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FIXED POINT RESULTS FOR MAPPINGS SATISFYING (ψ, φ)-WEAKLY CONTRACTIVE CONDITION IN ORDERED PARTIAL METRIC SPACES

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ABSTRACT. In [18], Matthews introduced a new class of metric spaces, that is, the concept of partial metric spaces, or equivalently, weightable quasi-metrics, are investigated to generalize metric spaces (X, d), to develop and to introduce a new fixed point theory. In partial metric spaces, the self-distance for any point need not be equal to zero. In this paper, we study some results for single map satisfying (ψ, φ)-weakly contractive condition in partial metric spaces endowed with partial order. An example is given to support the useability of our results.

1. Introduction

The Banach contraction principle [8] is a very popular tool in solving existence problems in many branches of mathematical analysis. This famous theorem can be stated as follows.

Theorem 1.1. ([8]) *Let (X, d) be a complete metric space and T be a mapping of X into itself satisfying:*

$$d(Tx, Ty) \leq kd(x, y) \qquad \text{for all} \quad x, y \in X, \tag{1.1}$$

where k is a some constant in $(0, 1)$. Then, T has a unique fixed point $x^* \in X$.

There is in the literature a great number of generalizations of the Banach contraction principle (see [2] and others).

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In recent years many works on domain theory have been made in order to equip semantics domain with a notion of distance. In particular, Matthews [18] introduced the notion of a partial metric space as a part of the study of denotational semantics of dataflow networks, and obtained, among other results, a nice relationship between partial metric spaces and the so-called weightable quasi-metric spaces. He showed that the Banach contraction mapping theorem can be generalized to the partial metric context for applications in program verification.

In partial metric spaces, the self-distance for any point need not be equal to zero. Specially, from the point of sequences, a convergent sequence need not have unique limit.

After then O'Neill defined the concept of dualistic partial metric, which is general then partial metric. In [26], Oltra and Valero gave a Banach fixed point theorem on complete dualistic partial metric spaces. Also in [26], it was showed that the contractive condition in Banach fixed point theorem on complete dualistic partial metric spaces can not be replaced by the contractive condition of Banach fixed point theorem for complete partial metric spaces. Later, Valero [34] has generalized the main theorem of [26] using nonlinear contractive condition instead of Banach contractive condition. As it can be understand above, fixed point theory on dualistic partial metric or partial metric spaces have been done for contractive or contractive type mappings. Altun and H. Simsek [4], Altun et al. [6], Romaguera [30], Oltra [25] and Samet et al. [32] also study fixed point theorem on partial metric space.

In present era, fixed point theory has developed rapidly in metric spaces endowed with a partial ordering (see detail in [5, 7, 9–11, 13, 14, 19–24, 29, 31, 33]). The first result in this direction was given by Ran and Reurings [28: Theorem 2.1] who presented its applications to matrix equation. Subsequently, Nieto and Rodŕiguez-López [22] extended the result of Ran and Reurings [28] for nondecreasing mappings and applied to obtain a unique solution for a first order ordinary differential equation with periodic boundary conditions. Further, in the year 2009, Harjani and Sadarangani [13] used the above discussed concept and proved some fixed point theorems for weakly contractive operators in ordered metric spaces. Subsequently, Harjani and Sadarangani [14] generalized their own results [13] by considering pair of altering functions (ψ, φ). Nashine and Altun [19] and Nashine and Bessem [20] generalized the results of Harjani and Sadarangani [13, 14]. Nashine, Samet and Vetro [21] also proved fixed point theorems for T-weakly isotone increasing mappings which satisfy a generalized nonlinear contractive condition in complete ordered metric spaces and gave an application to an existence theorem for a solution of some integral equations.

In paper [3], the idea of partial metric space and partial order is combined and give some fixed point theorems on ordered partial metric spaces.

The aim of this paper is to study partial metric spaces and partial order, and prove fixed point results for single map satisfying (ψ, φ)-weakly contractive condition. An example is given to support the useability of our results.

2. Main results

At first, we introduce some notations and definitions that will be used later.

2.1. Notations and definitions

Definition 2.1. ([16]) $\varphi\colon [0,+\infty) \to [0,+\infty)$ is called an altering distance function if the following properties are satisfied:

(a) φ is continuous and non-decreasing,

(b) $\varphi(t) = 0 \Longleftrightarrow t = 0$.

The following definition was introduced in [4, 15, 18, 26, 34]:

Definition 2.2. A partial metric on a nonempty set X is a function $p\colon X \times X \to \mathbb{R}^+ := [0,\infty)$ such that for all $x, y, z \in X$:

(p_1) $x = y \Longleftrightarrow p(x,x) = p(x,y) = p(y,y)$,

(p_2) $p(x,x) \le p(x,y)$,

(p_3) $p(x,y) = p(y,x)$,

(p_4) $p(x,y) \le p(x,z) + p(z,y) - p(z,z)$.

A partial metric space is a pair (X,p) such that X is a nonempty set and p is a partial metric on X. It is clear that, if $p(x,y) = 0$, then from (p_1) and (p_2) $x = y$. But if $x = y$, $p(x,y)$ may not be 0. A basic example of a partial metric space is the pair $(\mathbb{R}^+, p)$, where $p(x,y) = \max\{x,y\}$ for all $x, y \in \mathbb{R}^+$. Other examples of partial metric spaces which are interesting from a computational point of view may be found in [12, 18].

Each partial metric p on X generates a T_0 topology τ_p on X which has as a base the family open p-balls $\{B_p(x,\varepsilon) :\ x \in X,\ \varepsilon > 0\}$, where $B_p(x,\varepsilon) = \{y \in X :\ p(x,y) < p(x,x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

If p is a partial metric on X, then the function $p^s\colon X \times X \to \mathbb{R}^+$ given by

$$p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y)$$

is a metric on X.

Definition 2.3. ([3]) Let (X,p) be a partial metric space. Then:

(1) A sequence $\{x_n\}$ in a partial metric space (X,p) converges to a point $x \in X$ if and only if $p(x,x) = \lim_{n\to\infty} p(x,x_n)$.

(2) A sequence $\{x_n\}$ in a partial metric space (X, p) is called a Cauchy sequence iff $\lim_{n,m\to\infty} p(x_n, x_m)$ exists (and is finite).

(3) A partial metric space (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x, x) = \lim_{n,m\to\infty} p(x_n, x_m)$.

(4) A mapping $T\colon X \to X$ is said to be continuous at $x_0 \in X$, if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $T(B_p(x_0, \delta)) \subset B_p(Tx_0, \varepsilon)$.

Lemma 2.1. ([18,26]) *Let (X, p) be a partial metric space.*

(a) *$\{x_n\}$ is a Cauchy sequence in (X, p) if and only if it is a Cauchy sequence in the metric space (X, p^s).*

(b) *A partial metric space (X, p) is complete if and only if the metric space (X, p^s) is complete. Furthermore, $\lim_{n\to\infty} p^s(x_n, x) = 0$ if and only if*

$$p(x, x) = \lim_{n\to\infty} p(x_n, x) = \lim_{n,m\to\infty} p(x_n, x_m).$$

2.2. Results

Our first result is the following.

Theorem 2.1. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a partial metric p on X such that (X, p) is a complete partial metric space. Suppose $T\colon X \to X$ be a nondecreasing mapping such that*

$$\psi(p(Tx, Ty)) \leq \psi(p(x, y)) - \varphi(p(x, y)) \tag{2.1}$$

for all $x, y \in X$ with $y \preceq x$, where ψ and φ are altering distance functions. We suppose the following hypotheses:

(i) *T is continuous, or*

(ii) *$\{x_n\} \subset X$ is a nondecreasing sequence with $x_n \to x$ in X, then $x_n \preceq x$ for all $n \in \mathbb{N}$.*

If there exist $x_0 \in X$ such that $x_0 \preceq Tx_0$, then there exists $z \in X$ such that $z = Tz$. Moreover, $p(z, Tz) = 0$.

Proof. Let x_0 be an arbitrary point in X. If $x_0 = Tx_0$, then x_0 is a fixed point of T and then the proof is completed. Suppose $Tx_0 \neq x_0$. Now since $x_0 \preceq Tx_0$ and T is nondecreasing we have

$$x_0 \preceq Tx_0 \preceq T^2x_0 \preceq \cdots \preceq T^n x_0 \preceq T^{n+1}x_0 \cdots . \tag{2.2}$$

Put $x_n = T^n x_0$ and so $x_{n+1} = Tx_n$. If there exists $n_0 \in \{1, 2, \cdots\}$ such that $p(x_{n_0}, x_{n_0-1}) = 0$, then by (p_2) we have $p(x_{n_0-1}, x_{n_0-1}) = p(x_{n_0}, x_{n_0})$. Thus by (p_1), we get that $x_{n_0-1} = x_{n_0} = Tx_{n_0-1}$ and so we are finished. Now we can suppose

$$p(x_n, x_{n+1}) > 0 \qquad \text{for all} \quad n \geq 1. \tag{2.3}$$

First we will prove that $\lim_{n\to\infty} p(x_{n+1}, x_n) = 0$.

Now we claim that

$$p(x_{n+1}, x_{n+2}) \le p(x_n, x_{n+1}) \qquad \text{for all} \quad n \ge 1. \tag{2.4}$$

Now since $x_n \preceq x_{n+1}$, we can use (2.2) for these points, then we have for $n \ge 1$

$$\psi(p(x_{n+1}, x_{n+2})) \le \psi(p(x_n, x_{n+1})) - \varphi(p(x_n, x_{n+1})) \le \psi(p(x_n, x_{n+1})). \tag{2.5}$$

Since ψ is a nondecreasing function, we get that

$$p(x_{n+1}, x_{n+2}) \le p(x_n, x_{n+1}). \tag{2.6}$$

Therefore, (2.4) is true and so the sequence $\{p(x_n, x_{n+1})\}$ is nonincreasing and bounded below. Thus there exists $\rho \ge 0$ such that

$$\lim_{n\to+\infty} p(x_n, x_{n+1}) = \rho. \tag{2.7}$$

Now suppose that $\rho > 0$. Therefore, taking $n \to +\infty$ in (2.5) , then using (2.7) and the continuity of ψ and φ, we get that

$$\psi(\rho) \le \psi(\rho) - \varphi(\rho).$$

Hence $\varphi(\rho) = 0$. Therefore $\rho = 0$. So

$$\lim_{n\to+\infty} p(x_n, x_{n+1}) = 0. \tag{2.8}$$

By (p_2), we get that

$$\lim_{n\to+\infty} p(x_n, x_n) = 0. \tag{2.9}$$

Since p^s is a metric on X, then it is obvious that $p^s(x_n, x_n) = 0$ and so

$$\lim_{n\to+\infty} p^s(x_n, x_n) = 0 \tag{2.10}$$

Next, we claim that $\{x_n\}$ is a Cauchy sequence in the metric space (X, p^s).

Suppose to the contrary; that is, $\{x_n\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ for which we can find two subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ such that $n(k)$ is the smallest index for which

$$n(k) > m(k) > k, \qquad p^s(x_{m(k)}, x_{n(k)}) \ge \varepsilon. \tag{2.11}$$

This means that

$$p^s(x_{m(k)}, x_{n(k)-1}) < \varepsilon. \tag{2.12}$$

From (2.11), (2.12) and the triangular inequality, we get that

$$\begin{aligned}
\varepsilon &\leq p^s(x_{m(k)}, x_{n(k)}) \\
&\leq p^s(x_{m(k)}, x_{m(k)+1}) + p^s(x_{m(k)+1}, x_{n(k)-1}) + p^s(x_{n(k)-1}, x_{n(k)}) \\
&\leq p^s(x_{m(k)}, x_{m(k)+1}) + p^s(x_{m(k)+1}, x_{n(k)}) + 2p^s(x_{n(k)-1}, x_{n(k)}) \\
&\leq p^s(x_{m(k)}, x_{m(k)+1}) + p^s(x_{m(k)+1}, x_{m(k)}) + p^s(x_{m(k)}, x_{n(k)}) \\
&\quad + 2p^s(x_{n(k)-1}, x_{n(k)}) \\
&\leq 2p^s(x_{m(k)}, x_{m(k)+1}) + p^s(x_{m(k)+1}, x_{n(k)}) + 2p^s(x_{n(k)-1}, x_{n(k)}) \\
&\leq 2p^s(x_{m(k)}, x_{m(k)+1}) + p^s(x_{m(k)+1}, x_{n(k)-1}) + p^s(x_{n(k)-1}, x_{n(k)}) \\
&\quad + 2p^s(x_{n(k)-1}, x_{n(k)}) \\
&\leq 3p^s(x_{m(k)}, x_{m(k)+1}) + p^s(x_{m(k)}, x_{n(k)-1}) + 3p^s(x_{n(k)-1}, x_{n(k)}) \\
&< 2p^s(x_{m(k)}, x_{m(k)+1}) + \varepsilon + 3p^s(x_{n(k)-1}, x_{n(k)}).
\end{aligned}$$

Letting $k \to +\infty$ and using (2.10), we get that

$$\begin{aligned}
\lim_{k\to+\infty} p^s(x_{m(k)}, x_{n(k)}) &= \lim_{k\to+\infty} p^s(x_{m(k)+1}, x_{n(k)-1}) = \lim_{k\to+\infty} p^s(x_{m(k)+1}, x_{n(k)}) \\
&= \lim_{k\to+\infty} p^s(x_{m(k)}, x_{n(k)-1}) = \varepsilon.
\end{aligned}$$

Since

$$p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y)$$

for all $x, y \in X$, then by using (2.10) we conclude that

$$\begin{aligned}
\lim_{k\to+\infty} p(x_{m(k)}, x_{n(k)}) &= \lim_{k\to+\infty} p(x_{m(k)+1}, x_{n(k)-1}) = \lim_{k\to+\infty} p(x_{m(k)+1}, x_{n(k)}) \\
&= \lim_{k\to+\infty} p(x_{m(k)}, x_{n(k)-1}) = \frac{\varepsilon}{2}.
\end{aligned}$$

Now, since $x_{m(k)} \preceq x_{n(k)-1}$, we can use the inequality (2.1) for these points, then we have

$$\begin{aligned}
\psi(p(x_{m(k)+1}, x_{n(k)})) &= \psi(p(Tx_{m(k)}, Tx_{n(k)-1}) \\
&\leq \psi(p(x_{m(k)}, x_{n(k)-1})) - \varphi(p(x_{m(k)}, x_{n(k)-1})).
\end{aligned}$$

Letting $k \to +\infty$ and using the continuity of φ and ψ, we get that

$$\psi\left(\frac{\varepsilon}{2}\right) \leq \psi\left(\frac{\varepsilon}{2}\right) - \varphi\left(\frac{\varepsilon}{2}\right).$$

Therefore, we get that $\varphi(\frac{\varepsilon}{2}) = 0$. Hence $\varepsilon = 0$ a contradiction. Thus $\{x_n\}$ is a Cauchy sequence in (X, p^s). Since (X, p) is complete then from Lemma 2.1, the sequence $\{x_n\}$ converges in the metric space (X, p^s), say $\lim_{n\to\infty} p^s(x_n, z) = 0$. Again from Lemma 2.1, we have

$$p(z, z) = \lim_{n\to\infty} p(x_n, z) = \lim_{n,m\to\infty} p(x_n, x_m). \tag{2.13}$$

Moreover since $\{x_n\}$ is a Cauchy sequence in the metric space (X, p^s), we have $\lim_{n,m\to\infty} p^s(x_n, x_m) = 0$ and so we have $\lim_{n\to\infty} p(x_n, x_n)$, thus from definition p^s we have $\lim_{n,m\to\infty} p(x_n, x_m) = 0$. Therefore from (2.13), we have

$$p(z, z) = \lim_{n\to\infty} p(x_n, z) = \lim_{n,m\to\infty} p(x_n, x_m) = 0.$$

Now we claim that $Tz = z$. Suppose $p(z, Tz) > 0$.
Suppose that assumption (a) holds. Since T is continuous, then given $\varepsilon > 0$, there exists $\delta > 0$ such that $T(B_p(z, \delta)) \subseteq B_p(Tz, \varepsilon)$. Since $p(z, z) = \lim_{n\to\infty} p(x_n, z) = 0$, then there exists $k \in \mathbb{N}$ such that $p(x_n, z) < p(z, z) + \delta$ for all $n \geq k$. Therefore, we have $x_n \subset B_p(z, \delta)$ for all $n \geq k$. Thus $T(x_n) \in T(B_p(z, \delta)) \subset B_p(Tx, \varepsilon)$ and so $p(Tx_n, Tz) < p(Tz, Tz) + \varepsilon$ for all $n \geq k$. This shows that $p(Tz, Tz) = \lim_{n\to\infty} p(x_{n+1}, Tz)$. Now we use the inequality (2.1) for $x = y$, then we have

$$\psi(p(Tz, Tz)) \leq \psi(p(z, z)) - \varphi(p(z, z)) \leq \psi(p(z, Tz)).$$

Since ψ is a nondecreasing function, we get that

$$p(Tz, Tz) \leq p(z, Tz). \tag{2.14}$$

Therefore, we obtain

$$\begin{aligned} p(z, Tz) &\leq p(z, x_{n+1}) + p(x_{n+1}, Tz) - p(x_{n+1}, x_{n+1}) \\ &\leq p(z, x_{n+1}) + p(x_{n+1}, Tz) \end{aligned}$$

and letting $\lim_{n\to+\infty}$ we get

$$\begin{aligned} p(z, Tz) &\leq \lim_{n\to+\infty} p(z, x_{n+1}) + \lim_{n\to+\infty} p(x_{n+1}, Tz) \\ &\leq p(Tz, Tz) \leq p(z, Tz) \end{aligned}$$

which is a contradiction, $p(z, Tz) = 0$, and so $z = Tz$.
Suppose that assumption (b) holds.
Since $\{x_n\}$ is a nondecreasing sequence converges to z in (X, p), by the assumption on X, we get that $x_n \preceq z$ for all $n \in \mathbf{N}$. Then we can use (2.1) for $x = x_n$. Therefore, we obtain

$$\begin{aligned} \psi(p(x_{n+1}, Tz)) = \psi(p(Tx_n, Tz)) &\leq \psi(p(x_n, z)) - \varphi(p(x_n, z)) \\ &\leq \psi(p(Tx_{n+1}, z)) - \varphi(p(Tx_{n+1}, z)). \end{aligned}$$

Letting $n \to +\infty$ and using continuities of ψ and φ, we get

$$\lim_{n\to\infty} \psi(p(Tx_n, Tz)) \leq \quad \psi(p(z, Tz)) - \varphi(p(z, Tz)) \leq \psi(p(z, Tz)).$$

Therefore $\lim_{n\to\infty} p(Tx_n, Tz) \leq p(z, Tz)$.

Therefore, we obtain

$$\begin{aligned}p(z,Tz) &\leq \lim_{n\to+\infty} p(z,x_{n+1}) + \lim_{n\to+\infty} p(x_{n+1},Tz)\\ &\leq \lim_{n\to+\infty} p(z,x_{n+1}) + \lim_{n\to+\infty} p(Tx_n,Tz) \leq p(z,Tz)\end{aligned}$$

which is a contradiction. Thus $p(z,Tz)=0$. Hence $z=Tz$. Therefore z is a fixed point of T. □

Example 1. Let $X=[0,+\infty)$ endowed with the usual partial metric p defined by $p\colon X\times X\to[0,+\infty)$ with $p(x,y)=\max\{x,y\}$. We give the partial order on X by

$$x \preceq y \iff p(x,x)=p(x,y) \iff x=\max\{x,y\} \iff y\leq x.$$

It is clear that $(X,\preceq)$ is totally ordered. The partial metric space (X,p) is complete because (X,p^s) is complete. Indeed, for any $x,y\in X$,

$$p^s(x,y)=2p(x,y)-p(x,x)-p(y,y)=2\max\{x,y\}-(x+y)=|x-y|,$$

Thus, $(X,p^s)=([0,+\infty),|\cdot|)$ is the usual metric space, which is complete. Again, we define

$$T(t)=\frac{t}{2} \qquad \text{if} \quad t\geq 0.$$

The function T is continuous on (X,p). Indeed, let $\{x_n\}$ be a sequence converging to x in (X,p), then

$$\lim_{n\to+\infty}\max\{x_n,x\}=\lim_{n\to+\infty}p(x_n,x)=p(x,x)=x$$

hence by definition of T, we have

$$\begin{aligned}\lim_{n\to+\infty}p(Tx_n,Tx) &= \lim_{n\to+\infty}\max\{Tx_n,Tx\}\\ &= \lim_{n\to+\infty}\frac{1}{2}\max\{x_n,x\}=\frac{1}{2}x=p(Tx,Tx),\end{aligned} \tag{2.15}$$

that is $\{T(x_n)\}$ converges to $T(x)$ in (X,p). On the other hand, if $\{x_n\}$ converges properly to x in X, hence

$$\lim_{n\to+\infty}p^s(x_n,x)=0.$$

Thus, by definition of p^s and T, one can find

$$\lim_{n\to+\infty}p^s(Tx_n,Tx)=0. \tag{2.16}$$

Both convergence (2.15)–(2.16) yield that T is continuous on (X,p). Any $x,y\in X$ are comparable, so for example we take $x\preceq y$, and then $p(x,x)=p(x,y)$, so $y\leq x$. Since $T(y)\leq T(x)$, so $T(x)\preceq T(y)$, giving that T is monotone non-decreasing with respect to $\preceq$. In particular, for any $x\preceq y$, we have

$$p(x,y)=x, \qquad p(Tx,Ty)=T(x)=\frac{x}{2}. \tag{2.17}$$

Let us take $\varphi, \psi \colon [0,+\infty) \to [0,+\infty)$ such that

$$\psi(t) = t \qquad \text{and} \qquad \varphi(t) = \frac{t}{4}. \tag{2.18}$$

We have for any $x \in X$, $\frac{x}{2} \leq x - \frac{x}{4}$. Consequently, we get for any $x \preceq y$, and (2.17)

$$\psi(p(Tx,Ty)) \leq \psi(p(x,y)) - \varphi(p(x,y)),$$

that is (2.1) holds. All the hypotheses of Theorem 2.1 are satisfied, so T has a unique fixed point in X, which is $z = 0$.

Remark 1. In [28: Theorem 1] it is proved that if

$$\text{every pair of elements has a lower bound and upper bound,} \tag{2.19}$$

then for every $x \in X$,

$$\lim_{n\to\infty} T^n x = y,$$

where y is the fixed point of T such that

$$y = \lim_{n\to\infty} T^n x_0$$

and hence T has a unique fixed point. If condition (2.1) fails, it is possible to find examples of functions T with more than one fixed point. There exist some examples to illustrate this fact in [22].

In what follows, we give a sufficient condition for the uniqueness of the fixed point in Theorem 2.1, this condition is (2.19).

In [22], it was proved that condition (2.19) is equivalent to for

$$\text{for } x, y \in X \text{ there exists } z \in X \text{ which is comparable to } x \text{ and } y. \tag{2.20}$$

Theorem 2.2. *Adding condition* (2.20) *to the hypotheses of Theorem 2.1, one obtains uniqueness of the fixed point of T.*

Proof. Suppose that there exists z and that $y \in X$ are different fixed points of T, then $p(z,y) > 0$. Now, we consider the following two cases.

(i) If z and y are comparable, then $T^n(z) = z$ and $T^n(y) = y$ are comparable for $n = 0, 1, \cdots$. Therefore, we can use the condition (2.1), then we have

$$\begin{aligned} \psi(p(z,y)) &= \psi(p(T^n z, T^n y)) \\ &\leq \psi(p(T^{n-1}z, T^{n-1}y)) - \varphi(p(T^{n-1}z, T^{n-1}y)) \\ &\leq \psi(p(z,y)) - \varphi(p(z,y)) \end{aligned} \tag{2.21}$$

which is a contradiction.

(ii) If z and y are not comparable, then there exists $x \in X$ comparable to z and y. Since T is nondecreasing, then $T^n x$ is comparable to $T^n z = z$ and $T^n y = y$ for $n = 0, 1, \ldots$. Moreover,

$$\begin{aligned} \psi(p(z, T^n x)) &= \psi(p(T^n z, T^n x)) \\ &\leq \psi(p(T^{n-1} z, T^{n-1} x)) - \varphi(p(T^{n-1} z, T^{n-1} x)) \\ &= \psi(p(z, T^{n-1} x)) - \varphi(p(z, T^{n-1} x)) \\ &\leq \psi(p(z, T^{n-1} x)). \end{aligned} \tag{2.22}$$

Since ψ is a nondecreasing function, we get that

$$p(z, T^n x)) \leq p(z, T^{n-1} x). \tag{2.23}$$

This shows that $\{p(z, T^n x)\}$ is a nonnegative and nondecreasing sequence and so has a limit, say $\delta \geq 0$. Taking $n \to +\infty$ in (2.22) and using the continuity of ψ and φ, we get that

$$\psi(\delta) \leq \psi(\delta) - \varphi(\delta).$$

Hence $\varphi(\delta) = 0$. Therefore $\delta = 0$. So

$$\lim_{n \to +\infty} p(z, T^n x) = 0. \tag{2.24}$$

Similarly, it can be proven that, $\lim\limits_{n \to \infty} p(y, T^n x) = 0$. Finally,

$$\begin{aligned} p(z, y) &\leq p(z, T^n x) + p(T^n x, y) - p(T^n x, T^n x) \\ &\leq p(z, T^n x) + p(T^n x, y), \end{aligned} \tag{2.25}$$

and taking limit $n \to \infty$, we have $p(z, y) = 0$. This contradicts $p(z, y) > 0$. Consequently, T has no two fixed points. □

Corollary 2.2.1. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a partial metric p on X such that (X, p) is a complete partial metric space. Suppose $T\colon X \to X$ be a nondecreasing mapping such that*

$$p(Tx, Ty) \leq kp(x, y)$$

for all $x, y \in X$, where $k \in [0, 1)$. We suppose the following hypotheses:

(i) *T is continuous, or*

(ii) *$\{x_n\} \subset X$ is a nondecreasing sequence with $x_n \to x$ in X, then $x_n \preceq x$ for all $n \in \mathbb{N}$.*

If there exist $x_0 \in X$ such that $x_0 \preceq Tx_0$, then there exists $z \in X$ such that $z = Tz$. Moreover, $p(z, Tz) = 0$.

Proof. Define $\psi, \varphi\colon [0, +\infty) \to [0, +\infty)$ by $\psi(t) = t$ and $\varphi(t) = (1 - k)t$. Then φ and ψ satisfy all the hypotheses of Theorem 2.1. □

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