$. In this paper, we choose the guest graph to be the complete graph $\boldsymbol{K_n}$ and cycle $\boldsymbol{C_n}$ on $\boldsymbol{n}$ vertices, $\boldsymbol{n}$ odd, $\boldsymbol{n \geq 5}$ and the host graph to be the $\boldsymbol{2 \times \left\lceil \frac{n}{2} \right\rceil}$ grid on $\boldsymbol{n+1}$ vertices. The $\boldsymbol{2 \times \left\lceil \frac{n}{2} \right\rceil}$ grid is also known as a ladder graph.

3.1. Embedding $\boldsymbol{K_n}$, $\boldsymbol{n}$ odd, $\boldsymbol{n \geq 5}$ into $\boldsymbol{M\left(2, \left\lceil \frac{n}{2} \right\rceil\right)}$

Let $\boldsymbol{K_n}$ be the complete graph on $\boldsymbol{n}$-vertices, $\boldsymbol{n}$ odd, $\boldsymbol{n \geq 5}$ and $\boldsymbol{M\left(2, \left\lceil \frac{n}{2} \right\rceil\right)}$ be the grid with $\boldsymbol{n+1}$ vertices. Using Transmission Lemma [13], we compute the transmission of vertex $\boldsymbol{v}$. Then we find the wirelength of $\boldsymbol{G}$ into $\boldsymbol{H}$ for all the vertices, considering each vertex as a hole.

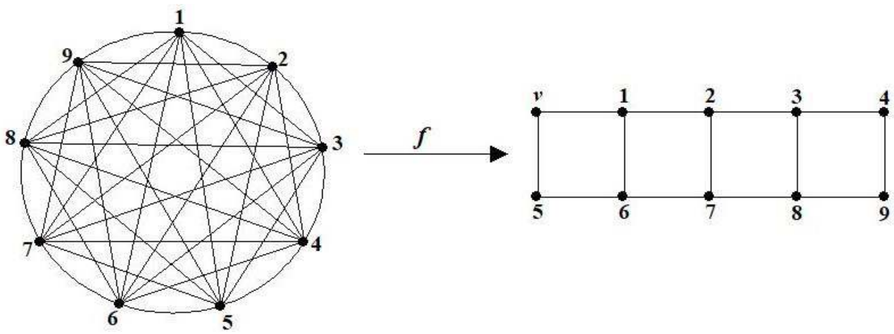

Fig.1. K_9 into Grid $M(2 \times 5)$.

In this section, we choose the guest graph to be the complete graph $\boldsymbol{K_n}$ on $\boldsymbol{n}$ vertices, $\boldsymbol{n}$ odd, $\boldsymbol{n \geq 5}$ and the host graph to be the $\boldsymbol{2 \times \left\lceil \frac{n}{2} \right\rceil}$ grid on $\boldsymbol{n+1}$ vertices. We have increased the number of processors in $\boldsymbol{H}$ by 1. We relate the wirelength problem to finding transmission of vertices in the Host Graph $\boldsymbol{H}$, where vertex $\boldsymbol{v}$ in $\boldsymbol{H}$ does not have a preimage. An embedding of $\boldsymbol{K_9}$ into grid $\boldsymbol{M(2 \times 5)}$ is shown in Fig.1.

Theorem 3.1.1. Let the complete graph K_n, n odd, $n \geq 5$ be embedded into the mesh $M\left(2, \left\lceil \frac{n}{2} \right\rceil\right)$ with exactly one hole in M. Then $WL_{v_i} + T(v_i) = WL_{v_j} + T(v_j), 1 \leq i, j \leq n$.

Proof: The wirelength WL_{v_i} is obtained by taking the sum of all shortest paths between all pairs of vertices except the paths with one end at v and the other end at every other vertex of M. This is nothing but the Wiener Index of M minus $T(v)$. Suppose WI_M denotes the Wiener Index of M, then $WL_{v_i} = WI_M - T(v_i), 1 \leq i \leq n$. This implies $WL_{v_i} + T(v_i) = WL_{v_j} + T(v_j), 1 \leq i, j \leq n$.

Theorem 3.1.2. Let the complete graph K_n, n odd, $n \geq 5$ be embedded into the mesh $M\left(2, \left\lceil \frac{n}{2} \right\rceil\right)$ with exactly one hole in M. Then, WL_v is minimum when v is a vertex of degree 2 in M.

Proof: It has been observed in [15] that T_x, x a vertex in M, is maximum when x is of degree 2. Since WI_M is a constant, by Theorem 3.1.1. WL_v is minimum, when $T(v)$ is maximum. In other words, WL_v is minimum when v is a corner vertex in M.

Theorem 3.1.3. Let the complete graph K_n, n odd, $n \geq 5$ be embedded into the mesh $M\left(2, \left\lceil\frac{n}{2}\right\rceil\right)$ with exactly one hole in M. Then $\sum_{i=1}^{n} WL_{v_i} = (n-2)\, WI_M = \left(\left\lceil\frac{n}{2}\right\rceil - 1\right)\sum T(v_i)$

Proof: By Theorem 3.2.1 $WL_{v_i} + T(v_i) = WI_M, 1 \leq i \leq n$,

Therefore,

$$\sum_{i=1}^{n} T(v_i) + \sum_{i=1}^{n} WL_{v_i} = n\, WI_M$$

But $\sum_{i=1}^{n} T(v_i) = 2\, WI_M$. Hence,

$$2\, WI_M + \sum_{i=1}^{n} WL_{v_i} = n\, WI_M$$

$$\sum_{i=1}^{n} WL_{v_i} = n\, WI_M - 2\, WI_M$$

$$\sum_{i=1}^{n} WL_{v_i} = (n-2)\, WI_M$$

$$\sum_{i=1}^{n} WL_{v_i} = (n-2)\frac{\sum_{i=1}^{n} T(v_i)}{2}$$

$$\sum_{i=1}^{n} WL_{v_i} = \left(\left\lceil\frac{n}{2}\right\rceil - 1\right)\sum_{i=1}^{n} T(v_i)$$

Therefore, $\sum_{i=1}^{n} WL_{v_i} = (n-2)\, WI_M = \left(\left\lceil\frac{n}{2}\right\rceil - 1\right)\sum T(v_i)$. Here, $WI_M, T(v_i)$ and WL_{v_i} denotes the Wiener Index of M, Transmission of a vertex v_i and Wirelength of K_n into M respectively.

3.2. Embedding C_n, n odd, $n \geq 5$ into $M\left(2, \left\lceil\frac{n}{2}\right\rceil\right)$

In this section, we choose the guest graph to be the cycle C_n on n vertices, n odd, $n \geq 5$ and the host graph to be the $2 \times \left\lceil\frac{n}{2}\right\rceil$ grid on $n+1$ vertices.

Theorem 3.3.1. Let C_n be the cycle on n-vertices, n odd, $n \geq 5$ and $M\left(2 \times \left\lceil\frac{n}{2}\right\rceil\right)$ be the grid with $n+1$ vertices. Then $WL(C_n, M) = 2\left\lceil\frac{n}{2}\right\rceil$.

Proof: Label the vertices of C_n with consecutive numbers $1, 2, \dots, n$ in the clockwise sense. Label the vertices of the ladder snakewise with the consecutive numbers $1, 2, \dots, n$, beginning with the first row from left to right and the second row from right to left, skipping the vertex v. Consider the embedding $f(x) = x$, $\forall\, x, 1 \leq x \leq n$. See Fig.2.

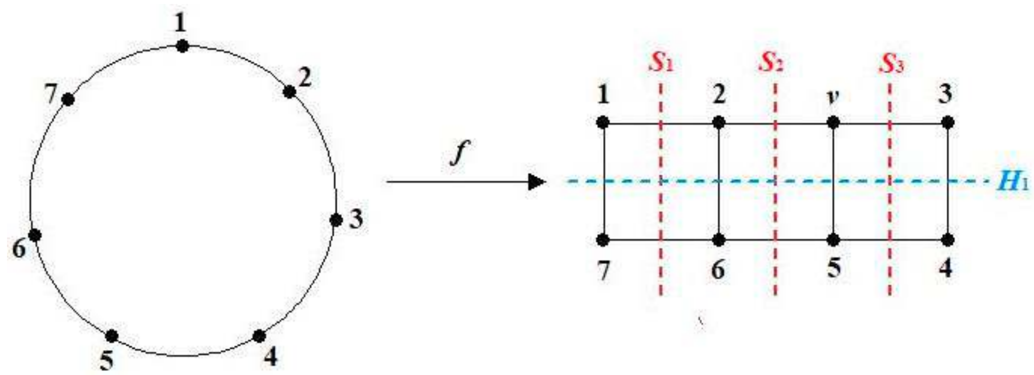

Fig.2. $\boldsymbol{C_7}$ into Grid $\boldsymbol{M(2 \times 4)}$.

The vertical edge cuts and the horizontal edge cut as shown in Fig.2. partition the graph M into two components, where the inverse images in C_n are maximum subgraphs. If a component consists of r vertices, the congestion on the corresponding cut is $2r - 2(r-1) = 2$. Hence the wirelength is $2\left\lceil \frac{n}{2} \right\rceil$.

Remark 1: It is interesting to note that the wirelength remains the same, irrespective of the position of the hole in M.
Remark 2: The wirelength of embedding C_n, n even, $n \geq 5$ into $M\left(2, \frac{n}{2}\right)$ is the same as embedding C_n, n odd, $n \geq 5$ into $M\left(2, \left\lceil \frac{n}{2} \right\rceil\right)$ with a hole.

4. Conclusion

In this paper we have studied the embedding of complete graph K_n and cycle C_n on odd vertices into grid network with one extra processor. We have also studied the relation between Wirelength and Transmission. This study brings out a number of open problems. To state a few, we have the following questions.

(i) What if Host is replaced by networks other than the mesh network?
(ii) What if the Guest graph is a network other than K_n?
(iii) What if the number of holes is increased in the mesh network, given the guest graph to be K_n?

References

[1] Paul Manuel, Indra Rajasingh, Bharati Rajan, Helda Mercy. (2009) "Exact wirelength of hypercubes on a grid." *Discrete Applied Mathematics* **157**: 1486–1495.

[2] S. L. Bezrukov, J. D. Chavez, L.H. Harper, M.Röttger, U. P. Schroeder. (1998) "Embedding of hypercubes into grids." *Mathematical Foundations of Computer Science* **1450**: 693–701.

[3] G. K. Wong, D. A. Coppersmith. (1994) "A combinatorial problem related to multimodule memory organization." *J. Assoc.Comput*. Mach. **21 (3)**: 392–401.

[4] Francis T. Boesch, Jhing-Fa Wang. (1985) "Reliable circulant networks with minimum transmission delay." *IEEE Transactions on Circuits and Systems* **32 (12)**: 1286–1291

[5] R. Beivide, E. Herrada, J. L. Balcazar, A. Arruabarrena. (1991) "Optimal distance networks of low degree for parallel computers." *Transactions on Computers* **40 (10)**: 1109–1124.

[6] J. C. Bermond, F. Comellas, D. F. Hsu. (1995) "Distributed loop computer networks, a survey." *Journal of Parallel Distributed Computing* **24 (1)**: 2–10.

[7] R. S. Wilkov. (1972) "Analysis and design of reliable computer networks." *IEEE Transactions on Communications* **20 (3)**: 660–678.

[8] Maurice. Karlin. (1969) "New binary coding results by circulants." *IEEE Transactions on Information Theory* **15 (1)**: 81–92.

[9] J. Xu. (2001) "Topological Structure and Analysis of Interconnection Networks." *Kluwer Academic Publishers*.

[10] Aida Abiad, Boris Brimkov, Aysel Erey, Lorinda Leshock, Xavier Martínez-Rivera, Suil O, Sung-Yell Song, Jason Williford. (2017) "On the Wiener index, distance cospectrality and transmission-regular graphs." *Discrete Applied Mathematics* **230**: 1–10.

[11] Matjaž Krnc, Riste Škrekovski. (2015) "Centralization of transmission in networks." *Discrete Mathematics* **338**: 2412–2420.

[12] L. Śoltès. (1991) "Transmission in graphs: A bound and vertex removing." *Mathematica Slovaca* **41 (1)**: 11–16.

[13] Indra Rajasingh, Paul Manuel, N. Parthiban, D. Azubha Jemilet, R. Sundara Rajan. (2016) "Transmission in Butterfly Networks." *The Computer Journal* **59 (8)**: 1174–1179.

[14] IndraRajasingh, R.SundaraRajana, N.Parthiban, T.M.Rajalaxmi. (2015) "Bothway embedding of circulant network into grid." *Journal of Discrete Algorithms* **33**: 2–9.

[15] Azubha Jemilet. (2019) "Tranmission in Interconnection Networks." Ph.D. Thesis.