IOP Conference Series: Materials Science and Engineering

PAPER • **OPEN ACCESS**

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To cite this article: B Mahaboob *et al* 2017 *IOP Conf. Ser.: Mater. Sci. Eng.* **263** 042121

IOP Conf. Series: Materials Science and Engineering **263** (2017) 042121 doi:10.1088/1757-899X/263/4/042121

Estimation of parameters of constant elasticity of substitution production functional model

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Abstract. Nonlinear model building has become an increasing important powerful tool in mathematical economics. In recent years the popularity of applications of nonlinear models has dramatically been rising up. Several researchers in econometrics are very often interested in the inferential aspects of nonlinear regression models [6]. The present research study gives a distinct method of estimation of more complicated and highly nonlinear model viz Constant Elasticity of Substitution (CES) production functional model. Henningen et.al [5] proposed three solutions to avoid serious problems when estimating CES functions in 2012 and they are i) removing discontinuities by using the limits of the CES function and its derivative. ii) Circumventing large rounding errors by local linear approximations iii) Handling ill-behaved objective functions by a multi-dimensional grid search. Joel Chongeh et.al [7] discussed the estimation of the impact of capital and labour inputs to the gris output agri-food products using constant elasticity of substitution production function in Tanzanian context. Pol Antras [8] presented new estimates of the elasticity of substitution between capital and labour using data from the private sector of the U.S. economy for the period 1948-1998.

1. Introduction

In Economics CES is properly of some utility functions and production functions. The CES production function is a neoclassical production function which exhibits CES. That is production technology has a constant percentage change in factor proposition due to a percentage change in marginal rate of technical substitution. American economist Robert Merton Solow first introduced two factor (labour, capital) CES production function and later it was popularized by renowned economists Kenneth Arrow, Chenery, Minhas, Solow [1].

2. Specification and properties of nonlinear constant elasticity of substitution production functional model

Consider the standard nonlinear regression model with usual assumptions in matrix notation as

$$Y_{n\times1} = f_{n\times1}(\beta) + \varepsilon_{n\times1} \tag{2.1}$$

and $\varepsilon \sim N_n\left(O, \sigma^2 I_n\right)$

Here, β is $(p\times1)$ vector of unknown parameters. Where a nonlinear hypothesis for the test procedure as $H_0 : h(\beta) = 0$ against $H_1 : h(\beta) \neq 0$

Where h is a q-vector valued nonlinear function and q < p and $h(\beta)$ is continuously first order differentiable function mapping $\mathbb{R}^p$ into $\mathbb{R}^q$ with Jacobian

$$H(\beta) = \frac{\partial}{\partial \beta'} h(\beta) \tag{2.2}$$

Here, $H(\beta)$ is of order $q \times p$ .By evaluating $H(\beta)$ at $\beta = \hat{\beta}_n$, where $\hat{\beta}_n$ is iterative NLLS estimator for β , one may write $\hat{H}_n = H(\hat{\beta}_n)$

Under characterizations of least squares estimators,

$$\hat{\beta}_n = \beta_n + \left[F'(\beta_n)F(\beta_n)\right]^{-1} F'(\beta_n)\varepsilon + O_p\left(\frac{1}{\sqrt{n}}\right) \tag{2.3}$$

Where, $F(\beta) = \frac{\partial}{\partial \beta'} f(\beta)$, $h(\hat{\beta}_n)$ may be characterized as

$$h(\hat{\beta}_n) = h(\beta) + H(\beta)\left[F'(\beta)F(\beta)\right]^{-1} F'(\beta)\varepsilon + O_p\left(\frac{1}{\sqrt{n}}\right) \tag{2.4}$$

Ignoring the remainder term, one may obtain,

$$h(\hat{\beta}_n) \overset{\text{approx}}{\sim} N_q\left[h(\beta),\ \sigma^2 H(\beta)\left[F'(\beta)F(\beta)\right]^{-1} H'(\beta)\right] \tag{2.5}$$

Thus, $$\left[\frac{h'(\hat{\beta}_n)\left[H(\beta)\left[F'(\beta)F(\beta)\right]^{-1} H'(\beta)\right]^{-1} h(\hat{\beta}_n)}{\sigma^2}\right] \overset{\text{approx}}{\sim} \text{Noncentral } \chi_q^2 \tag{2.6}$$

andnoncentrality parameter is given by

$$\lambda = \frac{h'(\beta)\left[H(\beta)\left[F'(\beta)F(\beta)\right]^{-1} H'(\beta)\right]^{-1} h(\beta)}{2\sigma^2} \tag{2.7}$$

Further, within the order of approximation, $\frac{(n-p)S^2}{\sigma^2}$ is distributed independently of $\hat{\beta}_n$ as the χ^2 distribution with $(n-p)$ degrees of freedom. Here, S^2 is an unbiased estimator of unknown error variance σ^2 . Hence, the ratio

$$\frac{h'(\hat{\beta}_n)\left[H(\beta)\left[F'(\beta)F(\beta)\right]^{-1} H'(\beta)\right]^{-1} h(\hat{\beta}_n) \Big/ (q\sigma^2)}{(n-p)S^2 \Big/ \left[(n-p)\sigma^2\right]}$$

or $$\frac{h'(\hat{\beta}_n)\left[H(\beta)\left[F'(\beta)F(\beta)\right]^{-1} H'(\beta)\right]^{-1} h(\hat{\beta}_n)}{qS^2} \sim \text{Noncentral}\, F_{[q,(n-p)]} \tag{2.8}$$

With noncentrality parameter λ

By substituting the estimates $\hat{H}_n$ and $\hat{D}_n$ for $H(\beta)$ and $\left[F'(\beta)F(\beta)\right]^{-1}$ respectively the Modified Wald test statistic for testing $H_0 : h(\beta)=0$ is given by

$$W^* = \frac{h'\left(\hat{\beta}_n\right)\left[\hat{H}_n\hat{D}_n\hat{H}_n^{'}\right]^{-1} h\left(\hat{\beta}_n\right)}{qS^2} \sim F_{[q,(n-p),\lambda]} \tag{2.9}$$

3. Estimation of parameters of CES production functional model

The constant elasticity of substitution (CES) production functional model form was first given by Dickinson and later popularized by Arrow, Chenery, Minhas and Solow [1]. This function cannot be linearized by logarithmic transformation and its estimation is more complex than the Cobb-Douglas [4] production functional model.

The general form of CES production functional model [5] is given by

$$Y = \gamma\left[\delta X_2^{-\rho} + (1-\delta)X_1^{-\rho}\right]^{-\nu/\rho}, \qquad \nu > 0 \tag{3.1}$$

Where, Y : output, X_2: Capital Input , X_1: Labor Input, γ : Efficiency parameter δ: Distribution parameter, ρ : Substitution parameter, ν : Returns to Scale parameter

The first order partial derivatives of Y with respect to inputs X_1 and X_2 yield,

$$\frac{\partial Y}{\partial X_2} = \nu\delta\gamma^{-\left(\frac{\rho}{\nu}\right)}\left[Y^{1+\frac{\rho}{\nu}} \Big/ X_2^{1+\rho}\right] \tag{3.2}$$

and $$\frac{\partial Y}{\partial X_1} = \nu(1-\delta)\gamma^{-\left(\frac{\rho}{\nu}\right)}\left[Y^{1+\frac{\rho}{\nu}} \Big/ X_1^{1+\rho}\right] \tag{3.3}$$

Here, $\frac{\partial Y}{\partial X_2}$ and $\frac{\partial Y}{\partial X_1}$ give the marginal products of capital and labor inputs respectively.

In practice, the prices of labor and capital are not directly measurable; the wage rate (w) and the rate of interest (r) may be taken as their corresponding prices respectively.

From the Mathematical Economics, under Theory of Firms, the marginal Productivity conditions give

$$\frac{\partial Y}{\partial X_1} = w \text{ and } \frac{\partial Y}{\partial X_2} = r$$

$$\Rightarrow \qquad w = \nu(1-\delta)\gamma^{-\left(\frac{\rho}{\nu}\right)}\left[Y^{1+\left(\frac{\rho}{\nu}\right)} \Big/ X_1^{1+\rho}\right] \tag{3.4}$$

$$\text{and} \qquad r = \nu\delta\gamma^{-\left(\frac{\rho}{\nu}\right)}\left[Y^{1+\left(\frac{\rho}{\nu}\right)} \Big/ X_2^{1+\rho}\right] \tag{3.5}$$

$$\Rightarrow \qquad \frac{w}{r} = \left[\frac{1-\delta}{\delta}\right]\left[\frac{X_2}{X_1}\right]^{1+\rho} \tag{3.6}$$

By taking logarithms on both sides of the equations (3.4), (3.5) and (3.6) and introducing the classical error variables, one may obtain

$$\mathrm{Ln\,w} = \mathrm{Ln}\left[\nu\left(1-\delta\right)\gamma^{\frac{-\rho}{\nu}}\right]+\left[1+\left(\frac{\rho}{\nu}\right)\right]\mathrm{Ln\,Y}-\left(1+\rho\right)\mathrm{LnX_1}+\mathrm{u_1} \tag{3.7}$$

$$\mathrm{Ln\,r} = \mathrm{Ln}\left[\nu\delta\gamma^{\frac{-\rho}{\nu}}\right]+\left[1+\left(\frac{\rho}{\nu}\right)\right]\mathrm{Ln\,Y}-\left(1+\rho\right)\mathrm{LnX_2}+\mathrm{u_2} \tag{3.8}$$

$$\mathrm{Ln}\left(\frac{\mathrm{w}}{\mathrm{r}}\right) = \mathrm{Ln}\left[\frac{1-\delta}{\delta}\right]+\left[1+\rho\right]\mathrm{Ln}\left(\frac{\mathrm{X_2}}{\mathrm{X_1}}\right)+\mathrm{u_3} \tag{3.9}$$

The equations (3.7), (3.8) and (3.9) may be written as

$$\mathrm{w}^* = \eta_0+\eta_1\mathrm{Y}^*+\eta_2\mathrm{X}_1^*+\mathrm{u}_1 \tag{3.10}$$

$$\mathrm{r}^* = \theta_0+\theta_1\mathrm{Y}^*+\theta_2\mathrm{X}_2^*+\mathrm{u}_2 \tag{3.11}$$

$$\mathrm{Q}^* = \mu_0+\mu_1\mathrm{Z}^*+\mathrm{u}_3 \tag{3.12}$$

where $\eta_0 = \mathrm{Ln}\left[\nu\left(1-\delta\right)\gamma^{-\frac{\rho}{\nu}}\right], \eta_1 = \left(1+\frac{\rho}{\nu}\right), \quad \eta_2 = -\left(1+\rho\right)$

$$\theta_0 = \mathrm{Ln}\left[\nu\delta\gamma^{-\frac{\rho}{\nu}}\right], \quad \theta_1 = \left(1+\frac{\rho}{\nu}\right), \quad \theta_3 = -\left(1+\rho\right)$$

$$\mu_0 = \mathrm{Ln}\left[\frac{1-\delta}{\delta}\right], \quad \mu_1 = \left[1+\rho\right]$$

$$\mathrm{w}^* = \mathrm{Ln\,w};\ \mathrm{r}^* = \mathrm{Ln\,r},\ \mathrm{Q}^* = \mathrm{Ln}\left(\frac{\mathrm{w}}{\mathrm{r}}\right),\ \mathrm{Z}^* = \mathrm{Ln}\left(\frac{\mathrm{X_2}}{\mathrm{X_1}}\right);\ \mathrm{Y}^* = \mathrm{LnY},\ \mathrm{X}_1^* = \mathrm{LnX_1},$$

$\mathrm{X}_2^* = \mathrm{LnX_2}$; $\mathrm{u_1}$, $\mathrm{u_2}$ and $\mathrm{u_3}$ are classical error variables.

By considering (3.10), (3.11) and (3.12) multiple linear regression models and applying ordinary least squares (OLS) method of estimation, one can obtain the OLS estimators of parameters in the equations respectively as,

(i) $\hat{\eta}_0, \hat{\eta}_1$ and $\hat{\eta}_2$ (ii) $\hat{\theta}_0, \hat{\theta}_1$ and $\hat{\theta}_2$ and (iii) $\hat{\mu}_0, \hat{\mu}_1$

Now, the estimators of ρ , δ , γ and ν are given by

$$\hat{\rho} = \left(1+\hat{\eta}_2\right) \tag{3.13}$$

$$\hat{\upsilon} = \left[\frac{\hat{\eta}_2+1}{\hat{\eta}_1-1}\right] \tag{3.14}$$

$$\hat{\delta} = \left[\frac{1}{1+\mathrm{Anti\,Ln}\,\hat{\mu}_0}\right] \tag{3.15}$$

4. Estimation of parameters of CES production functional model [2] by Taylor series expansion method

Consider the constant elasticity of substitution (CES) production functional model as

$$\mathrm{Y} = \gamma\left[\eta\mathrm{X}_2^{-\rho}+\left(1-\eta\right)\mathrm{X}_1^{-\rho}\right]^{-\nu/\rho}\mathrm{e}^{\varepsilon}, \qquad \nu>0 \tag{4.1}$$

Here, ε is a classical error variable, by taking logarithms on both sides of equation (4.1) gives

$$\mathrm{Ln\,Y} = \mathrm{Ln}\,\gamma-\left(\frac{\nu}{\rho}\right)\mathrm{Ln}\left[\eta\mathrm{X}_2^{-\rho}+\left(1-\eta\right)\mathrm{X}_1^{-\rho}\right]+\varepsilon \tag{4.2}$$

Expanding Ln Y by Taylor series formula for expansion about $\rho = 0$, after discarding the terms of third and higher orders, the expansion yields,

$$\mathrm{Ln\,Y} = \mathrm{Ln}\,\gamma + \nu\eta\,\mathrm{Ln\,X}_2 + \nu(1-\eta)\,\mathrm{Ln\,X}_1 - \left(\frac{1}{2}\right)\rho\nu\eta(1-\eta)\left[\mathrm{Ln\,X}_2 - \mathrm{Ln\,X}_1\right]^2 + \varepsilon \tag{4.3}$$

$$\Rightarrow \mathrm{Y}^* = \beta_0 + \beta_1\mathrm{X}_1^* + \beta_2\mathrm{X}_2^* + \beta_3\mathrm{X}_3^* + \varepsilon \tag{4.4}$$

where, $\mathrm{Y}^* = \mathrm{Ln\,Y}$, $\mathrm{X}_1^* = \mathrm{Ln\,X}_1$, $\mathrm{X}_2^* = \mathrm{Ln\,X}_2$ $\mathrm{X}_3^* = \left[\mathrm{Ln\,X}_2 - \mathrm{Ln\,X}_1\right]^2$

$$\beta_0 = \mathrm{Ln}\,\gamma,\ \beta_1 = \nu(1-\eta),\ \beta_2 = \nu\eta,\ \beta_3 = -\left(\frac{1}{2}\right)\rho\nu\eta(1-\eta)$$

Thus, the nonlinear CES production functional model (4.1) reduces to a multiple linear regression model (4.4).The application of Ordinary Least Squares (OLS) method [3] of estimation to the model (4.4) yields the OLS estimators of parameters $\beta_0, \beta_1, \beta_2$ and β_3 as $\hat{\beta}_0$ $\hat{\beta}_1, \hat{\beta}_2$ and $\hat{\beta}_3$ respectively. Now the estimators of parameters of the CES production functional model are given by

$$\hat{\gamma} = \mathrm{Anti\,Ln}\left(\hat{\beta}_0\right),\ \hat{\eta} = \frac{\hat{\beta}_2}{\hat{\beta}_1 + \hat{\beta}_2},\ \hat{\nu} = \hat{\beta}_1 + \hat{\beta}_2 \text{ and } \hat{\rho} = -2\left[\frac{\hat{\beta}_1 + \hat{\beta}_2}{\hat{\beta}_1\hat{\beta}_2}\right]\hat{\beta}_3$$

5. Conclusions

The CES function is popular in several areas of economics but it is rarely used in economics analysis because it cannot be estimated by standard linear regression techniques. In the above research work one of the inferential aspects (estimation of parameters) of highly nonlinear model namely CES production model has been discussed. Though a large number of estimating methods of this has been specified in the existing literature, a new namely OLS method of estimation has been discussed.

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