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FIXED POINT THEOREMS FOR GENERALIZED WEAKLY S-CONTRACTIVE MAPPINGS IN PARTIAL METRIC SPACES

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Abstract In this paper, we establish some unique fixed point theorems for generalized weakly S-contractive with nondecreasing and weakly increasing mappings in complete partial metric space. Also, we give some examples for strengthens of our main results.

Keywords Fixed point, complete partial metric space, weakly S-contractive mapping, decreasing mapping, weakly increasing mappings.

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1. Introduction and Preliminaries

In 1906, Frechet [13] was first presented the concept of metric space. The notion of partial metric space was introduced by Matthew [17, 18], which is generalization of the metric space. Matthews [17] generalized the Banach contraction principle to the class of complete partial metric space as follows: a self mapping f on a complete partial metric space (X, p) has a unique fixed point, if there exist $0 \leq k < 1$ such that

$$p(fx, fy) \leq kp(x, y), \ \forall \ x, y \in X.$$

Several authors have been focused on partial metric spaces and its topological properties (see [4, 5, 7, 19, 23, 28] and references therein). Some interesting work related to generalization of the contraction mapping and metric space can be seen in [10–12, 20–22, 26, 27] and its references. Also, some authors have to generalized fixed point theorems from class of metric space to the class of partial metric space.

First, we recall some useful definitions and results, which is useful throughout the paper.

Definition 1.1 ([17, 18]). A partial metric p on a nonempty set X is a function $p : X \times X \to [0, \infty)$ such that:

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(P_1) $p(x,x) = p(x,y) = p(y,y)$ if and only if $x = y$ (equality),
(P_2) $p(x,x) \leq p(x,y)$ (small self distance),
(P_3) $p(x,y) = p(y,x)$ (symmetry),
(P_4) $p(x,y) + p(z,z) \leq p(x,z) + p(z,y)$ (triangularity)
for all $x, y, z \in X$. Then the pair (X,p) is called a partial metric space. Throughout this paper, (X,p) represent a partial metric space equipped with a partial metric p, unless or otherwise stated.

Example 1.1 ([5,19])**.** Consider a mapping $p : [0,\infty) \times [0,\infty) \to [0,\infty)$ such that $p(x,y) = max\{x,y\}$ for all $x, y \in [0,\infty)$. Then p will be satisfy all the property of partial metric, and hence $([0,\infty), p)$ is a partial metric space, but fails to the condition of $p(x,x) = 0$ for all non zero $x \in [0,\infty)$. Therefore $([0,\infty), p)$ is not a metric space.

Remark 1.1 ([14])**.** (1) If $p(x,y) = 0$, then $x = y$ but if $x = y$, then $p(x,y)$ may not be zero. (2) $x \neq y$, then $p(x,y) > 0$.

Also, each partial metric p on X generates a T_0 topology τ_p on X with a base of the family of open p-balls $\{B_p(x,\varepsilon), x \in X, \varepsilon > 0\}$, where $B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x) + \varepsilon\}$, for all $x \in X$ and $\varepsilon > 0$ [7]. Let (X,p) be a partial metric space on X, then a function $d^p : X \times X \to [0,\infty)$ defined as

$$d^p(x,y) = 2p(x,y) - p(x,x) - p(y,y)$$

is a (usual) metric on X [7]. Furthermore, it is possible to observe that the following

$$\begin{aligned} d^m(x,y) &= \max\{p(x,y) - p(x,x), p(x,y) - p(y,y)\} \\ &= p(x,y) - \min\{p(x,x), p(y,y)\}, \end{aligned}$$

also defines a metric on X. In fact d^p and d^m are equivalent [5].

Definition 1.2 ([15])**.** In a partial metric space (X,p),
(1) a sequence $\{x_n\}$ is said to be convergent to a point $x \in X$ if and only if $\lim_{n\to\infty} p(x_n, x) = p(x,x)$.
(2) a sequence $\{x_n\}$ is called Cauchy sequence if and only if $\lim_{n,m\to\infty} p(x_n, x_m)$ is finite.
(3) if every Cauchy sequence $\{x_n\}$ is converges to a point $x \in X$ such that

$$\lim_{n,m\to\infty} p(x_n, x_m) = p(x,x),$$

then (X,p) is known as complete partial metric space.

Chatterjea [6] has been generalized the concept of contractive mappings into C-contractive mappings and Choudhury [9] introduced the weak C-contraction or weakly C-contractive mapping, which is more generalization of C-contractive mappings, as follows:

Definition 1.3 ([9,24])**.** Let (X,d) be a metric space, then a self mapping f on X, satisfying

$$d(fx,fy) \leq \frac{1}{2}[d(x,fy) + d(fx,y)] - \phi(d(x,fy), d(fx,y))$$

for all $x, y \in X$ and $\phi : [0,\infty)^2 \to [0,\infty)$ is a continuous function with $\phi(x,y) = 0$ if and only if $x = y = 0$, is called weakly C-contractive.

In the parallel consequences of weakly C-contractive mapping [9, 24], Shukla and Tiwari [25] has been introduced the concept of S-contractive mapping and generalization its, known as weakly S-contractive mapping, as follows:

Definition 1.4 ([25])**.** Let (X, d) be a metric space, then a self mapping f on X is said to be weakly S-contractive mapping or a weak S-contraction, if the following inequality holds:

$$d(fx, fy) \leq \frac{1}{3}[d(x, fy) + d(fx, y) + d(x, y)] - \phi(d(x, fy), d(fx, y), d(x, y))$$

for all $x, y \in X$ and $\phi : [0, \infty)^3 \to [0, \infty)$ is a continuous mapping with $\phi(x, y, z) = 0$ if and only if $x = y = z = 0$.

Definition 1.5. A self mapping f on a partial metric space X is called nondecreasing, if for all $x_1, x_2 \in X$, such that

$$x_1 \leq x_2 \Rightarrow f(x_1) \leq f(x_2).$$

Definition 1.6 ([3])**.** In a partial metric space (X, p), two self mappings f and g are said to be weakly increasing mappings, if for all $x \in X$ such that

$$g(x) \leq fg(x) \quad \text{and} \quad f(x) \leq gf(x).$$

Lemma 1.1 ([1,8])**.** *In a partial metric space (X, p), if a sequence $\{x_n\}$ convergent to a point $x \in X$, then $\lim_{n\to\infty} p(x_n, x) \leq p(x, z)$ for all $z \in X$. Also, if $p(x, x) = 0$, then*

$$\lim_{n\to\infty} p(x_n, z) = p(x, z), \quad \forall\, z \ \in X.$$

Lemma 1.2 ([7])**.** *If $\{x_{2n}\}$ is not a Cauchy sequence in (X, p), and two sequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers such that $n(k) > m(k) > k$, then the following four sequences*

$$p(x_{2m(k)}, x_{2n(k)+1}), p(x_{2m(k)}, x_{2n(k)}), p(x_{2m(k)-1}, x_{2n(k)+1}), p(x_{2m(k)-1}, x_{2n(k)})$$

tend to $\varepsilon > 0$, when $k \to \infty$.

Lemma 1.3 ([5, 19])**.** *In a partial metric space (X, p),*
(a) a sequence $\{x_n\}$ is a Cauchy if and only if, it is a Cauchy (X, d^p).
(b) X is complete if and only if it is complete in (X, d^p). Also, $\lim_{n\to\infty} d^p(x_n, x) = 0$ if and only if

$$\lim_{n,m\to\infty} p(x_n, x_m) = \lim_{n\to\infty} p(x_n, x) = p(x, x).$$

(c) if $\{x_n\}$ is a Cauchy sequence in the metric space (X, d^p), we have

$$\lim_{n,m\to\infty} d^p(x_n, x_m) = 0$$

and therefore by definition of d^p, we have

$$\lim_{n,m\to\infty} p(x_n, x_m) = 0.$$

The aim of this paper, is to prove some fixed point results in complete partial metric space for more generalization of weakly S-contractive mappings described in equations (2.1) and (2.16).

2. Main results

Theorem 2.1. *In a complete partial metric space (X, p), a self continuous nondecreasing mapping f on X, satisfying the condition*

$$p(fx, fy) \leq M(x, y), \tag{2.1}$$

$$M(x,y) = \max\Big\{\phi(p(x,y)), \phi(p(fx,y)), \phi(p(x,fy)), \phi\Big(\frac{p(x,y)+p(fx,y)+p(x,fy)}{3}\Big)\Big\} \tag{2.2}$$

for all $x, y \in X$ and $\phi : [0,\infty) \to [0,\infty)$ is a continuous function with $\phi(t) < t$ $\forall$ $t > 0$ and $\phi(0) = 0$, has a unique fixed point in X.

Proof. We choose an arbitrary point $x_0 \in X$, if $x_0 = fx_0$, then theorem follows trivially. Now, we discuss about $x_0 < fx_0$, then we can choose $x_1 \in X$ such that $fx_0 = x_1$. Since f is nondecreasing function, then we have

$$x_0 < x_1 = fx_0.$$

Again, let $x_2 = fx_1$, then we get

$$x_0 < x_1 = fx_0 \leq fx_1 = x_2 \leq fx_2.$$

Similarly, proceeding this work, we can construct a sequence $\{x_n\}$ in X, such that $x_{n+1} = fx_n$ with

$$x_0 < x_1 \leq x_2 \leq x_3 \leq ..... \leq x_n \leq x_{n+1} \leq ......$$

Suppose that $p(x_{n_0}, x_{n_0+1}) = 0$ for some $n_0 \geq 0$, then by Remark 1.1, $x_{n_0} = x_{n_0+1} = fx_{n_0}$, that is, x_{n_0} is a fixed point of f. So we assume that $p(x_n, x_{n+1}) \geq 0$ for all $n \geq 0$.

Next, we prove that $\{p(x_n, x_{n+1})\}$ is non increasing sequence in X. Now, we putting $x = x_{2n+1}, y = x_{2n}$ in (2.1) we get

$$p(fx_{2n}, fx_{2n+1}) = p(x_{2n+1}, x_{2n+2}) \leq M(x_{2n}, x_{2n+1}). \tag{2.3}$$

By (2.2), we have

$$\begin{aligned} M(x_{2n}, x_{2n+1}) &= \max\Big\{\phi(p(x_{2n}, x_{2n+1})), \phi(p(fx_{2n}, x_{2n+1})), \phi(p(x_{2n}, fx_{2n+1})), \\ &\qquad \phi\Big(\frac{p(x_{2n}, x_{2n+1}) + p(fx_{2n}, x_{2n+1}) + p(x_{2n}, fx_{2n+1})}{3}\Big)\Big\} \\ &= \max\Big\{\phi(p(x_{2n}, x_{2n+1})), \phi(p(x_{2n+1}, x_{2n+1})), \phi(p(x_{2n}, x_{2n+2})), \\ &\qquad \phi\Big(\frac{p(x_{2n}, x_{2n+1}) + p(x_{2n+1}, x_{2n+1}) + p(x_{2n}, x_{2n+2})}{3}\Big)\Big\}. \end{aligned} \tag{2.4}$$

Case-I: If $M(x_{2n}, x_{2n+1}) = \phi(p(x_{2n}, x_{2n+1}))$, then by equation (2.3) and using the fact $\phi(t) < t$ for all $t > 0$, we have

$$p(x_{2n+1}, x_{2n+2}) < p(x_{2n}, x_{2n+1}).$$

Case-II: If $M(x_{2n}, x_{2n+1}) = \phi(p(x_{2n+1}, x_{2n+1}))$, then by similar argument of case-I, we get

$$p(x_{2n+1}, x_{2n+2}) < p(x_{2n+1}, x_{2n+1}),$$

which is contradiction of (P_2).

Case-III: If $M(x_{2n}, x_{2n+1}) = \phi(p(x_{2n}, x_{2n+2}))$, then by similar argument of case-I and applying the property of (P_4) and (P_2), we have

$$p(x_{2n+1}, x_{2n+1}) < p(x_{2n}, x_{2n+1}).$$

Case-IV: If $M(x_{2n}, x_{2n+1}) = \phi\Big(\frac{p(x_{2n},x_{2n+1})+p(x_{2n+1},x_{2n+1})+p(x_{2n},x_{2n+2})}{3}\Big)$, using similar argument of case-III, we have

$$p(x_{2n+1}, x_{2n+2}) < p(x_{2n}, x_{2n+1}).$$

Thus, in all possible cases, we say that $\{p(x_{2n}, x_{2n+1})\}$ for all $n \geq 0$, is a monotonically decreasing sequence in X. Since, a monotonic decreasing bounded below sequence convergent to its greatest lower bound [16]. Thus,

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = 0 \text{ and } \lim_{n\to\infty} p(x_n, x_n) = 0, \qquad \forall\ n \geq 0. \tag{2.5}$$

Now, we have required to prove that the sequence $\{x_n\}$ is a Cauchy sequence in the partial metric space (X, p) and so in (X, d^p) (by Lemma 1.3(a). On the contrary, we suppose that the sequence $\{x_n\}$ is not a Cauchy sequence in (X, p), then sequences in Lemma 1.2 tend to $\varepsilon > 0$, when $k \to \infty$. Now, we putting $x = x_{2n(k)-1}$ and $y = x_{2m(k)}$ in equation (2.1), we get

$$\begin{aligned}
&p(x_{2n(k)}, x_{2m(k)+1})\\
=&p(fx_{2n(k)-1}, fx_{2m(k)}) \leq M(x_{2n(k)-1,x_{2m(k)}})\\
=&\max\Big\{\phi(p(x_{2n(k)-1}, x_{2m(k)})), \phi(p(fx_{2n(k)-1}, x_{2m(k)})), \phi(p(x_{2n(k)-1}, fx_{2m(k)})),\\
&\phi\Big(\frac{p(x_{2n(k)-1}, x_{2m(k)}) + p(fx_{2n(k)-1}, x_{2m(k)}) + p(x_{2n(k)-1}, fx_{2m(k)})}{3}\Big)\Big\}\\
=&\max\Big\{\phi(p(x_{2n(k)-1}, x_{2m(k)})), \phi(p(x_{2n(k)}, x_{2m(k)})), \phi(p(x_{2n(k)-1}, x_{2m(k)+1})),\\
&\phi\Big(\frac{p(x_{2n(k)-1}, x_{2m(k)}) + p(x_{2n(k)}, x_{2m(k)}) + p(x_{2n(k)-1}, x_{2m(k)+1})}{3}\Big)\Big\}.
\end{aligned}$$

Taking $k \to \infty$ and using Lemma 1.2 in above inequality, we have

$$\varepsilon \leq \max\big\{\phi(\varepsilon), \phi(\varepsilon), \phi(\varepsilon), \phi(\varepsilon)\big\}.$$

Using the fact $\phi(t) < t$ for all $t > 0$, we get

$$\varepsilon \leq \phi(\varepsilon) < \varepsilon,$$

which is a contradiction with respect to $\varepsilon > 0$. Thus $\{x_{2n}\}$ is a Cauchy sequence in (X, p) and so, in (X, d^p) (by Lemma 1.3(a)). Since (X, p) is complete, so (X, d^p) is also complete (by Lemma 1.3(b)). Therefore, the Cauchy sequence $\{x_n\}$ converges in the metric space (X, d^p), say $\lim_{n\to\infty} d^p(x_n, z) = 0$ then again applying Lemma 1.3(b), we have

$$p(z, z) = \lim_{n\to\infty} p(x_n, z) = \lim_{n,m\to\infty} p(x_n, x_m). \tag{2.6}$$

By Lemma 1.3(c) and definition of d^p, we get

$$\lim_{n,m\to\infty} d^p(x_n, x_m) = 0 \quad \text{and}$$
$$d^p(x_n, x_m) = 2p(x_n, x_m) - p(x_n, x_n) - p(x_m, x_m).$$

Taking $n, m \to \infty$ and using equation (2.5) in above inequality, we obtain

$$\lim_{n,m\to\infty} p(x_n, x_m) = 0. \tag{2.7}$$

From equations (2.6) and (2.7), we get

$$p(z, z) = \lim_{n\to\infty} p(x_n, z) = 0. \tag{2.8}$$

By (P_4), we have

$$p(z, fz) \leq p(z, x_n) + p(x_n, fz) - p(x_n, x_n).$$
$$p(z, fz) \leq p(z, x_n) + p(x_n, fz) = p(z, x_n) + p(fx_{n-1}, fz).$$

Taking $n \to \infty$ in above inequality and using equation (2.8) and Lemma 1.1, we have

$$p(z, fz) \leq p(fz, fz). \tag{2.9}$$

From (P_2), we have

$$p(fz, fz) \leq p(z, fz). \tag{2.10}$$

From (2.9) and (2.10), we get

$$p(z, fz) = p(fz, fz). \tag{2.11}$$

Next, we show that $p(fz, fz) = 0$. On contrary, we suppose that $p(fz, fz) > 0$, then by equation (2.1) and (2.8), we have

$$\begin{aligned} p(fz, fz) \leq & M(z, z) \\ &= \max\Big\{\phi(p(z, z)), \phi(p(fz, z)), \phi(p(z, fz)) \\ &\qquad \phi\Big(\frac{p(z, z) + p(fz, z) + p(z, fz)}{3}\Big)\Big\} \\ &< \max\Big\{0, p(fz, fz), p(fz, fz), \frac{2}{3}p(fz, fz)\Big\} \\ &= p(fz, fz), \end{aligned}$$

which is a contradiction, so

$$p(fz, fz) = 0.$$

Using equation (2.11) in above, we get

$$p(z, fz) = 0 \Rightarrow fz = z. \tag{2.12}$$

Hence z is a fixed point of f. Now, we claim that the fixed point of f is unique. Against of our claim, we assume that $u, v \in X$ be two fixed points of f with $u \neq v$,

such that $fu = u$ and $fv = v$.
From equation (2.1), we have

$$\begin{aligned} p(u,v) =& p(fu,fv) \leq M(u,v) \\ =& \max\Big\{\phi(p(u,v)), \phi(p(fu,v)), \phi(p(u,fv)), \phi\Big(\frac{p(u,v)+p(fu,v)+p(u,fv)}{3}\Big)\Big\} \\ <& p(u,v), \end{aligned} \tag{2.13}$$

which is a contradiction. Hence $u = v$, that is, fixed point of f is unique.

□

Example 2.1. Consider a complete partial metric space $(X = [0,1], p)$, such that $p(x,y) = max\{x,y\}$ for all $x, y \in X$. We define a self mapping f on X in such a way

$$f(x) = \begin{cases} x - 0.91x^3 : & \forall \quad x \leq 0.605227532, \\ \quad \alpha \quad : & \forall \quad \alpha \in [\frac{1}{2}, \infty], x > 0.605227532, \end{cases}$$

and a map $\phi : [0,\infty) \to [0,\infty)$ such that

$$\phi(t) = \begin{cases} t - \frac{t^2}{2} - \frac{t^3}{3} : & \forall \quad t \in [0,1], \\ \quad \beta \quad : & \forall \quad \beta \in [0,1], t \geq 1. \end{cases}$$

Then equation (2.1) holds for all comparable $x, y \in X$ $(thatis,\ x \leq y \text{ or } x \geq y)$, and satisfies all the requirements of Theorem 2.1. Therefore f has a unique fixed point 0 in X.

Corollary 2.1. *In a complete partial metric space (X, p), a self continuous non-decreasing mapping f on X, satisfying the condition*

$$p(fx, fy) \leq \phi(M(x,y)), \tag{2.14}$$

$$M(x,y) = \max\Big\{p(x,y), p(fx,y), p(x,fy), \frac{p(x,y)+p(fx,y)+p(x,fy)}{3}\Big\} \tag{2.15}$$

for all $x, y \in X$ and all other are same as in theorem 2.1, then f has a unique fixed point in X.

Corollary 2.2. *In Corollary 2.1, if we replaced second and third ordinates by $p(x, fx)$ and $p(y, fy)$ respectively, and fourth ordinate is replace by average of only $p(x, fy)$ and $p(fx, y)$ on $M(x, y)$, then it reduces to the similar results of [2, 4].*

Theorem 2.2. *In a complete partial metric space (X, p), two self weakly increasing continuous mappings f and g on X, satisfying the condition*

$$p(fx, gy) \leq M(x,y), \tag{2.16}$$

$$M(x,y) = \max\Big\{\phi(p(x,y)), \phi(p(fx,y)), \phi(p(x,gy)), \phi\Big(\frac{p(x,y)+p(fx,y)+p(x,gy)}{3}\Big)\Big\} \tag{2.17}$$

for all $x, y \in X$ and $\phi : [0,\infty) \to [0,\infty)$ is a continuous function with $\phi(t) < t$ $\forall$ $t > 0$ and $\phi(0) = 0$, then f and g have a unique common fixed point in X.

Proof. We choose an arbitrary point $x_0 \in X$, such that $x_0 = fx_0$ and $x_0 = gx_0$, then theorem follows trivially. Therefore, we discuss about $x_0 \neq fx_0$ and $x_0 \neq gx_0$. Now, we choose $x_1 \in X$, such that $fx_0 = x_1$ and $x_2 \in X$, such that $gx_1 = x_2$. Then, we can construct a sequence $\{x_n\}$ in X, in such a way

$$x_{2n+1} = fx_{2n}, \;\; x_{2n+2} = gx_{2n+1}.$$

Since f and g are weakly increasing mappings on X, then we have

$$\begin{aligned} x_1 &= fx_0 \leq gfx_0 = gx_1 = x_2, \\ x_2 &= gx_1 \leq fgx_1 = fx_2 = x_3, \\ x_3 &= fx_2 \leq gfx_2 = gx_3 = x_4 \end{aligned}$$

and proceeding this work, we get

$$x_0 \leq x_1 \leq x_2 \leq x_3 \leq \cdots \leq x_n \leq x_{n+1} \leq \cdots .$$

Thus $\{x_n\}$ is a nondecreasing sequence in X. If we suppose that $p(x_{n_0}, x_{n_0+1}) = 0$ for some $n_0 \geq 0$, then by Remark 1.1 $x_{n_0} = x_{n_0+1} = fx_{n_0}$, that is, x_{n_0} is a fixed point of f and g. So we assume that $p(x_n, x_{n+1}) \geq 0$ for all $n \geq 0$.

Firstly, we show that the sequence $\{p(x_n, x_{n+1})\}$ is non-increasing in X. Now, we putting $x = x_{2n}, y = x_{2n+1}$ in (2.16), we get

$$p(fx_{2n}, gx_{2n+1}) = p(x_{2n+1}, x_{2n+2}) \leq M(x_{2n}, x_{2n+1}). \tag{2.18}$$

By (2.16), we have

$$\begin{aligned} M(x_{2n}, x_{2n+1}) = &\max\Big\{\phi(p(x_{2n}, x_{2n+1})), \phi(p(fx_{2n}, x_{2n+1})), \phi(p(x_{2n}, gx_{2n+1})), \\ &\phi\Big(\frac{p(x_{2n}, x_{2n+1}) + p(fx_{2n}, x_{2n+1}) + p(x_{2n}, gx_{2n+1})}{3}\Big)\Big\} \\ = &\max\Big\{\phi(p(x_{2n}, x_{2n+1})), \phi(p(x_{2n+1}, x_{2n+1})), \phi(p(x_{2n}, x_{2n+2})), \\ &\phi\Big(\frac{p(x_{2n}, x_{2n+1}) + p(x_{2n+1}, x_{2n+1}) + p(x_{2n}, x_{2n+2})}{3}\Big)\Big\}. \end{aligned} \tag{2.19}$$

Case-I: If $M(x_{2n}, x_{2n+1}) = \phi(p(x_{2n}, x_{2n+1}))$, then by equation (2.18) and using the fact $\phi(t) < t$ for all $t > 0$, we have

$$p(x_{2n+1}, x_{2n+2}) < p(x_{2n}, x_{2n+1}).$$

Case-II: If $M(x_{2n}, x_{2n+1}) = \phi(p(x_{2n+1}, x_{2n+1}))$, then by similar argument of case-I, we get

$$p(x_{2n+1}, x_{2n+2}) < p(x_{2n+1}, x_{2n+1}),$$

which is contradiction of (P_2).

Case-III: If $M(x_{2n}, x_{2n+1}) = \phi(p(x_{2n}, x_{2n+2}))$, using the similar argument of case-I and applying the property of (P_4) and (P_2), we have

$$p(x_{2n+1}, x_{2n+1}) < p(x_{2n}, x_{2n+1}).$$

Case-IV: If $M(x_{2n}, x_{2n+1}) = \phi\Big(\frac{p(x_{2n},x_{2n+1})+p(x_{2n+1},x_{2n+1})+p(x_{2n},x_{2n+2})}{3}\Big)$, then by similar argument of case-III, we obtain

$$p(x_{2n+1}, x_{2n+2}) < p(x_{2n}, x_{2n+1}).$$

Thus, in all possible cases, we observe that $\{p(x_{2n}, x_{2n+1})\}$ for all $n \geq 0$, is a monotonically decreasing sequence in X. Then, by the similar argument as in Theorem 2.1, we have

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = 0 \text{ and } \lim_{n\to\infty} p(x_n, x_n) = 0, \qquad \forall\ n \geq 0. \tag{2.20}$$

Also, we will show that $\{x_n\}$ is a Cauchy sequence in the partial metric space (X, p). By similar arguments as used in case of proving Theorem 2.1, we find that the sequence $\{x_n\}$ is a Cauchy sequence. Now, we putting $x = x_{2n(k)}$ and $y = x_{2m(k)-1}$ in equation (2.16). We get

$$\begin{aligned}
&p(x_{2n(k)+1}, x_{2m(k)})\\
&=p(fx_{2n(k)}, gx_{2m(k)-1}) \leq M(x_{2n(k),x_{2m(k)-1}})\\
&= \max\Big\{\phi(p(x_{2n(k)}, x_{2m(k)-1})), \phi(p(fx_{2n(k)}, x_{2m(k)-1})), \phi(p(x_{2n(k)}, gx_{2m(k)-1})),\\
&\qquad \phi\Big(\frac{p(x_{2n(k)}, x_{2m(k)-1}) + p(fx_{2n(k)}, x_{2m(k)-1}) + p(x_{2n(k)}, gx_{2m(k)-1})}{3}\Big)\Big\}\\
&= \max\Big\{\phi(p(x_{2n(k)}, x_{2m(k)-1})), \phi(p(x_{2n(k)+1}, x_{2m(k)-1})), \phi(p(x_{2n(k)}, x_{2m(k)})),\\
&\qquad \phi\Big(\frac{p(x_{2n(k)}, x_{2m(k)-1}) + p(x_{2n(k)+1}, x_{2m(k)-1}) + p(x_{2n(k)}, x_{2m(k)})}{3}\Big)\Big\}.
\end{aligned}$$

Taking $k \to \infty$ and applying Lemma 1.2 in above inequality, we have

$$\varepsilon \leq \max\big\{\phi(\varepsilon), \phi(\varepsilon), \phi(\varepsilon), \phi(\varepsilon)\big\}.$$

Using the fact $\phi(t) < t$ for all $t > 0$, we get

$$\varepsilon \leq \phi(\varepsilon) < \varepsilon,$$

which is a contradiction with respect to $\varepsilon > 0$. Thus $\{x_{2n}\}$ is a Cauchy sequence in (X, p) and so, in (X, d^p) (by Lemma 1.3(a)). Further, the similar argument of Theorem 2.1, we have $\lim_{n\to\infty} d^p(x_n, z) = 0$ then by Lemma 1.3(b), gives that

$$p(z, z) = \lim_{n\to\infty} p(x_n, z) = \lim_{n,m\to\infty} p(x_n, x_m). \tag{2.21}$$

From Lemma 1.3(c), we have

$$\lim_{n,m\to\infty} d^p(x_n, x_m) = 0.$$

So, by definition of d^p, we get

$$d^p(x_n, x_m) = 2p(x_n, x_m) - p(x_n, x_n) - p(x_m, x_m).$$

Taking $n, m \to \infty$ and using equation (2.20) in above, we obtain

$$\lim_{n,m\to\infty} p(x_n, x_m) = 0. \tag{2.22}$$

From equations (2.21) and (2.22), we get

$$p(z,z) = \lim_{n\to\infty} p(x_n, z) = 0. \tag{2.23}$$

By (P_4), we have

$$\begin{aligned}p(z,fz) &\le p(z,x_{2n+1}) + p(x_{2n+1},fz) - p(x_{2n+1},x_{2n+1})\\ &\le p(z,x_{2n+1}) + p(x_{2n+1},fz) = p(z,x_n) + p(fx_{2n},fz).\end{aligned}$$

Taking $n \to \infty$ and using equation (2.23), we obtain

$$p(z,fz) \le p(fz,fz). \tag{2.24}$$

By (P_2), we have

$$p(fz,fz) \le p(z,fz). \tag{2.25}$$

From (2.24) and (2.25), we get

$$p(z,fz) = p(fz,fz). \tag{2.26}$$

Similarly,

$$p(z,gz) = p(gz,gz). \tag{2.27}$$

Using equations (2.26) and (2.27) and applying property of (P_4), we have

$$\begin{aligned}&p(z,gz) \le p(z,fz) + p(fz,gz) - p(fz,fz),\\ &p(z,gz) \le p(fz,gz).\end{aligned} \tag{2.28}$$

Similarly,

$$p(z,fz) \le p(fz,gz). \tag{2.29}$$

Now, we prove that $p(fz,gz) = 0$. On contrary, we suppose that $p(fz,gz) > 0$, then by equations (2.16), (2.23) and using above inequality, we get

$$\begin{aligned}&p(fz,gz)\\ &\le M(z,z)\\ &= \max\left\{\phi(p(z,z)), \phi(p(fz,z)), \phi(p(z,gz)), \phi\Big(\frac{p(z,z)+p(fz,z)+p(z,gz)}{3}\Big)\right\}\\ &< \max\left\{p(z,z), p(fz,z), p(z,gz), \frac{p(z,z)+p(fz,z)+p(z,gz)}{3}\right\}\\ &= \max\left\{p(fz,z), p(z,gz), \frac{p(fz,z)+p(z,gz)}{3}\right\}\\ &= \max\left\{p(fz,gz), p(fz,gz), \frac{2p(fz,gz)}{3}\right\} = p(fz,gz),\\ &\Rightarrow p(fz,gz) < p(fz,gz),\end{aligned}$$

which is a contradiction to our assumption. Thus we get

$$p(fz,gz) = 0 \Rightarrow p(z,fz) = 0 \ \text{ and } \ p(z,gz) = 0.$$

Applying Remark 1.1, we have

$$fz = gz, fz = z \ \text{ and } \ gz = z. \tag{2.30}$$

Hence, z is a common fixed point of f and g. Next, we prove that the common fixed point of f and g is unique. Let us suppose that $u, v \in X$ be two fixed points of f and g with $u \neq v$, such that $fu = gu = u$ and $fv = gv = v$.

From equation (2.16), we have

$$\begin{aligned} & p(u,v) \\ =& p(fu,gv) \leq M(u,v) \\ =& \max\Big\{\phi(p(u,v)), \phi(p(fu,v)), \phi(p(u,gv)), \phi\Big(\frac{p(u,v)+p(fu,v)+p(u,gv)}{3}\Big)\Big\} \\ <& p(u,v), \end{aligned} \tag{2.31}$$

which is a contradiction with respect to $u \neq v$. Therefore, $u = v$. Thus the proof is complete.

□

Corollary 2.3. *In a complete partial metric space (X, p), two self weakly increasing continuous mappings f and g on X, satisfying the condition*

$$p(fx,gy) \leq \phi(M(x,y)), \tag{2.32}$$

$$M(x,y) = \max\Big\{p(x,y), p(fx,y), p(x,gy), \frac{p(x,y)+p(fx,y)+p(x,gy)}{3}\Big\} \tag{2.33}$$

for all $x, y \in X$ and all other are same as in Theorem 2.2, then f and g have a unique common fixed point in X.

Corollary 2.4. *In Theorem 2.2, if we replaced second and third ordinates by $p(x, fx)$ and $p(y, fy)$ respectively, and fourth ordinate is replaced by average of only $p(x, fy)$ and $p(fx, y)$ on $M(x, y)$, where f and g are weakly isotone increasing mappings such that $fx \leq gfx \leq fgfx$ and $gx \leq fgx \leq gfgx$ instead of weakly increasing mappings, then it reduces to the main result of [23].*

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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