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Embedding of Hypercube into Extended Rooted Theta Mesh

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Abstract

The hypercube is one of the most popular interconnection networks due to its structural regularity, potential for parallel computation of various algorithms, and the high degree of fault tolerance. In this paper, we introduce a graph called extended rooted theta mesh and we compute the exact wirelength of embedding r-dimensional hypercube into r-dimensional extended rooted theta mesh, $r \geq 2$.

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*Keywords:*Embedding; dilation, congestion; hypercube; extended rooted theta mesh

1. Introduction

Interconnection networks play a major role in the performance of distributed-memory multiprocessors and the one primary concern for choosing an appropriate interconnection network is the graph embedding ability. Graph embedding is the mapping of a topological structure (guest graph) into another topological structure (host graph) that preserves certain required topological properties and the graph embedding ability reflects how efficiently a parallel algorithm with a guest graph can be executed on a host graph and the utilization of system resources in the host graph [2]. Many applications, such as architectural simulations and processor allocations, can be modeled as graph embedding [3–10].

An embedding of a guest graph G into a host graph H is a one-to-one mapping of the vertex set of G into that of H. The quality of an embedding can be measured by certain cost criteria. One of these criteria which is considered very often is the *dilation*. The dilation of an embedding is defined as the maximum distance between a pair of vertices of H that are images of adjacent vertices of G. It is a measure for the communication time needed

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when simulation one network on another [11].

Another important cost criteria is the *wirelength*. The wirelength of a graph embedding arises from VLSI designs, data structures and data representations, networks for parallel computer systems, biological models that deal with cloning and visual stimuli, parallel architecture, structural engineering and so on [1, 12].

In recent years, among many interconnection networks, the hypercube has been the focus of many researchers due to its structural regularity, potential for parallel computation of various algorithms, and the high degree of fault tolerance [13]. Hypercubes are known to simulate other structures such as grids and binary trees [14, 15].

Graph embeddings have been well studied for meshes into crossed cubes [16], binary trees into paths [12], binary trees into hypercubes [11, 15], complete binary trees into hypercubes [17], incomplete hypercube in books [13], tori and grids into twisted cubes [2], meshes into locally twisted cubes [18], meshes into faulty crossed cubes [19], meshes into crossed cubes [16], generalized ladders into hypercubes [20], grids into grids [21], binary trees into grids [22], hypercubes into cycles [23, 24], star graph into path [25], snarks into torus [26], generalized wheels into arbitrary trees [27], hypercubes into grids [14], m-sequencial k-ary trees into hypercubes [28], meshes into möbius cubes [29], ternary tree into hypercube [30], enhanced and augmented hypercube into complete binary tree [31], circulant into arbitrary trees, cycles, certain multicyclic graphs and ladders [32], hypercubes into cylinders, snakes and caterpillars [33], hypercubes into necklace, windmill and snake graphs [34].

Even though there are numerous results and discussions on the wirelength problem, most of them deal with only approximate results and the estimation of lower bounds [23, 35]. In this paper, we produce exact wirelength of embedding hypercube into extended rooted theta mesh.

2. Basic concepts

In this section we give the basic definitions and preliminaries that are required for the study.

Definition 2.1 [35] *Let G and H be finite graphs with n vertices. An embedding f of G into H is defined as follows:*

1. f is a bijective map from $V(G) \to V(H)$
2. f is a one-to-one map from $E(G)$ to $\{P_f(u,v): P_f(u,v)$ is a path in H between $f(u)$ and $f(v)$ for $(u,v) \in E(G)\}$.

The *edge congestion* of an embedding f of G into H is the maximum number of edges of the graph G that are embedded on any single edge of H. Let $EC_f(G,H(e))$ denote the number of edges(u,v) of G such that e is in the path $P_f(u,v)$between $f(u)$ and $f(v)$ in H. In other words,

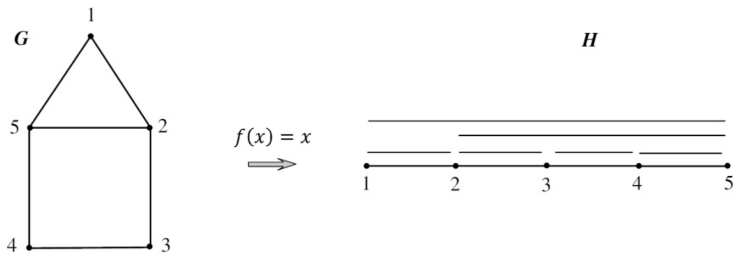

Figure 1: Wiring diagram of a graph G into path H with $WL_f(G,H) = 11$

$$EC_f(G,H(e)) = |\{(u,v) \in E(G): e \in P_f(u,v)\}|$$

where $P_f(u,v)$ denotes the path between $f(u)$ and $f(v)$ in H with respect to f.

If we think of G as representing the wiring diagram of an electronic circuit, with the vertices representing components and the edges representing wires connecting them, then the edge congestion $EC(G,H)$ is the minimum, over all embeddings $f: V(G) \to V(H)$, of the maximum number of wires that cross any edge of H [36]. See Figure 1.

Definition 2.2 [14] *The wirelength of an embedding f of G into H is given by*

$$WL_f(G,H) = \sum_{(u,v)\epsilon E(G)} d_H\big(f(u),f(v)\big) = \sum_{e\epsilon E(H)} EC_f(G,H(e))$$

where $d_H(f(u),f(v))$ denotes the length of the path $P_f(u,v)$ in H. Then, the *wirelength* of G into H is defined as

$$WL(G,H) = \min WL_f(G,H)$$

where the minimum is taken over all embeddings f of G into H.

The *wirelength problem* [14, 22, 23, 27, 35, 36] of a graph G into H is to find an embedding of G into H that induces the minimum wirelength $WL(G,H)$. The following two versions of the edge isoperimetric problem of a graph $G(V,E)$ have been considered in the literature [37], and are NP-complete [38].

Problem 1 : Find a subset of vertices of a given graph, such that the edge cut separating this subset from its complement has minimal size among all subsets of the same cardinality. Mathematically, for a given m, if $\theta_G(m) = \min_{A\subseteq V,|A|=m}|\theta_G(A)|$ where $\theta_G(A) = \{(u,v) \in E: u \in A, v \notin A\}$, then the problem is to find $A \subseteq V$ such that $|A| = m$ and $\theta_G(m) = |\theta_G(A)|$.

Problem 2 : Find a subset of vertices of a given graph, such that the number of edges in the subgraph induced by this subset is maximal among all induced subgraphs with the same number of vertices. Mathematically, for a given m, if $I_G(m) = \max_{A\subseteq V,|A|=m}|\, I_G(m)|$ where $I_G(A) = \{(u,v) \in E: u,v \in A\}$, then the problem is to find $A \subseteq V$ such that $|A| = m$ and $I_G(m) = |\, I_G(A)|$.

For a given m, where $m = 1,2,\dots,n,$ we consider the problem of finding a subset A of vertices of G such that $|A| = m$ and $|\,\theta_G(A)| = \theta_G(m)$. Such subsets are called optimal. We say that optimal subsets are nested if there exists a total order $\mathcal{O}$ on the set V such that for any $m = 1,2,\dots,n,$ the first m vertices in this order is an optimal subset. In this case we call the order $\mathcal{O}$ an optimal order [37, 39]. This implies that $WL(G,P_n) = \sum_{m=0}^{n}\theta_G\,(m)$.
.

Further, if a subset of vertices is optimal with respect to Problem 1, then its complement is also an optimal set. But, it is not true for Problem 2 in general. However for regular graphs a subset of vertices S is optimal with respect to Problem 1 if and only if S is optimal for Problem 2 [37]. In the literature, Problem 2 is defined as the maximum subgraph problem.

Lemma 2.3 (Congestion Lemma) [14] *Let G be an r-regular graph and f be an embedding of G into H. Let S be an edge cut of H such that the removal of edges of S leaves H into 2 components H_1 and H_2 and let $G_1 = f^{-1}(H_1)$ and $G_2 = f^{-1}(H_2)$. Also S satisfies the following conditions:*

1. *For every edge $(a,b) \in G_i, i = 1,2, P_f(a,b)$has no edges in S.*
2. *For every edge (a,b) in G with $a \in G_1$ and $b\in G_2$, $P_f(a,b)$ has exactly one edge in S.*
3. *G_1 is an optimal set.*

Then $EC_f(S)$ is minimum and $EC_f(S) = r|V(G_1)| - 2|E(G_1)|$.

Lemma 2.4 (Partition Lemma) [14] *Let $f: G \to H$ be an embedding. Let$\{S_1, S_2, \dots, S_p\}$ be a partition of $E(H)$ such that each S_i is an edge cut of H. Then*

$$WL_f(G,H) = \sum_{i=1}^{p} EC_f(S_i).$$

Definition 2.5 [1] *For $r \geq 1$, let Q^r denote the graph of r-dimensional hypercube. The vertex set of Q^r is formed by the collection of all r-dimensional binary strings. Two vertices $x, y \in V(Q^r)$ are adjacent if and only if the corresponding binary strings differ exactly in one bit.*

Equivalently if $n = 2^r$ then the vertices of Q^r can also be identified with integers $0, 1, \ldots, n-1$ so that if a pair of vertices i and j are adjacent then $i - j = \pm 2^p$ for some $p \geq 0$.

Definition 2.6 [40] *An incomplete hypercube on i vertices of Q^r is the subcube induced by $\{0, 1, \ldots, i-1\}$ and is denoted by L_i, $1 \leq i \leq 2^r$.*

Theorem 2.7 [39, 41, 42] *Let Q^r be an r-dimensional hypercube. For $1 \leq i \leq 2^r$, L_i is an optimal set on i vertices.*

Lemma 2.8 [14] *Let Q^r be an r-dimensional hypercube. Let $m = 2^{t_1} + 2^{t_2} + \ldots + 2^{t_l}$ such that $r \geq t_1 > t_2 > \ldots > t_l \geq 0$. Then* $|E(Q^r[L_m])| = [t_1 . 2^{t_1 - 1} + t_2 . 2^{t_2 - 1} + \ldots + t_l . 2^{t_l - 1}] + [2^{t_2} + 2. 2^{t_3} + \ldots + (l-1)2^{t_l}]$.

3. Embedding of Hypercube into Extended Rooted Theta Mesh

A tree is a connected graph that contains no cycles. The most common type of tree is the binary tree. It is so named because each node can have at most two descendents. A binary tree is said to be a complete binary tree if each internal node has exactly two descendents. These descendents are described as left and right children of the parent node. Binary trees are widely used in data structures because they are easily stored, easily manipulated, and easily retrieved. Also, many operations such as searching and storing can be easily performed on tree data structures. Furthermore, binary trees appear in communication pattern of divided-and-conquer type algorithms, functional and logic programming, and graph algorithms. A rooted tree represents a data structure with a hierarchical relationship among its various elements [1].

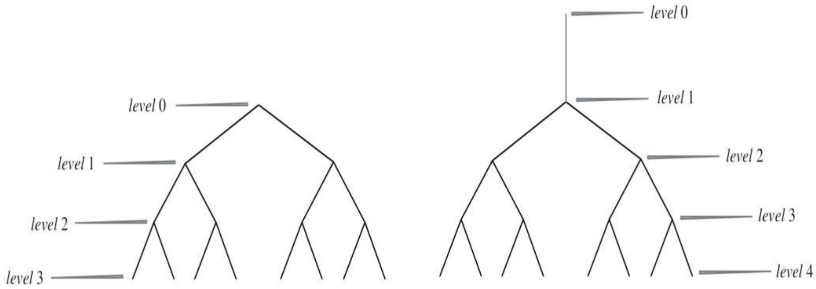

Figure 2: Complete binary tree T_4 and Rooted complete binary tree RT_5

For any non-negative integer r, the complete binary tree of height $r-1$, denoted by T_r, is the binary tree where each internal vertex has exactly two children and all the leaves are at the same level. Clearly, a complete binary tree T_r has $r-1$ levels and level i, $0 \leq i \leq r-1$, contains 2^i vertices. Thus T_r has exactly $2^r - 1$ vertices. The rooted complete binary tree RT_r is obtained from a complete binary tree T_{r-1} by attaching to its root a pendant edge. The new vertex is called the root of RT_r and is considered to be at level 0 and level i in T_{r-1} becomes $i+1$ in RT_r, where $0 \leq i \leq r-1$. See Figure 2.

Definition 3.1 *Let T_r be a complete binary tree, $r \geq 1$. A graph which is obtained from two copies of complete binary tree T_r say T_r^1, T_r^2 by joining P_2 to each vertex of T_r^1 and the corresponding vertex of T_r^2 is called a extended theta mesh and denoted by $ETM(r)$. See Figure* 3.

Definition 3.2 *Let RT_r be a rooted complete binary tree, $r \geq 1$. A graph which is obtained from two copies of rooted complete binary tree RT_r, say RT_r^1, RT_r^2 by joining P_2 to each vertex of RT_r^1 and the corresponding vertex of*

RT_r^2 *except level 0 is called a extended rooted theta mesh and denoted by* $ERTM(r)$. *See Figure* 3.

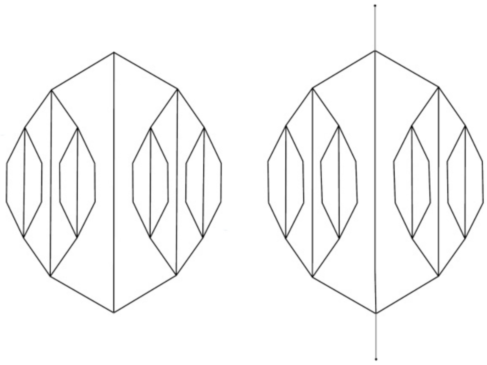

Figure 3: Extended theta mesh $ETM(4)$ and extended rooted theta mesh $ERTM(5)$

In this section, we compute the exact wirelength of embedding hypercube into extended rooted theta mesh. For proving the main result, we need the following results.

Lemma 3.3 *For* $i = 1,2,\ldots,r-1$, $NcutS_i^{2^i} = \{2^i, 2^i+1, \ldots, 2^{i+1}-1\}$ *is an optimal set in* Q^r.

Proof. Define $\varphi : NcutS_i^{2^i} \to L_{2^i}$ by $\varphi(2^i + k) = k$. If the binary representation of $2^i + k$ is $\alpha_1\alpha_2\cdots\alpha_r$ then the binary representation of k is $\underbrace{00\cdots00}_{r-i\text{times}}\alpha_{r-i+1}\alpha_{r-i+2}\cdots\alpha_r$. Thus the binary representation of two numbers x and y differ in exactly one bit $\Leftrightarrow$ the binary representation of $\varphi(x)$ and $\varphi(y)$ differ in exactly one bit. Therefore (x, y) is an edge in $NcutS_i^{2^i}$ $\Leftrightarrow$ $(\varphi(x), \varphi(y))$ is an edge in L_{2^i}. Hence $NcutS_i^{2^i}$ and L_{2^i} are isomorphic. By Theorem 2.7, $NcutS_i^{2^i}$ is an optimal set in Q^r.

Lemma 3.4 *For* $i = 1,2,\ldots,r-2$, $NcutS_1^i = \{0,1,2,\ldots,2^i-2, 2^{r-1}, 2^{r-1}+1, \ldots, 2^{r-1}+2^i-2\}$ *is an optimal set in* Q^r.

Proof. By Theorem 2.7, the set $\{0,1,2,\ldots,2^i-2\}$ is optimal and by Lemma 3.3, the set $\{2^{r-1}, 2^{r-1}+1, \ldots, 2^{r-1}+2^i-2\}$ is optimal in Q^r. Also the binary representation of k and $2^{r-1}+k$, $0 \le k \le 2^i - 2$, differ exactly in one bit. Therefore $\left|E(Q^r[NcutS_1^i])\right| = 2\left|E(Q^r[L_{2^i-1}])\right| + 2^i - 1 = 2i(2^{i-1} - i) + 2^i - 1 = (i+1)2^i - 2i - 1$. But by Lemma 2.8, $\left|E(Q^r[L_{2(2^i-1)}])\right| = (i+1)2^i - 2i - 1$ and hence by Theorem 2.7, $NcutS_1^i$ is an optimal set in Q^r.

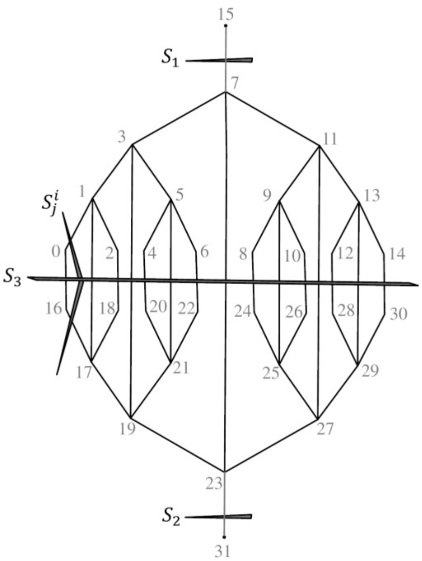

Figure 4: The edge cuts of $ERTM(5)$

Embedding Algorithm

Input : The r -dimensional hypercube Q^r and the r -dimensional extended rooted theta mesh $ERTM(r)$, $r \geq 2$
Algorithm : Label the vertices of Q^r by lexicographic order [35] from 0 to $2^r - 1$. Label the vertices of RT_r^1 and RT_r^2 in $ERTM(r)$ by in-order labeling starting from 0 to $2^{r-1} - 1$ and 2^{r-1} to $2^r - 1$ respectively, such that the label 2^{r-1} is adjacent to the label 0. See Figure 4.

Output : An embedding f of Q^r into $ERTM(r)$ given by $f(x) = x$ with minimum wirelength.

Theorem 3.5 *The exact wirelength of* Q^r *into* $ERTM(r)$, $r \geq 2$ *is given by*

$$WL(Q^r, ERTM(r)) = 2^{r-1}(r^2 - 5r + 11) - 2r.$$

4. Concluding Remark

In this paper, we compute the exact wirelength of embedding hypercube into extended rooted theta mesh. Finding the dilation of an embedding extended theta mesh into hypercube is under investigation.

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