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RESEARCH PAPER

Exploration on initial structures of extrasolar protoplanets via new explicit RKAHeM(4,4) method

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Abstract In this paper, a newly proposed embedded Runge–Kutta fourth order with four stages arithmetic Heronian mean method is employed and verified in determining the distribution of thermodynamic variables inside the extra-solar protoplanets formed through gravitational instability at their initial stages. In specific, the case of conduction–radiation is considered regarding the transference of heat inside the protoplanets. A general brief theoretical framework for the proposed numerical method is stated in addition to pseudo code followed by error estimation description. The results based on newly proposed explicit RKAHeM(4,4) method are found to be optimal and efficient in comparison with the ones obtained with classical fourth order Runge–Kutta method.

1. Introduction

Indeed numerical computations play an indispensable and significant role in solving real time mathematical, physical and engineering problems to provide optimal and efficient solutions. In the case of numerical computation, three phases are of importance, such as construction of appropriate numerical technique, implementation of the method to obtain effective solution and validation of the obtained results. But, before selection and/or construction of new techniques, one needs to consider different factors, namely types of equations, types of availability of machines/systems, programing and maintenance, execution speed, accuracy of the obtained solutions and their validity, etc. The proposed explicit RKAHeM(4,4) is an embedded hybrid Runge–Kutta method to solve various real world problems involving ordinary differential equations arising in some of the scientific areas such as

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celestial mechanics, weather modeling, reaction rates, infectious diseases, genetic variation, population competition and stock trends, interest rates and the market equilibrium price changes. Any type of Runge–Kutta method including RKAHeM(4,4) is said to be consistent, if the truncation error tends to zero when globally the step size tends to zero.

Shampine and Gordon (1975) discussed the normal order of a Runge–Kutta algorithm having the approximate number of leading terms of an infinite Taylor series, which calculates the trajectory of a moving point. Evans and Yaacob (1995) introduced a new fourth order Runge–Kutta method based on the Heronian mean formula for solving initial value problem in numerical analysis. Bader (1987, 1998) introduced the RK–Butcher algorithm for finding the truncation error estimates, intrinsic accuracies and the early detection of stiffness in coupled differential equations that arises in theoretical chemistry problems. A new fourth order embedded RKAHeM(4,4) method and algorithm based on Runge–Kutta arithmetic Heronian mean with error control is proposed to solve the real time application problem under raster scheme by Ponalagusamy and Senthilkumar (2009) and through time-multiplexing approach by Ponalagusamy and Senthilkumar (2011) in image processing under CNN environment efficiently. A detailed illustration related to local truncation error (LTE), global truncation error (GTE) and error estimates (ERREST) and control for fourth order with four stages Runge–Kutta numerical algorithms is eventually reported by Senthilkumar (2009).

The formation of the planetary system has been a topic of interest to the mankind ever since the dawn of civilization. However, scientific theories for the formation of the system largely date from Descartes (1644) when he proposed his vortex theory of planetary formation. Since then many theories with regard to the planetary formation both inside and outside the solar system have been advanced through the works of many authors (e.g., McCrea and Williams, 1965; Cameron, 1978; Boss 1997; Boley et al., 2010; Cha and Nayakshin, 2011). The viable mechanism, disk instability, advocated in the past, in principle, can form giant planets and are believed to be the promising route for the rapid formation of giant planets both in our solar system and elsewhere (Boss, 1997, 1998). This theory with disk instability and the gravitational collapse of an unsegregated protoplanet was actually in vogue during 1970s when a great deal of now forgotten work was carried out has been reformulated with fragmentation from massive protoplanetary disks (see, Paul et al., 2012) and has been advanced through the works of many authors (see, e.g., Nayakshin, 2010; Mayer et al., 2002, 2004; Cai et al., 2006; Boley et al., 2010; Cha and Nayakshin, 2011). But despite substantial study and progress in recent decades, the initial structures of isolated gaseous giant protoplanets formed via disk instability are still unknown and different models predict different initial characteristics (see, Boss, 1997; Helled and Schubert, 2008; Senthilkumar and Paul, 2012; Paul and Bhattacharjee, 2013). Boss (1997) in his investigation assumed an initial protoplanet to be fully radiative, Helled and Schubert (2008) found such protoplanets to be fully convective with a thin outer radiative zone. Paul et al. (2012) and Senthilkumar and Paul (2012) in their investigations neglected the radiative thin zone and assumed initial protoplanets to be fully convective, while Paul and Bhattacharjee (2013) analyzed initial structures of protoplanets assuming them to be in conductive-radiative equilibrium.

In this study, we intend to reinvestigate the problem of Paul and Bhattacharjee (2013) employing the newly proposed explicit RKAHeM(4,4) algorithm to see how the obtained results compare the ones obtained in Paul and Bhattacharjee (2013) and those obtained through other investigations. Further, it is our interest to exhibit that if masses and radii are prescribed, then the distribution of thermodynamic variables can uniquely be determined suggesting that formation of protoplanets via disk instability is a reasonable hypothesis.

The remainder of the article is structured as follows. Section 2 discusses theoretical foundation of the problem in addition to boundary conditions. Section 3 addresses a notion on structure determination including numerical approach employed. A brief note on the analysis of explicit embedded Runge–Kutta fourth order with four stages arithmetic Heronian mean method is presented in Section 4 along with pseudo code, local truncation error and error control. Results and discussion are presented in Section 5. Finally, conclusion is given in Section 6.

2. Theoretical foundation

As in Paul and Bhattacharjee (2013), our model assumes a non-rotating, non-magnetic spherical giant gaseous object in the mass range 0.3–10 M_J, where M_J represents the mass of Jupiter. The object being formed via disk instability and is assumed to be of solar composition, which is in a steady state of quasi-static equilibrium. The mode of heat transport inside such an object is assumed to be conductive-radiative. Then the structure of the object, following Paul et al. (2008) and Paul and Bhattacharjee (2013), can be given by the following set of equations.

The equation of hydrostatic equilibrium,

$$\frac{dP(r)}{dr} = -\frac{GM(r)}{r^2}\rho(r). \tag{1}$$

The equation of conservation of mass,

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho(r). \tag{2}$$

The equation of conductive-radiative heat flux,

$$\left(\frac{8\sigma H}{3\times 10^{-24}}\frac{T^3(r)}{\rho(r)} + \eta\right)\frac{dT(r)}{dr} = -\frac{C_R}{4\pi R}\frac{GM^2(r)}{r^3}. \tag{3}$$

The gas law,

$$P(r) = \frac{k}{\mu H}\rho(r)T(r). \tag{4}$$

In Eqs. 1–4, $P(r)$, $M(r)$, $T(r)$ and $\rho(r)$ represent pressure, mass, temperature and density, respectively, inside a protoplanet with r measuring the distance from the center; H is the mass of a hydrogen atom; η is the thermal conductivity of the gas of the protoplanet; G the universal gravitational constant; μ is the mean molecular weight; C_R is an arbitrary constant; σ is the Stefan–Boltzmann constant, and k is the Boltzmann constant.

2.1. Boundary conditions

The mass inside a sphere of infinitesimal radius r at the center of a protoplanet treating ρ sensibly constant in this sphere is given by $M(r) = \frac{4}{3}\pi r^3\rho$. Hence, as $r \to 0$, $M(r) \to 0$. Clearly, $M(r) = M$ at the surface. Also, central temperature of an initially formed protoplanet should fairly be low (Paul et al., 2011) and hence it must have low surface temperature. We assume in the first approximation that the surface temperature is zero. Furthermore, the mass of the atmosphere of a protoplanet is just a minute fraction of its total mass, so we may take the pressure on its surface as approximately equal to zero. Therefore, the approximate boundary conditions can be given by

$$\left.\begin{array}{lll} T=0, \quad P=0 & \text{at} & r=R \\ M(r)=M & \text{at} & r=R \\ M(r)=0 & \text{at} & r=0 \end{array}\right\}. \tag{5}$$

3. Structure determination

3.1. Non-dimensionalization

We introduce the non-dimensional variables x, p, t, and q following Schwarzschild (1958) with the help of the transformations $r = (1-x)R$, $P(r) = \frac{GM^2}{4\pi R^4}p(x)$, $T(r) = \frac{\mu HGM}{kR}t(x)$, $M(r) = q(x)M$. By means of the above transformations, Eqs. (1)–(3) with the help of Eq. (4) can be, respectively, put in the form

$$\frac{dp}{dx} = \frac{pq}{t(1-x)^2}, \tag{6}$$

$$\frac{dq}{dx} = -\frac{p(1-x)^2}{t} \tag{7}$$

and

$$\frac{dt}{dx} = C_R \frac{cpq^2}{(1-x)^3(at^4+bp)}, \tag{8}$$

as ρ, given by Eq. (4), with the help of the above transformations can be put to the form

$$\rho = \frac{M}{4\pi R^3}\frac{p}{t}. \tag{9}$$

In Eq. (8), $a = \frac{8\sigma H}{3\times 10^{-24}}\left(\frac{\mu HGM}{kR}\right)^3$, $b = \frac{M\eta}{4\pi R^3}$, and $c = \frac{M^2k}{16\pi^2R^5\mu H}$.

The boundary conditions given by Eq. (5), then in terms of the non-dimensional variables are transformed to

$$\left.\begin{array}{lll} t(x)=0, \quad p(x)=0 & \text{at} & x=0 \\ q(x)=1 & \text{at} & x=0 \\ q(x)=0 & \text{at} & x=1 \end{array}\right\}. \tag{10}$$

3.2. Numerical values used

Countable numbers of parameters are involved in our calculations. The values of masses and radii in this study are similar to those used in the study of Paul and Bhattacharjee (2013). Besides the values of masses and radii, we have used $\mu = 2.3$ (Dullemond and Dominik, 2004), $\sigma = 5.6686 \times 10^{-5}$ erg cm^{-2} deg^{-2} sec, $\eta = 1.2684 \times 10^4$ and the remaining parameters involved in the study have been assumed to have their standard values.

3.3. Numerical approach

It is obvious that analytic solution of Eqs. 6–8 as they stand is nevertheless possible. Therefore, any suitable numerical method and its algorithm should be adopted to yield respective solutions. It is pertinent to point out here that the numerical calculations cannot be started right either from the surface or from the center for the existence of vanishing denominators in the basic set of equations. These factors indicate that one needs to develop solution near either of the boundaries. Based on the boundary conditions, we have developed the solution near the surface, which can be put in the form

$$p \approx a_0\frac{y^4}{(1-x)^4}, \quad t \approx \frac{c_0 y}{(1-x)}, \quad q \approx 1, \quad \text{as } x \to 0,$$

where $c_0 = 0.25$ and $a_0 = \frac{ac_o^5}{C_R c - bc_0}$.

With the approximated values mentioned above as initial conditions, inserting the values of the required parameters involved, we have solved Eqs. 6–8 numerically by the proposed new explicit RKAHeM(4,4) method from $x = 0.01$ downwards to the point 0.99 to obtain the distribution of p, q, and t. The proposed explicit RKAHeM(4,4) method employed in the study is discussed in Section 4. As the distribution of p and t for varying x are known, we have determined the density distribution using Eq. (9). The structures of the protoplanets are found to be dependent on the parameter C_R and best values of C_R for the prescribed protoplanetary masses 0.3 M_J, 1 M_J, 3 M_J, 5 M_J, 7 M_J and 10 M_J satisfying the third condition of Eq. (10) can be found alike those obtained in the study of Paul and Bhattacharjee (2013) as 0.026, 0.2, 1.27, 2.43, 4.03 and 8.4, respectively. The results of our calculations are shown diagrammatically through Figs. 1–4.

4. Analysis about newly proposed explicit RKAHeM(4,4) numerical method

The formulation of RK(4,4) for solving the initial value problem $y' = f(x, y(x))$, $x_0 \leqslant x \leqslant x_n$ subject to $y(x_0) = y_0$ is based on the general form of the extrapolation equation

$$y_{n+1} = y_n + \Delta y$$

or

$$y_{n+1} = y_n + \text{slope} \times \text{interval size}$$

or

$$y_{n+1} = y_n + mh, \tag{11}$$

where 'm' represents the slope, i.e., weighted average of the slopes at various points in the interval with interval size h.

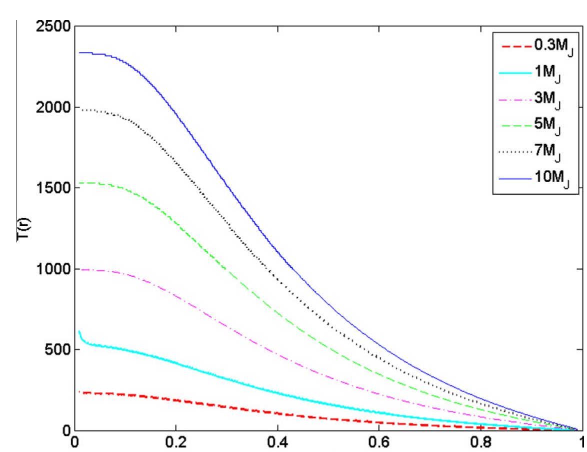

Figure 1 Temperature profiles inside some initial protoplanets.

Suppose, if one estimates 'm' using slopes at 'r' points in the interval $[x_n, x_{n+1}]$, then 'm' can be written as $m = w_1m_1 + w_2m_2 + \cdots + w_rm_r$, where w_1, $w_2,\ldots$, w_r are weights of the slopes at various points.

Therefore, $y_{n+1} = y_n + w_1m_1h + \cdots + w_rm_rh$

or

$$y_{n+1} = y_n + \sum_{i=1}^{r} w_im_ih$$

or

$$y_{n+1} = y_n + \sum_{i=1}^{r} w_ik_i \tag{12}$$

By considering the above initial value problem without loss of generality,

$$\frac{dy}{dx} = f(x,y),\ \ x_0 \leqslant x \leqslant x_n \text{ subject to, } y(x_0) = y_0, \tag{13}$$

the general s-stage Runge–Kutta method can be written as follows.

$$k_1 = hf(x_n, y_n);\ k_i = hf(x_n + c_ih, y_n + \sum_{j=1}^{i-1} a_{ij}k_j);\ i = 2, \ldots, s, \tag{14}$$

$$y_{n+1} = y_n + \sum_{i=1}^{s} w_ik_i. \tag{15}$$

The Runge–Kutta formula approximates the integrand $f(x_n, \varphi(x_n))$ with a weighted average of its values at the two endpoints and at the midpoint, where φ represents slope. Runge–Kutta methods are a specialization of one-step numerical methods which essentially characterize that the error is of the form $E_i = Ch^k$, where C is a positive real constant, the number k is called the order of the method. In case of Runge–Kutta method, number of stages implies that number of times the function is evaluated at each one step i, which is important concept because, the evaluation of the function requires a computational cost (sometimes higher) and therefore preferred methods with minimum number of stages as possible.

4.1. Pseudo code of explicit RKAHeM(4,4) technique for solving IVPs

The pseudo code for newly proposed explicit RKAHeM(4,4) method to solve any initial value problem prescribed by Eq. (13) is as follows:

Input:	initial values x_0, y_0 and number of steps n
Output:	approximation to y at the n + 1 values of x
Step 1:	Set $h = (x_n - x_0)/n$
	$x = x_0$
	$y = y_0$
	Output (x, y, y^*)
Step 2:	For $i = 1$ to n do steps 3-5
Step 3:	Set $k_1 = f(x, y) = k_1^*$
	$k_2 = f(x + \frac{h}{2}, y + \frac{hk_1}{2}) = k_2^*$
	$k_3 = f(x + \frac{h}{2}, y + \frac{hk_2}{2})$
	$k_4 = f(x + h, y + hk_3)$
	$k_3 = f(x + \frac{1}{2}h, y - \frac{1}{48}hk_1 + \frac{25}{48}hk_2) = k_3^*$
	$k_4 = f(x + h, y - \frac{1}{24}hk_1 + \frac{47}{600}hk_2 + \frac{289}{300}hk_3) = k_4^*$
Step 4:	Set $y = y + \frac{h}{3}[\frac{k_1+k_2}{2} + \frac{k_2+k_3}{2} + \frac{k_3+k_4}{2}]$ (Compute y_i)
	$y^* = y + \frac{h}{9}[k_1^* + 2(k_2^* + k_3^*) + k_4^* + \sqrt{\lvert k_1^*k_2^*\rvert}$
	$+ \sqrt{\lvert k_2^*k_3^*\rvert} + \sqrt{\lvert k_3^*k_4^*\rvert}]$ (Compute y_i^*)
	$x = x + ih$ (Compute x_i)
Step 5:	Output (x, y, y^*)
Step 6:	Stop

The new embedded RKAHeM(4,4) is represented by Butcher array form as

$$\begin{array}{c|ccc}
0 & & & \\
\frac{1}{2} & \frac{1}{2} & & \\
\frac{1}{2} & 0 & \frac{1}{2} & \\
1 & 0 & 0 & 1 \\
 & \cdots & \cdots & \cdots \\
 & \cdots & \cdots & \cdots \\
\frac{1}{2} & \frac{-1}{48} & \frac{25}{48} & \\
1 & \frac{-1}{24} & \frac{47}{600} & \frac{289}{300} \\
\hline
 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
\hline
 & \mathrm{E^T} & &
\end{array} \tag{16}$$

where

$$b^T = y_{n+1}^{\mathrm{AM}} = y_n + \frac{h}{3}\left[\frac{k_1+k_2}{2} + \frac{k_2+k_3}{2} + \frac{k_3+k_4}{2}\right]. \tag{17}$$

$$\hat{b}^T = y_{n+1}^{\mathrm{HeM}} = y_n + \frac{h}{9}\left[k_1^* + 2(k_2^* + k_3^*) + k_4^* + \sqrt{\lvert k_1^*k_2^*\rvert} + \sqrt{\lvert k_2^*k_3^*\rvert} + \sqrt{\lvert k_3^*k_4^*\rvert}\right] \tag{18}$$

and the estimation of the local truncation error, $E^T = \lvert b^T - \hat{b}^T\rvert$. In the RKAHeM(4,4) method, four stages are required to obtain the solution, which share the same set of vectors k_1 and k_2 using b^T and $\hat{b}^T$ approximately, but k_3 and k_4 use b^T while k_3^* and k_4^* use $\hat{b}^T$.

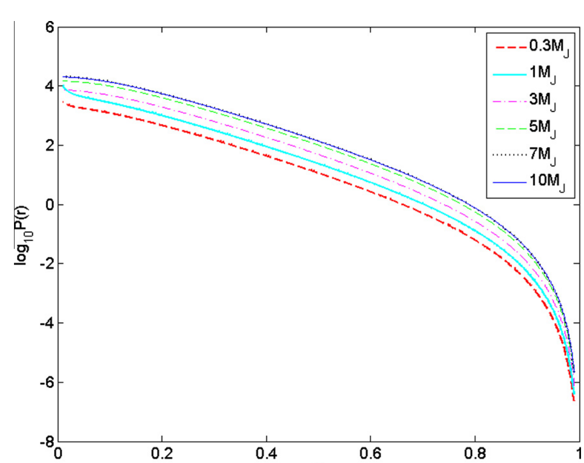

Figure 2 Pressure profiles inside some initial protoplanets.

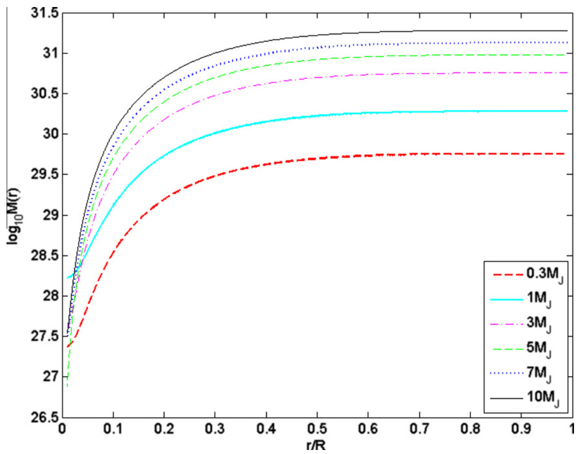

Figure 3 Mass distribution inside some initial protoplanets.

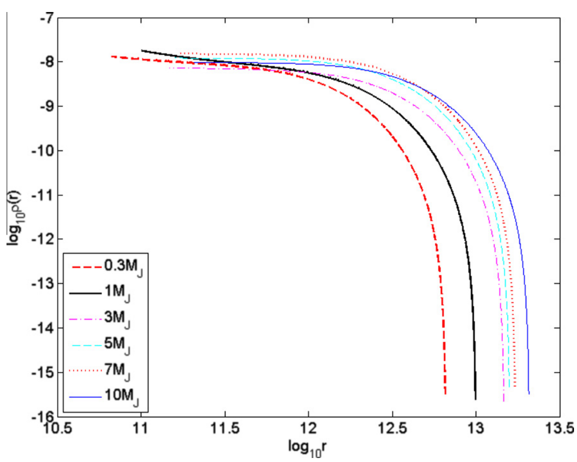

Figure 4 Density distribution inside some initial protoplanets.

4.2. Derivation of LTE for explicit RKAHeM(4,4) technique

According to Lotkin (1951), Ralston (1957) and Lambert (1973, 1980), the error estimate for the fourth order RK scheme is given by $|\psi(x_n, y_n : h)| \leqslant (73/720)ML^4$, where M and L are positive constants. To control the value of step-size (h), we use the new formation of RKAHeM(4,4) to obtain an estimate of the local truncation error (LTE) for the RKAHeM(4,4) as LTE $= y_{n+1} - y^*_{n+1}$. The LTE for the classical fourth order Runge–Kutta method based on arithmetic mean (RKAM) is given by

$$y^{\mathrm{AM}}_{n+1} = y_n + \mathrm{LTE}_{\mathrm{AM}}, \tag{19}$$

and the LTE for the Runge–Kutta method based on Heronian mean (RKHeM) is given by

$$y^{\mathrm{HeM}}_{n+1} = y_n + \mathrm{LTE}_{\mathrm{HeM}}, \tag{20}$$

where y^{AM}_{n+1} and $y^{\mathrm{HeM}}_{n+1}$are numerical approximations at x_{n+1} obtained by AM and HeM, respectively, but $\mathrm{LTE}_{\mathrm{AM}}$ and $\mathrm{LTE}_{\mathrm{HeM}}$ are the LTEs in RKAM and RKHeM, respectively. Then at x_{n+1},

$$y^{\mathrm{AM}}_{n+1} - y^{\mathrm{HeM}}_{n+1} = \mathrm{LTE}_{\mathrm{AM}} - \mathrm{LTE}_{\mathrm{HeM}}. \tag{21}$$

But the LTE for RKAM and RKAHeM are, respectively, given by

$$\mathrm{LTE}_{\mathrm{AM}} = \frac{h^5}{2880}\left[-24ff_y^4 + f^4 f_{yyyy} + 2f^3 f_y f_{yyy} - 6f^3 f_{yy}^2 + 36 f^2 f_y^2 f_{yy}\right], \tag{22}$$

$$\mathrm{LTE}_{\mathrm{HeM}} = \frac{h^5}{1,658,880}\left[19,759ff_y^4 - 19,926f^2 f_y^2 f_{yy} - 36,576 f^3 f_{yy}^2 - 13,292 f^3 f_y f_{yyy} + 576 f^4 f_{yyyy}\right]. \tag{23}$$

The absolute difference between $\mathrm{LTE}_{\mathrm{AM}}$ and $\mathrm{LTE}_{\mathrm{HeM}}$ is given by

$$|\mathrm{LTE}_{\mathrm{AM}} - \mathrm{LTE}_{\mathrm{HeM}}| = \frac{h^5}{1,658,880}\left[33,583ff_y^4 + 40,662f^2 f_y^2 f_{yy} + 33,120 f^3 f_{yy}^2 + 14,444 f^3 f_y f_{yyy}\right]. \tag{24}$$

According to Lotkin (1951), the step-size (h) selection can be determined in order to control the error as

$$\frac{121,809}{1,658,880} P^4 Q h^5 < \mathrm{Tol} \text{ or } h < \left[\frac{13.618698 \times \mathrm{Tol}}{P^4 Q}\right]^{1/5}, \tag{25}$$

where Tol = 0.00005, P and Q are positive constants.

4.2.1. Error estimation for RKAHeM(4,4) technique

It is significant to point out that in explicit RKAHeM(4,4) technique with error control program, we choose error estimate as the difference between the fourth order Arithmetic Mean (RKAM) method and the Heronian Mean (HeM) method. From Eq. (25), the error estimation (ERREST) can be expressed as (see, Senthilkumar and Paul, 2012)

$$\mathrm{ERREST} = |Y_{AM} - Y_{HeM}| \times \frac{121,809}{1,658,880}. \tag{26}$$

5. Results and discussion

We have determined the structures of the protoplanets formed by disk instability in the mass range 0.3–10 Jovian masses in their initial stages under approximate zero boundary condition by the novel explicit RKAHeM(4,4) method. Following Paul and Bhattacharjee (2013), the protoplanets have been assumed to be spheres of solar composition each of which is in a steady state of quasi-static equilibrium, where the energy equation assumes the conduction–radiation heat transport. Fig. 1 depicts temperature profile inside the protoplanets with masses 0.3 M_J, 1 M_J, 3 M_J, 5 M_J, 7 M_J and 10 M_J. It can be seen from the figure that the more massive the object the more hotter is its interior. The distribution for the temperature that came out through the study can be found to be in excellent agreement with the ones presented in Paul and Bhattacharjee (2013). Our presented temperature distribution for the protoplanets with prescribed masses and radii can also be found to be compared well with the ones presented in Helled and Schubert (2008), Nayakshin (2010), Senthilkumar and Paul (2012). Fig. 2 depicts our calculated results for pressure distribution inside the protoplanets with the prescribed masses. It can be realized from the Fig. 2 that from a point little depth of the surface down to the core region, the pressures at a corresponding depth inside the protoplanets increase with their

Table 1 Comparative distribution of thermodynamic variables inside a 1 Jupiter mass protoplanet.

r/R	Classical 4th order Runge–Kutta method			New explicit RKAHeM(4,4) algorithm		
	P (dyne cm^{-2})	T (°K)	ρ (gm cm^{-3})	P (dyne cm^{-2})	T (°K)	ρ (gm cm^{-3})
0.99	3.8954527×10^{-07}	1.6102493×10^{00}	6.4512556×10^{-15}	3.8954527×10^{-07}	1.6102493×10^{00}	6.4512556×10^{-15}
0.90	5.4656957×10^{-03}	1.7901342×10^{01}	8.1421533×10^{-12}	5.4657120×10^{-03}	1.7901356×10^{01}	8.1421716×10^{-12}
0.80	1.3290122×10^{-01}	4.0709788×10^{01}	8.7058176×10^{-11}	1.3290142×10^{-01}	4.0709803×10^{01}	8.7058273×10^{-11}
0.70	1.0706637×10^{00}	7.0401377×10^{01}	4.0555650×10^{-10}	1.0706647×10^{00}	7.0401392×10^{01}	4.0555678×10^{-10}
0.60	5.6569018×10^{00}	1.0969770×10^{02}	1.3751831×10^{-09}	5.6569052×10^{00}	1.0969772×10^{02}	1.3751838×10^{-09}
0.50	2.4262787×10^{01}	1.6205682×10^{02}	3.9925723×10^{-09}	2.4262796×10^{01}	1.6205684×10^{02}	3.9925736×10^{-09}
0.40	9.2395087×10^{01}	2.3108465×10^{02}	1.0662455×10^{-08}	9.2395105×10^{01}	2.3108466×10^{02}	1.0662457×10^{-08}
0.30	3.2328971×10^{02}	3.1822282×10^{02}	2.7091933×10^{-08}	3.2328972×10^{02}	3.1822281×10^{02}	2.7091934×10^{-08}
0.20	1.0297073×10^{03}	4.1622901×10^{02}	6.5972184×10^{-08}	1.0297070×10^{03}	4.1622896×10^{02}	6.5972177×10^{-08}
0.10	2.7969290×10^{03}	4.9796902×10^{02}	1.4978163×10^{-07}	2.7969272×10^{03}	4.9796890×10^{02}	1.4978157×10^{-07}
0.01	1.0954682×10^{04}	6.1391756×10^{02}	4.7584900×10^{-07}	1.0954650×10^{04}	6.1391687×10^{02}	4.7584812×10^{-07}

Table 2 Comparison of LTE, GTE, EE for the fourth order four stage explicit RK-embedded HeM technique.

Explicit RK-embedded technique	Local truncation error (LTE)	Global truncation error (GTE)	Error estimation (EE)
Explicit RK-embedded arithmetic Heronian mean (present paper)	$LTE_{AM}-LTE_{HeM}\leqslant(\frac{121{,}809}{1{,}658{,}880})P^4\cdot Qh^5$ $=\lvert y_{n+1}^{AM}-y_{n+1}^{HeM}\rvert\leqslant\frac{121{,}809}{1{,}658{,}880}P^4\cdot Qh^5$	$\lvert\varepsilon_n\rvert\leqslant(\frac{h^4}{1{,}658{,}880LD})\times M(e^{DL(x_n-x_0)}-1)$	$\mathrm{ERREST}=\lvert Y_{AM}-Y_{HeM}\rvert\times\frac{121{,}809}{1{,}658{,}880}$

increasing masses, except for the protoplanet with mass 10 M_J. The pressure distribution obtained in our investigation agrees fairly well with the corresponding result presented in Paul and Bhattacharjee (2013) but the central pressures obtained in the investigation of corresponding protoplanets can be found to be higher than the ones presented in Helled and Schubert (2008), Senthilkumar and Paul (2012). It is to be noted here that Helled and Schubert (2008) assumed the initial protoplanets to be convective with a thin outer radiative zone, while Senthilkumar and Paul (2012) assumed such protoplanets to be fully convective, which is consistent with Helled et al. (2005). Fig. 3 shows our mass distribution inside the protoplanets with the assumed masses. It is obvious from the figure that in the protoplanetary atmosphere, the matter is not distributed uniformly and gravitational stratification may be the reasons of variation in parameters. This is as to be expected for initially formed unsegregated protoplanets. Fig. 4 depicts our calculated density distribution inside the protoplanets having assumed masses. It is obvious from the figure that the more massive the protoplanet, the greater the density of the material except for the protoplanets with masses 0.3 M_J and 10 M_J. The results in this case are also found to be in general agreement with the ones found in Paul and Bhattacharjee (2013). But our study presents centrally dense and surfacically rarer protoplanets with the ones presented in Senthilkumar and Paul (2012). The distribution of thermodynamic variables obtained employing the new explicit RKAHeM(4,4) method with the ones obtained with the classical Runge–Kutta method for a 1 Jupiter protoplanet is presented in Table 1 for direct comparison. It is obvious from Table 1 that the distribution of the thermodynamic variables obtained in the study inside a 1 Jupiter mass protoplanet agrees fairly well with the ones obtained with classical Runge–Kutta method. The distribution of thermodynamic variables inside other protoplanetary masses can also be found to be compared well with the results obtained with the classical Runge–Kutta method. But to show all results in tabular form would only be space consuming, they have been avoided. Likewise Paul and Bhattacharjee (2013), we have also tested our results with the ones obtained with varying end points. The results are found to be insensitive to the choice of the end points. However, the system possesses unique solution suggesting that the protoplanetary formation via disk instability is a reasonable hypothesis. Finally, Table 2 shows the comparison of the Local Truncation Error (LTE), the Global Truncation Error (GTE) and the Error Estimation (EE) of a newly proposed explicit RKAHeM(4,4) method.

6. Conclusion

In order to determine the distribution of thermodynamic variables inside extra-solar protoplanets, formed by gravitational instability, at their initial stages, a newly proposed explicit RKAHeM(4,4) algorithm is employed, verified and the obtained results are validated. A general brief theoretical framework for the proposed numerical method is stated in addition to pseudo code followed by error estimation description. The results based on newly proposed embedded Runge–Kutta fourth order with four stages arithmetic Heronian mean method are found to be optimal and efficient in comparison with the ones obtained with the classical Runge–Kutta method. In our calculation, the effect of radiative heating has not been considered. Also, we have used the Clapeyron equation, which is valid for gas at not very high pressure. But our research work concentrates on the investigation of initial structures of protoplanets formed via disk instability considering an appropriate equation of state valid for protoplanetary atmosphere taking into the factor mentioned to see how the results compare our calculated results.

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