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Hankel Determinants of Non-Zero Modulus Dixon Elliptic Functions via Quasi C Fractions

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Abstract: The Sumudu transform of the Dixon elliptic function with non-zero modulus $\alpha \neq 0$ for arbitrary powers N is given by the product of quasi C fractions. Next, by assuming the denominators of quasi C fractions as one and applying the Heliermanncorrespondence relating formal power series (Maclaurin series of the Dixon elliptic function) and the regular C fraction, the Hankel determinants are calculated for the non-zero Dixon elliptic functions and shown by taking $\alpha = 0$ to give the Hankel determinants of the Dixon elliptic function with zero modulus. The derived results were back-tracked to the Laplace transform of Dixon elliptic functions.

Keywords: Dixon elliptic functions; Sumudu transform; Hankel determinants; continued fractions; quasi C fractions

MSC: 33E05; 44A10; 11A55; 11C20

1. Introduction

To determine the coefficients of the Maclaurin series of Jacobi elliptic functions, Hankel determinants, and determinants of Bernoulli numbers, continued fractions and Heilermann correspondence were employed in [1]. By using the Fourier series expansions of Jacobi elliptic functions and continued fractions, orthogonal polynomials were calculated and related to each other through the multiplication formulas of the Jacobi elliptic functions in [2]. The Laplace transform of Jacobi elliptic functions was expanded to continued fractions, and it was shown that their coefficients were orthogonal polynomials and the derived dual Hahn polynomials in [3]. The Fourier series and continued fractions expansions of ratios of Jacobi elliptic functions and their Hankel determinants were used in different ways for representing the sum of square numbers derived in the determinant forms in [4]. The Laplace transform of bimodular Jacobi elliptic functions were solved as continued fractions, and then, by using the modular transformation, the results were shown for unimodular Jacobi elliptic functions in [5].

Dixon studied the cubic curve $x^3 + y^3 - 3\alpha xy = 1\,;\ \alpha \neq -1$ for the orthogonal polynomials, where the curve has a double period, which gives rise to the two set of functions $sm\,(x,\alpha)$ and $cm\,(x,\alpha)$, now known as the Dixon elliptic functions in [6]. The examples, the relation to hypergeometric series, modular transformation, and formulae for their ratio were given in [7]. When $\alpha = 0$ in the above cubic curve, their series expansions and transformations were studied in [8]. The Dixon elliptic functions were used in the study of the conformal mapping and geographical structure of world maps, and addition and multiplication formulae for Dixon elliptic functions were derived in [9]. The Laplace transform was applied for the Dixon elliptic functions of both cases $\alpha = 0$ and $\alpha \neq 0$ to expand as the set of continued fractions (in [5]). The above cubic curve and its relation to the Fermat curve were studied for the urn representation and combinatorics in [10]. Number theory-related results (in [4]) followed by the factorial of numbers using the Dixon elliptic functions were given in [11]. The Dixon elliptic functions' relation to trefoil curves and to Weierstrass functions and their derivatives was shown in [12].

The fractional heat equations were solved using the Sumudu transform in [13]. The Sumudu transform embedded with the decomposition method in [14] and the homotopy perturbation method were used to solve the Klein–Gordon equations in [15]. The fractional order Maxwell's equations and ordinary differential equations were solved by the Sumudu transform in [16,17]. The fractional gas dynamics differential equations were solved using the Sumudu transform in [18]. The Sumudu transform calculation, the new definition for trigonometric functions, and their expansion to infinite series were proven with illustrations comprising tables and properties in [19]. The Maxwell's coupled equations were solved by the Sumudu transform for magnetic field solutions in TEMPwaves given in [20]. Without using any of the decomposition, perturbation, or analysis techniques, the Sumudu transform of the functions calculated by differentiating the function and the Symbolic C++ program were given in [21]. The Sumudu transform was applied to the bimodular Jacobi elliptic function [5,8] for arbitrary powers and given as the associated continued fraction and their Hankel determinants, and next, by using the modular transformations, the Sumudu transform of $\tan(x)$ and $\sec(x)$ was derived in [22]. The Sumudu transform was applied to the Dixon elliptic functions with non-zero modulus and obtained the quasi-associated continued fractions and Hankel determinants $H_m^{(1)}(.)$ in [23]. The Sumudu decomposition technique was applied to solve systems of partial differential equations in [24] and systems of ordinary differential equations in [25]. The detailed theory and applications about the continued fractions, elliptic functions, and determinants can be seen in [26–31]. Recently, the Sumudu transform was applied to the Dixon elliptic functions with modulus $\alpha = 0$ and expanded into the associated continued fractions followed by the Hankel determinants $H_m^{(1)}(.)$ in [32]. The discrete inverse Sumudu transform was applied for the first time to solve the Whittaker, Zettl, and algebrogeometric equations and their new exact solutions obtained, and tables comprising elementary functions and their inverse Sumudu transforms were given in [33,34].

The Sumudu transform of the function $f(x)$ defined in the set,

$$A = \left\{ f(x) \,|\, \exists M, \tau_1, \tau_2 > 0, |f(x)| < M e^{\frac{|x|}{\tau_j}}, \text{if } x \in (-1)^j \times [0, \infty) \right\},$$

is given by the integral equation.

$$\mathbb{S}[f(x)](u) \stackrel{\text{def}}{=} F(u) :\stackrel{\text{def}}{=} \int_0^\infty e^{-x} f(ux)\, dx \,;\; u \in (-\tau_1, \tau_2). \tag{1}$$

Through this research communication, the Sumudu transform is applied for the Dixon elliptic functions of arbitrary powers and expanded as the quasi C fractions. Using the numerator coefficients of the quasi C fractions, Hankel determinants are calculated by the correspondence connecting formal power series and the regular C fractions.

2. Preliminaries

The derivative of the Dixon elliptic functions [23] (Equations (1) and (3), page 171 [6], and Equations (1.18) and (1.19), page 9 [5]) takes the following definition,

$$\frac{d}{dx}sm\left(x,\alpha\right)=cm^{2}\left(x,\alpha\right)-\alpha sm\left(x,\alpha\right)\quad\text{and}\quad\frac{d}{dx}cm\left(x,\alpha\right)=-sm^{2}\left(x,\alpha\right)+\alpha cm\left(x,\alpha\right), \tag{2}$$

and has [23] (Equation (1.21), page 10 [5]),

$$sm\left(0,\alpha\right)=0\quad\text{and}\quad cm\left(0,\alpha\right)=1. \tag{3}$$

These functions satisfy the aforesaid cubic curve, and hence (Equation (2), page 171 [6], and Equation (1.22), page 10 [5]) [23]:

$$sm^{3}\left(x,\alpha\right)+cm^{3}\left(x,\alpha\right)-3\alpha sm\left(x,\alpha\right)cm\left(x,\alpha\right)=1. \tag{4}$$

The infinite continued fraction is represented by [23] Equation (2.1.4b, page 18 [26], and Equation (1.2.5$'$), page 8 [27].

$$\cfrac{a_1}{b_1+\cfrac{a_2}{b_2+\cfrac{a_3}{b_3+\cfrac{a_4}{b_4+\ddots}}}}\overset{\text{def}}{\equiv}\frac{a_1}{b_1+}\frac{a_2}{b_2+}\frac{a_3}{b_3+\cdots}\overset{\text{def}}{\equiv}\mathop{\mathbf{K}}_{n=1}^{\infty}\frac{a_n}{b_n}.$$

Definition 1. *Let $a=\{a_n\}$, $b=\{b_n\}$ and u be indeterminate, then the C-fraction (Equation (7.1.1), page 221 [26]) [28] and Equation (54.2) (page 208 [29]) are defined by [23].*

$$1+\mathop{\mathbf{K}}_{n=1}^{\infty}\frac{a_n u^{\beta(n)}}{1}.$$

When the sequence $\beta\left(n\right)$ is constant, then the C-fraction is called the regular C fraction, while the quasi C fraction has the following form.

$$\frac{a_0}{b_0\left(u\right)+}\mathop{\mathbf{K}}_{n=1}^{\infty}\frac{a_n u}{b_n\left(u\right)}.$$

Sometimes, the coefficients $a_n=a_n\left(u\right)$ are functions of u.

Definition 2. *[4,5,23,26] Let $c=\{c_v\}_{v=1}^{\infty}$ be a sequence in $\mathbb{C}$. Then, the Hankel determinants $H_m^{(n)}\left(.\right)$ and $\chi_m\left(.\right)$ are defined by,*

$$H_m^{(n)}\overset{\text{def}}{\equiv}H_m^{(n)}\left(c_v\right):\overset{\text{def}}{\equiv}det\begin{pmatrix}c_n & c_{n+1} & \cdots & c_{m+n-2} & c_{n+m-1}\\ c_{n+1} & c_{n+2} & \cdots & c_{m+n-1} & c_{m+n}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ c_{m+n-1} & c_{m+n} & \cdots & c_{2m+n-3} & c_{2m+n-2}\end{pmatrix}.$$

$$\chi_m \overset{\text{def}}{=} \chi_m\left(c_v\right) :\overset{\text{def}}{=} det \begin{pmatrix} c_1 & c_2 & \cdots & c_{m-1} & c_{m+1} \\ c_2 & c_3 & \cdots & c_m & c_{m+2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_m & c_{m+1} & \cdots & c_{2m-2} & c_{2m} \end{pmatrix}.$$

Remark 1. *[4,5,23,26] $\chi_m(.)$ are obtained from $H_{m+1}^{(1)}(.)$ by deleting the last row and last, but one column. When $n=1$, $H_1^{(1)}(.)=c_1$ and $\chi_1(.)=c_2$.*

The formal power series and regular C fractions are related by the following [4,5,23] (Theorem 7.2, pp. 223–226, [26]) lemma.

Lemma 1. *When the regular C fraction converges to the formal power series:*

$$1+\sum_{v=1}^{\infty} c_v z^v = 1+\underset{n=1}{\overset{\infty}{\mathbf{K}}} \frac{a_n u}{1}\ ;\ \left(a_n \neq 0\right). \tag{5}$$

then,

$$H_m^{(1)}\left(\left[c_v\right]\right) \neq 0 \quad , \quad H_m^{(2)}\left(\left[c_v\right]\right) \neq 0 \quad and \quad a_1 = H_1^{(1)}\left(\left[c_v\right]\right)\ ;\ (m \geq 1). \tag{6}$$

$$a_{2m} := -\frac{H_{m-1}^{(1)}(.)\, H_m^{(2)}(.)}{H_m^{(1)}(.)\, H_{m-1}^{(2)}(.)} \quad and \quad a_{2m+1} := -\frac{H_{m+1}^{(1)}(.)\, H_{m-1}^{(2)}(.)}{H_m^{(1)}(.)\, H_m^{(2)}(.)}\ ;\ (m \geq 1). \tag{7}$$

where $H_0^{(1)}(.)=H_0^{(2)}(.)=1$. Conversely, if Equations (6) and (7) hold, then Equation (5) holds true. Furthermore,

$$H_m^{(2)}\left(\left[c_v\right]\right) := (-1)^m H_m^{(1)}\left(\left[c_v\right]\right) \prod_{j=1}^{m} a_{2j} = (-1)^m H_{m+1}^{(1)}\left(\left[c_v\right]\right) \prod_{j=1}^{m} \frac{1}{a_{2j+1}}\ ;\ (m \geq 1). \tag{8}$$

3. Quasi C Fractions' Expansions of the Dixon Elliptic Functions $(\alpha \neq 0)$

The Laplace transform of the Dixon elliptic functions for modulus non-zero is given as quasi C fractions in [5]. Here, we apply the Sumudu transform Equation (1) for the Dixon elliptic functions given by $sm^N(x,\alpha)\,;\ N \geq 1$, $sm^N(x,\alpha)\, cm(x,\alpha)\,;\ N \geq 0$ and $sm^N(x,\alpha)\, cm^2(x,\alpha)\,;\ N \geq 0$ for the arbitrary powers derived, which leads to the difference equation, then expanding to the quasi C fractions. Next by assuming the denominator of the quasi C fractions as one, using Lemma 1, the Hankel determinants are calculated for the non-zero modulus Dixon elliptic functions.

Theorem 1. *For $j \geq 1$, $\mathbb{S}\left[sm^N(x,\alpha)\right]$ is given by the following quasi C fractions:*

1.

$$\mathbb{S}\left[sm(x,\alpha)\right] := \frac{u}{(1-2\alpha u)(1+\alpha u)+} \underset{n=2}{\overset{\infty}{\mathbf{K}}} \frac{a_n u^3}{b_n(u)} \begin{cases} a_{2j} = (3j-2)(3j-1)^2 \\ b_{2j}(u) = (1-(6j-1)\alpha u) \\ a_{2j+1} = (3j)^2(3j+1) \\ b_{2j+1}(u) = (1-2(3j+1)\alpha u)(1+(3j+1)\alpha u). \end{cases} \tag{9}$$

2.

$$\mathbb{S}\left[sm^2(x,\alpha)\right] := \frac{1}{(1-\alpha u)+} \mathop{\mathbf{K}}_{n=2}^{\infty} \frac{a_n u^3}{b_n(u)} \begin{cases} a_{2j} = (3j-2)^2(3j-1) \\ b_{2j}(u) = (1-2(3j-1)\alpha u)(1+(3j-1)\alpha u) \\ a_{2j+1} = (3j-1)(3j)^2 \\ b_{2j+1}(u) = (1-(6j+1)\alpha u) \end{cases}$$
$$\times \frac{2u^2}{(1-4\alpha u)(1+2\alpha u)+} \mathop{\mathbf{K}}_{n=2}^{\infty} \frac{a_n u^3}{b_n(u)} \begin{cases} a_{2j} = (3j-1)(3j)^2 \\ b_{2j}(u) = (1-(6j+1)\alpha u) \\ a_{2j+1} = (3j+1)^2(3j+2) \\ b_{2j+1}(u) = (1-2(3j+2)\alpha u)(1+(3j+2)\alpha u). \end{cases} \tag{10}$$

3. *For* $N = 3, 6, 9, 12, \cdots$.

$$\mathbb{S}\left[sm^N(x,\alpha)\right] := \prod_{i=1}^{\frac{N}{3}} \frac{(3i-2)u}{(1-(6i-3)\alpha u)+} \mathop{\mathbf{K}}_{n=2}^{\infty} \frac{a_n u^3}{b_n(u)} \begin{cases} a_{2j} = (3j+3i-4)^2(3j+3i-3) \\ b_{2j}(u) = (1-2(3j+3i-3)\alpha u)(1+(3j+3i-3)\alpha u) \\ a_{2j+1} = (3j+3i-3)(3j+3i-2)^2 \\ b_{2j+1}(u) = (1-(6j+6i-3)\alpha u) \end{cases}$$
$$\times \prod_{i=1}^{\frac{N}{3}} \frac{(3i-1)(3i)u^2}{(1-2(3i)\alpha u)(1+(3i)\alpha u)+} \mathop{\mathbf{K}}_{n=2}^{\infty} \frac{a_n u^3}{b_n(u)} \begin{cases} a_{2j} = (3j+3i-3)(3j+3i-2)^2 \\ b_{2j}(u) = (1-(6j+6i-3)\alpha u) \\ a_{2j+1} = (3j+3i-1)^2(3j+3i) \\ b_{2j+1}(u) = (1-2(3j+3i)\alpha u)(1+(3j+3i)\alpha u). \end{cases} \tag{11}$$

4. *For* $N = 4, 7, 10, 13, \cdots$.

$$\mathbb{S}\left[sm^N(x,\alpha)\right] := \frac{u}{(1-2\alpha u)(1+\alpha u)+} \mathop{\mathbf{K}}_{n=2}^{\infty} \frac{a_n u^3}{b_n(u)} \begin{cases} a_{2j} = (3j-2)(3j-1)^2 \\ b_{2j}(u) = (1-(6j-1)\alpha u) \\ a_{2j+1} = (3j)^2(3j+1) \\ b_{2j+1}(u) = (1-2(3j+1)\alpha u)(1+(3j+1)\alpha u) \end{cases}$$
$$\times \prod_{i=1}^{\frac{N-1}{3}} \frac{(3i-1)u}{(1-(6i-1)\alpha u)+} \mathop{\mathbf{K}}_{n=2}^{\infty} \frac{a_n u^3}{b_n(u)} \begin{cases} a_{2j} = (3j+3i-3)^2(3j+3i-2) \\ b_{2j}(u) = (1-2(3j+3i-2)\alpha u)(1+(3j+3i-2)\alpha u) \\ a_{2j+1} = (3j+3i-2)(3j+3i-1)^2 \\ b_{2j+1}(u) = (1-(6j+6i-1)\alpha u) \end{cases} \tag{12}$$
$$\times \prod_{i=1}^{\frac{N-1}{3}} \frac{(3i)(3i+1)u^2}{(1-2(3i+1)\alpha u)(1+(3i+1)\alpha u)+} \mathop{\mathbf{K}}_{n=2}^{\infty} \frac{a_n u^3}{b_n(u)} \begin{cases} a_{2j} = (3j+3i-2)(3j+3i-1)^2 \\ b_{2j}(u) = (1-(6j+6i-1)\alpha u) \\ a_{2j+1} = (3j+3i)^2(3j+3i+1) \\ b_{2j+1}(u) = (1-2(3j+3i+1)\alpha u)(1+(3j+3i+1)\alpha u). \end{cases}$$

5. *For* $N = 5, 8, 11, 14, \cdots$.

$$
\begin{aligned}
\mathbb{S}\left[sm^N(x,\alpha)\right] &:= \frac{1}{(1-\alpha u)+} \underset{n=2}{\overset{\infty}{\mathbf{K}}} \frac{a_n u^3}{b_n(u)} \begin{cases} a_{2j} = (3j-2)^2(3j-1) \\ b_{2j}(u) = (1-2(3j-1)\alpha u)(1+(3j-1)\alpha u) \\ a_{2j+1} = (3j-1)(3j)^2 \\ b_{2j+1}(u) = (1-(6j+1)\alpha u) \end{cases} \\
&\times \frac{2u^2}{(1-4\alpha u)(1+2\alpha u)+} \underset{n=2}{\overset{\infty}{\mathbf{K}}} \frac{a_n u^3}{b_n(u)} \begin{cases} a_{2j} = (3j-1)(3j)^2 \\ b_{2j}(u) = (1-(6j+1)\alpha u) \\ a_{2j+1} = (3j+1)^2(3j+2) \\ b_{2j+1}(u) = (1-2(3j+2)\alpha u)(1+(3j+2)\alpha u) \end{cases} \\
&\times \prod_{i=1}^{\frac{N-2}{3}} \frac{(3i)u}{(1-(6i+1)\alpha u)+} \underset{n=2}{\overset{\infty}{\mathbf{K}}} \frac{a_n u^3}{b_n(u)} \begin{cases} a_{2j} = (3j+3i-2)^2(3j+3i-1) \\ b_{2j}(u) = (1-2(3j+3i-1)\alpha u)(1+(3j+3i-1)\alpha u) \\ a_{2j+1} = (3j+3i-1)(3j+3i)^2 \\ b_{2j+1}(u) = (1-(6j+6i+1)\alpha u) \end{cases} \\
&\times \prod_{i=1}^{\frac{N-2}{3}} \frac{(3i+1)(3i+2)u^2}{(1-2(3i+2)\alpha u)(1+(3i+2)\alpha u)+} \underset{n=2}{\overset{\infty}{\mathbf{K}}} \frac{a_n u^3}{b_n(u)} \begin{cases} a_{2j} = (3j+3i-1)(3j+3i)^2 \\ b_{2j}(u) = (1-(6j+6i+1)\alpha u) \\ a_{2j+1} = (3j+3i+1)^2(3j+3i+2) \\ b_{2j+1}(u) = (1-2(3j+3i+2)\alpha u)(1+(3j+3i+2)\alpha u). \end{cases}
\end{aligned} \tag{13}
$$

Proof. Defining the Sumudu transform of the Dixon elliptic functions by the integral equations, let $N = 0, 1, 2, \cdots$.

$$\mathbb{S}\left[sm^N(x,\alpha)\right] = A_N := \int_0^\infty e^{-x} sm^N(xu,\alpha)\,dx. \tag{14}$$

$$\mathbb{S}\left[sm^N(x,\alpha)\,cm(x,\alpha)\right] = B_N := \int_0^\infty e^{-x} sm^N(xu,\alpha)\,cm(xu,\alpha)\,dx. \tag{15}$$

$$\mathbb{S}\left[sm^N(x,\alpha)\,cm^2(x,\alpha)\right] = C_N := \int_0^\infty e^{-x} sm^N(xu,\alpha)\,cm^2(xu,\alpha)\,dx. \tag{16}$$

Evaluating by parts, using Equations (2)–(4), with $A_0 = 1$, leads to the following:

$$
\begin{aligned}
A_1 &:= uC_0 - \alpha u A_1. \\
A_2 &:= 2uC_1 - 2\alpha u A_2. \\
A_3 &:= 3uC_2 - 3\alpha u A_3. \\
A_N &:= NuC_{N-1} - N\alpha u A_N.
\end{aligned}
$$

Solving with the recurrences of Equations (15) and (16) yields the following quasi C fractions:

$$\frac{A_N}{B_{N-2}} := \frac{(N-1)Nu^2}{(1-2N\alpha u)(1+N\alpha u) + N(N+1)u^2 \frac{B_{N+1}}{A_N}}\ ;\ (N \geq 2). \tag{17}$$

$$\frac{B_N}{A_{N-1}} := \frac{Nu}{(1-(2N+1)\alpha u) + (N+1)u \frac{A_{N+2}}{B_N}}\ ;\ (N \geq 2). \tag{18}$$

When $N = 1, 2$, and 3:

$$A_1 := \frac{u}{(1-2\alpha u)(1+\alpha u) + 2u^2 \frac{B_2}{A_1}}. \tag{19}$$

$$A_2 := B_0 \times \frac{2u^2}{(1-4\alpha u)(1+2\alpha u) + 6u^2 \frac{B_3}{A_2}}. \tag{20}$$

$$A_3 := B_1 \times \frac{6u^2}{(1-6\alpha u)(1+3\alpha u) + 12u^2 \frac{B_4}{A_3}}. \tag{21}$$

Now, Equation (9) is obtained from Equation (19) by iterating with Equations (17) and (18). Next, Equation (10) is obtained from Equation (20) by iterating with Equations (17) and (18) where B_0 is derived from Equation (15). Following the same procedure, Equations (11)–(14) are derived upon continuous iteration of Equations (17) and (18) and after the mathematical simplifications. □

Theorem 2. *For $j \geq 1$, $\mathbb{S}\left[sm^N(x,\alpha)\, cm(x,\alpha)\right]$, given by the following quasi C fractions:*

1.

$$\mathbb{S}\left[cm(x,\alpha)\right] := \frac{1}{(1-\alpha u)+} \mathop{\mathbf{K}}_{n=2}^{\infty} \frac{a_n u^3}{b_n(u)} \begin{cases} a_{2j} = (3j-2)^2(3j-1) \\ b_{2j}(u) = (1-2(3j-1)\alpha u)(1+(3j-1)\alpha u) \\ a_{2j+1} = (3j-1)(3j)^2 \\ b_{2j+1}(u) = (1-(6j+1)\alpha u). \end{cases} \tag{22}$$

2.

$$\mathbb{S}\left[sm(x,\alpha)\, cm(x,\alpha)\right] := \frac{u}{(1-3\alpha u)+} \mathop{\mathbf{K}}_{n=2}^{\infty} \frac{a_n u^3}{b_n(u)} \begin{cases} a_{2j} = (3j-1)^2(3j) \\ b_{2j}(u) = (1-2(3j)\alpha u)(1+(3j)\alpha u) \\ a_{2j+1} = (3j)(3j+1)^2 \\ b_{2j+1}(u) = (1-(6j+3)\alpha u). \end{cases} \tag{23}$$

3.

$$\begin{aligned} \mathbb{S}\left[sm^2(x,\alpha)\, cm(x,\alpha)\right] :=& \frac{u}{(1-2\alpha u)(1+\alpha u)+} \mathop{\mathbf{K}}_{n=2}^{\infty} \frac{a_n u^3}{b_n(u)} \begin{cases} a_{2j} = (3j-2)(3j-1)^2 \\ b_{2j}(u) = (1-(6j-1)\alpha u) \\ a_{2j+1} = (3j)^2(3j+1) \\ b_{2j+1}(u) = (1-2(3j+1)\alpha u)(1+(3j+1)\alpha u) \end{cases} \\ &\times \frac{2u}{(1-5\alpha u)+} \mathop{\mathbf{K}}_{n=2}^{\infty} \frac{a_n u^3}{b_n(u)} \begin{cases} a_{2j} = (3j)^2(3j+1) \\ b_{2j}(u) = (1-2(3j+1)\alpha u)(1+(3j+1)\alpha u) \\ a_{2j+1} = (3j+1)(3j+2)^2 \\ b_{2j+1}(u) = (1-(6j+5)\alpha u). \end{cases} \end{aligned} \tag{24}$$

4. *For* $N = 3, 6, 9, 12, \cdots$.

$$
\begin{aligned}
\mathbb{S}\left[sm^{N}(x,\alpha)\,cm(x,\alpha)\right] &:= \frac{1}{(1-\alpha u)+} \mathop{\mathbf{K}}_{n=2}^{\infty} \frac{a_n u^3}{b_n(u)} \begin{cases} a_{2j} = (3j-2)^2(3j-1) \\ b_{2j}(u) = (1-2(3j-1)\alpha u)(1+(3j-1)\alpha u) \\ a_{2j+1} = (3j-1)(3j)^2 \\ b_{2j+1}(u) = (1-(6j+1)\alpha u) \end{cases} \\
&\times \prod_{i=1}^{\frac{N}{3}} \frac{(3i-2)(3i-1)u^2}{(1-2(3i-1)\alpha u)(1+(3i-1)\alpha u)+} \mathop{\mathbf{K}}_{n=2}^{\infty} \frac{a_n u^3}{b_n(u)} \begin{cases} a_{2j} = (3j+3i-4)(3j+3i-3)^2 \\ b_{2j}(u) = (1-(6j+6i-5)\alpha u) \\ a_{2j+1} = (3j+3i-2)^2(3j+3i-1) \\ b_{2j+1}(u) = (1-2(3j+3i-1)\alpha u)(1+(3j+3i-1)\alpha u) \end{cases} \\
&\times \prod_{i=1}^{\frac{N}{3}} \frac{(3i)u}{(1-(6i+1)\alpha u)+} \mathop{\mathbf{K}}_{n=2}^{\infty} \frac{a_n u^3}{b_n(u)} \begin{cases} a_{2j} = (3j+3i-2)^2(3j+3i-1) \\ b_{2j}(u) = (1-2(3j+3i-1)\alpha u)(1+(3j+3i-1)\alpha u) \\ a_{2j+1} = (3j+3i-1)(3j+3i)^2 \\ b_{2j+1}(u) = (1-(6j+6i+1)\alpha u). \end{cases}
\end{aligned} \tag{25}
$$

5. *For* $N = 4, 7, 10, 13, \cdots$.

$$
\begin{aligned}
\mathbb{S}\left[sm^{N}(x,\alpha)\,cm(x,\alpha)\right] &:= \frac{u}{(1-3\alpha u)+} \mathop{\mathbf{K}}_{n=2}^{\infty} \frac{a_n u^3}{b_n(u)} \begin{cases} a_{2j} = (3j-1)^2(3j) \\ b_{2j}(u) = (1-2(3j)\alpha u)(1+(3j)\alpha u) \\ a_{2j+1} = (3j)(3j+1)^2 \\ b_{2j+1}(u) = (1-(6j+3)\alpha u) \end{cases} \\
&\times \prod_{i=1}^{\frac{N-1}{3}} \frac{(3i-1)(3i)u^2}{(1-2(3i)\alpha u)(1+(3i)\alpha u)+} \mathop{\mathbf{K}}_{n=2}^{\infty} \frac{a_n u^3}{b_n(u)} \begin{cases} a_{2j} = (3j+3i-3)(3j+3i-2)^2 \\ b_{2j}(u) = (1-(6j+6i-3)\alpha u) \\ a_{2j+1} = (3j+3i-1)^2(3j+3i) \\ b_{2j+1}(u) = (1-2(3j+3i)\alpha u)(1+(3j+3i)\alpha u) \end{cases} \\
&\times \prod_{i=1}^{\frac{N-1}{3}} \frac{(3i+1)u}{(1-(6i+3)\alpha u)+} \mathop{\mathbf{K}}_{n=2}^{\infty} \frac{a_n u^3}{b_n(u)} \begin{cases} a_{2j} = (3j+3i-1)^2(3j+3i) \\ b_{2j}(u) = (1-2(3j+3i)\alpha u)(1+(3j+3i)\alpha u) \\ a_{2j+1} = (3j+3i)(3j+3i+1)^2 \\ b_{2j+1}(u) = (1-(6j+6i+3)\alpha u). \end{cases}
\end{aligned} \tag{26}
$$

6. *For* $N = 5, 8, 11, 14, \cdots$.

$$
\begin{aligned}
\mathbb{S}\left[sm^{N}(x,\alpha)\,cm(x,\alpha)\right] &:= \frac{u}{(1-2\alpha u)(1+\alpha u)+} \mathop{\mathbf{K}}_{n=2}^{\infty} \frac{a_n u^3}{b_n(u)} \begin{cases} a_{2j} = (3j-2)(3j-1)^2 \\ b_{2j}(u) = (1-(6j-1)\alpha u) \\ a_{2j+1} = (3j)^2(3j+1) \\ b_{2j+1}(u) = (1-2(3j+1)\alpha u)(1+(3j+1)\alpha u) \end{cases} \\
&\times \frac{2u}{(1-5\alpha u)+} \mathop{\mathbf{K}}_{n=2}^{\infty} \frac{a_n u^3}{b_n(u)} \begin{cases} a_{2j} = (3j)^2(3j+1) \\ b_{2j}(u) = (1-2(3j+1)\alpha u)(1+(3j+1)\alpha u) \\ a_{2j+1} = (3j+1)(3j+2)^2 \\ b_{2j+1}(u) = (1-(6j+5)\alpha u) \end{cases} \\
&\times \prod_{i=1}^{\frac{N-2}{3}} \frac{(3i)(3i+1)u^2}{(1-2(3i+1)\alpha u)(1+(3i+1)\alpha u)+} \mathop{\mathbf{K}}_{n=2}^{\infty} \frac{a_n u^3}{b_n(u)} \begin{cases} a_{2j} = (3j+3i-2)(3j+3i-1)^2 \\ b_{2j}(u) = (1-(6j+6i-1)\alpha u) \\ a_{2j+1} = (3j+3i)^2(3j+3i+1) \\ b_{2j+1}(u) = (1-2(3j+3i+1)\alpha u)(1+(3j+3i+1)\alpha u) \end{cases} \\
&\times \prod_{i=1}^{\frac{N-2}{3}} \frac{(3i+2)u}{(1-(6i+5)\alpha u)+} \mathop{\mathbf{K}}_{n=2}^{\infty} \frac{a_n u^3}{b_n(u)} \begin{cases} a_{2j} = (3j+3i)^2(3j+3i+1) \\ b_{2j}(u) = (1-2(3j+3i+1)\alpha u)(1+(3j+3i+1)\alpha u) \\ a_{2j+1} = (3j+3i+1)(3j+3i+2)^2 \\ b_{2j+1}(u) = (1-(6j+6i+5)\alpha u). \end{cases}
\end{aligned} \tag{27}
$$

Proof. Solving the recurrences of Equation (15):

$$\begin{aligned} B_0 &:= 1 - uA_2 + \alpha u B_0. \\ B_1 &:= u - 2uA_3 + 3\alpha u B_1. \\ B_2 &:= 2uA_1 - 3uA_4 + 5\alpha u B_2. \\ B_3 &:= 3uA_2 - 4uA_5 + 7\alpha u B_3. \\ B_N &:= NuA_{N-1} - (N+1)uA_{N+2} + (2N+1)\alpha u B_N. \end{aligned}$$

For $N = 0$, 1, and 2 in Equation (15), after solving with the recurrences of Equations (14) and (16):

$$B_0 := \frac{1}{(1-\alpha u) + u\frac{A_2}{B_0}}. \tag{28}$$

$$B_1 := \frac{u}{(1-3\alpha u) + 2u\frac{A_3}{B_1}}. \tag{29}$$

$$B_2 := A_1 \times \frac{2u}{(1-5\alpha u) + 3u\frac{A_4}{B_2}}. \tag{30}$$

Now, Equation (22) is derived from Equation (28) upon iterating with Equations (17) and (18). Equation (23) is derived from Equation (29) upon iterating with Equations (17) and (18). Equation (24) is derived from Equation (30) where A_1 given by Equation (19), and both are iterated with Equations (17) and (18). Continuing in the same way, Equations (25)–(27) are obtained by iterations, mathematical calculations and simplification. □

Theorem 3. *For $j \geq 1$. $\mathbb{S}\left[sm^N(x,\alpha)\, cm^2(x,\alpha)\right]$ is given by the following Quasi C fractions:*

1.

$$\mathbb{S}\left[cm^2(x,\alpha)\right] := \frac{1}{(1-2\alpha u)+} \mathop{\boldsymbol{K}}_{n=2}^{\infty} \frac{a_n(u)u^3}{b_n(u)} \begin{cases} a_{2j}(u) = (3j-2)(3j-1)^2(1+(3j+1)\alpha u) \\ b_{2j}(u) = \Upsilon_{3j-1} \\ a_{2j+1}(u) = (3j)^2(3j+1)(1+(3j-2)\alpha u) \\ b_{2j+1}(u) = (1-(6j+2)\alpha u). \end{cases} \tag{31}$$

2.

$$\begin{aligned} \mathbb{S}\left[sm(x,\alpha)\, cm^2(x,\alpha)\right] :=& \frac{1}{(1-\alpha u)+} \mathop{\boldsymbol{K}}_{n=2}^{\infty} \frac{a_n u^3}{b_n(u)} \begin{cases} a_{2j} = (3j-2)^2(3j-1) \\ b_{2j}(u) = (1-2(3j-1)\alpha u)(1+(3j-1)\alpha u) \\ a_{2j+1} = (3j-1)(3j)^2 \\ b_{2j+1}(u) = (1-(6j+1)\alpha u) \end{cases} \\ &\times \frac{u}{(1-4\alpha u)+} \mathop{\boldsymbol{K}}_{n=2}^{\infty} \frac{a_n(u)u^3}{b_n(u)} \begin{cases} a_{2j}(u) = (3j-1)(3j)^2(1+(3j+2)\alpha u) \\ b_{2j}(u) = \Upsilon_{3j} \\ a_{2j+1}(u) = (3j+1)^2(3j+2)(1+(3j-1)\alpha u) \\ b_{2j+1}(u) = (1-(6j+4)\alpha u). \end{cases} \end{aligned} \tag{32}$$

3.

$$\mathbb{S}\left[sm^2(x,\alpha)\,cm^2(x,\alpha)\right] := \frac{u}{(1-3\alpha u)+}\mathop{\mathbf{K}}_{n=2}^{\infty}\frac{a_n u^3}{b_n(u)}\begin{cases}a_{2j}=(3j-1)^2(3j)\\ b_{2j}(u)=(1-2(3j)\alpha u)(1+(3j)\alpha u)\\ a_{2j+1}=(3j)(3j+1)^2\\ b_{2j+1}(u)=(1-(6j+3)\alpha u)\end{cases}$$
$$\times\frac{2u}{(1-6\alpha u)+}\mathop{\mathbf{K}}_{n=2}^{\infty}\frac{a_n(u) u^3}{b_n(u)}\begin{cases}a_{2j}(u)=(3j)(3j+1)^2(1+(3j+3)\alpha u)\\ b_{2j}(u)=\Upsilon_{3j+1}\\ a_{2j+1}(u)=(3j+2)^2(3j+3)(1+(3j)\alpha u)\\ b_{2j+1}(u)=(1-(6j+6)\alpha u).\end{cases} \tag{33}$$

4.

$$\mathbb{S}\left[sm^3(x,\alpha)\,cm^2(x,\alpha)\right] := \frac{u}{(1-2\alpha u)(1+\alpha u)+}\mathop{\mathbf{K}}_{n=2}^{\infty}\frac{a_n u^3}{b_n(u)}\begin{cases}a_{2j}=(3j-2)(3j-1)^2\\ b_{2j}(u)=(1-(6j-1)\alpha u)\\ a_{2j+1}=(3j)^2(3j+1)\\ b_{2j+1}(u)=(1-2(3j+1)\alpha u)(1+(3j+1)\alpha u)\end{cases}$$
$$\times\frac{2u}{(1-5\alpha u)+}\mathop{\mathbf{K}}_{n=2}^{\infty}\frac{a_n u^3}{b_n(u)}\begin{cases}a_{2j}=(3j)^2(3j+1)\\ b_{2j}(u)=(1-2(3j+1)\alpha u)(1+(3j+1)\alpha u)\\ a_{2j+1}=(3j+1)(3j+2)^2\\ b_{2j+1}(u)=(1-(6j+5)\alpha u)\end{cases} \tag{34}$$
$$\times\frac{3u}{(1-8\alpha u)+}\mathop{\mathbf{K}}_{n=2}^{\infty}\frac{a_n(u) u^3}{b_n(u)}\begin{cases}a_{2j}(u)=(3j+1)(3j+2)^2(1+(3j+4)\alpha u)\\ b_{2j}(u)=\Upsilon_{3j+2}\\ a_{2j+1}(u)=(3j+3)^2(3j+4)(1+(3j+1)\alpha u)\\ b_{2j+1}(u)=(1-(6j+8)\alpha u).\end{cases}$$

5. *For $N = 4, 7, 10, 13, \cdots$.*

$$\mathbb{S}\left[sm^N(x,\alpha)\,cm^2(x,\alpha)\right] := \frac{1}{(1-\alpha u)+}\mathop{\mathbf{K}}_{n=2}^{\infty}\frac{a_n u^3}{b_n(u)}\begin{cases}a_{2j}=(3j-2)^2(3j-1)\\ b_{2j}(u)=(1-2(3j-1)\alpha u)(1+(3j-1)\alpha u)\\ a_{2j+1}=(3j-1)(3j)^2\\ b_{2j+1}(u)=(1-(6j+1)\alpha u)\end{cases}$$
$$\times\prod_{i=1}^{\frac{N-1}{3}}\frac{(3i-2)(3i-1)u^2}{(1-2(3i-1)\alpha u)(1+(3i-1)\alpha u)+}\mathop{\mathbf{K}}_{n=2}^{\infty}\frac{a_n u^3}{b_n(u)}\begin{cases}a_{2j}=(3j+3i-4)(3j+3i-3)^2\\ b_{2j}(u)=(1-(6j+6i-5)\alpha u)\\ a_{2j+1}=(3j+3i-2)^2(3j+3i-1)\\ b_{2j+1}(u)=(1-2(3j+3i-1)\alpha u)(1+(3j+3i-1)\alpha u)\end{cases}$$
$$\times\prod_{i=1}^{\frac{N-1}{3}}\frac{(3i)u}{(1-(6i+1)\alpha u)+}\mathop{\mathbf{K}}_{n=2}^{\infty}\frac{a_n u^3}{b_n(u)}\begin{cases}a_{2j}=(3j+3i-2)^2(3j+3i-1)\\ b_{2j}(u)=(1-2(3j+3i-1)\alpha u)(1+(3j+3i-1)\alpha u)\\ a_{2j+1}=(3j+3i-1)(3j+3i)^2\\ b_{2j+1}(u)=(1-(6j+6i+1)\alpha u)\end{cases} \tag{35}$$
$$\times\frac{Nu}{(1-(2(N+1))\alpha u)+}\mathop{\mathbf{K}}_{n=2}^{\infty}\frac{a_n(u) u^3}{b_n(u)}\begin{cases}a_{2j}(u)=(3j+N-2)(3j+N-1)^2(1+(3j+N+1)\alpha u)\\ b_{2j}(u)=\Upsilon_{3j+N-1}\\ a_{2j+1}(u)=(3j+N)^2(3j+N+1)(1+(3j+N-2)\alpha u)\\ b_{2j+1}(u)=(1-(6j+2(N+1))\alpha u).\end{cases}$$

6. For $N = 5, 8, 11, 14, \cdots$.

$$\mathbb{S}\left[sm^N(x,\alpha)\,cm^2(x,\alpha)\right] := \frac{u}{(1-3\alpha u)+} \overset{\infty}{\underset{n=2}{\boldsymbol{K}}} \frac{a_n u^3}{b_n(u)} \begin{cases} a_{2j} = (3j-1)^2(3j) \\ b_{2j}(u) = (1-2(3j)\alpha u)(1+(3j)\alpha u) \\ a_{2j+1} = (3j)(3j+1)^2 \\ b_{2j+1}(u) = (1-(6j+3)\alpha u) \end{cases}$$
$$\times \prod_{i=1}^{\frac{N-2}{3}} \frac{(3i-1)(3i)u^2}{(1-2(3i)\alpha u)(1+(3i)\alpha u)+} \overset{\infty}{\underset{n=2}{\boldsymbol{K}}} \frac{a_n u^3}{b_n(u)} \begin{cases} a_{2j} = (3j+3i-3)(3j+3i-2)^2 \\ b_{2j}(u) = (1-(6j+6i-3)\alpha u) \\ a_{2j+1} = (3j+3i-1)^2(3j+3i) \\ b_{2j+1}(u) = (1-2(3j+3i)\alpha u)(1+(3j+3i)\alpha u) \end{cases}$$
$$\times \prod_{i=1}^{\frac{N-2}{3}} \frac{(3i+1)u}{(1-(6i+3)\alpha u)+} \overset{\infty}{\underset{n=2}{\boldsymbol{K}}} \frac{a_n u^3}{b_n(u)} \begin{cases} a_{2j} = (3j+3i-1)^2(3j+3i) \\ b_{2j}(u) = (1-2(3j+3i)\alpha u)(1+(3j+3i)\alpha u) \\ a_{2j+1} = (3j+3i)(3j+3i+1)^2 \\ b_{2j+1}(u) = (1-(6j+6i+3)\alpha u) \end{cases} \tag{36}$$
$$\times \frac{Nu}{(1-(2(N+1))\alpha u)+} \overset{\infty}{\underset{n=2}{\boldsymbol{K}}} \frac{a_n(u) u^3}{b_n(u)} \begin{cases} a_{2j}(u) = (3j+N-2)(3j+N-1)^2(1+(3j+N+1)\alpha u) \\ b_{2j}(u) = Y_{3j+N-1} \\ a_{2j+1}(u) = (3j+N)^2(3j+N+1)(1+(3j+N-2)\alpha u) \\ b_{2j+1}(u) = (1-(6j+2(N+1))\alpha u). \end{cases}$$

7. For $N = 6, 9, 12, 15, \cdots$.

$$\mathbb{S}\left[sm^N(x,\alpha)\,cm^2(x,\alpha)\right] := \frac{u}{(1-2\alpha u)(1+\alpha u)+} \overset{\infty}{\underset{n=2}{\boldsymbol{K}}} \frac{a_n u^3}{b_n(u)} \begin{cases} a_{2j} = (3j-2)(3j-1)^2 \\ b_{2j}(u) = (1-(6j-1)\alpha u) \\ a_{2j+1} = (3j)^2(3j+1) \\ b_{2j+1}(u) = (1-2(3j+1)\alpha u)(1+(3j+1)\alpha u) \end{cases}$$
$$\times \frac{2u}{(1-5\alpha u)+} \overset{\infty}{\underset{n=2}{\boldsymbol{K}}} \frac{a_n u^3}{b_n(u)} \begin{cases} a_{2j} = (3j)^2(3j+1) \\ b_{2j}(u) = (1-2(3j+1)\alpha u)(1+(3j+1)\alpha u) \\ a_{2j+1} = (3j+1)(3j+2)^2 \\ b_{2j+1}(u) = (1-(6j+5)\alpha u) \end{cases}$$
$$\times \prod_{i=1}^{\frac{N-3}{3}} \frac{(3i)(3i+1)u^2}{(1-2(3i+1)\alpha u)(1+(3i+1)\alpha u)+} \overset{\infty}{\underset{n=2}{\boldsymbol{K}}} \frac{a_n u^3}{b_n(u)} \begin{cases} a_{2j} = (3j+3i-2)(3j+3i-1)^2 \\ b_{2j}(u) = (1-(6j+6i-1)\alpha u) \\ a_{2j+1} = (3j+3i)^2(3j+3i+1) \\ b_{2j+1}(u) = (1-2(3j+3i+1)\alpha u)(1+(3j+3i+1)\alpha u) \end{cases} \tag{37}$$
$$\times \prod_{i=1}^{\frac{N-3}{3}} \frac{(3i+2)u}{(1-(6i+5)\alpha u)+} \overset{\infty}{\underset{n=2}{\boldsymbol{K}}} \frac{a_n u^3}{b_n(u)} \begin{cases} a_{2j} = (3j+3i)^2(3j+3i+1) \\ b_{2j}(u) = (1-2(3j+3i+1)\alpha u)(1+(3j+3i+1)\alpha u) \\ a_{2j+1} = (3j+3i+1)(3j+3i+2)^2 \\ b_{2j+1}(u) = (1-(6j+6i+5)\alpha u) \end{cases}$$
$$\times \frac{Nu}{(1-(2(N+1))\alpha u)+} \overset{\infty}{\underset{n=2}{\boldsymbol{K}}} \frac{a_n(u) u^3}{b_n(u)} \begin{cases} a_{2j}(u) = (3j+N-2)(3j+N-1)^2(1+(3j+N+1)\alpha u) \\ b_{2j}(u) = Y_{3j+N-1} \\ a_{2j+1}(u) = (3j+N)^2(3j+N+1)(1+(3j+N-2)\alpha u) \\ b_{2j+1}(u) = (1-(6j+2(N+1))\alpha u). \end{cases}$$

Proof. Evaluating by parts, Equation (16) gives:

$$\begin{aligned} C_0 &:= 1 - 2uB_2 + 2\alpha u C_0. \\ C_1 &:= uB_0 - 3uB_3 + 4\alpha u C_1. \\ C_2 &:= 2uB_1 - 4uB_4 + 6\alpha u C_2. \\ C_3 &:= 3uB_2 - 5uB_5 + 8\alpha u C_3. \\ C_N &:= NuB_{N-1} - (N+2)uB_{N+2} + (2N+2)\alpha u C_N. \end{aligned}$$

Solving with recurrences of Equations (14) and (15) yields the quasi C fractions:

$$\frac{C_N}{B_{N-1}} := \frac{Nu}{(1-(2N+2)\alpha u) + (N+2)u\frac{B_{N+2}}{C_N}}\ ;\ (N \geq 1). \tag{38}$$

$$\frac{B_N}{C_{N-2}} := \frac{(N-1)N(1+(N+2)\alpha u)u^2}{Y_N + (N+1)(N+2)(1+(N-1)\alpha u)u^2\frac{C_{N+1}}{B_N}}\ ;\ (N \geq 2). \tag{39}$$

where,

$$Y_N = Y_N(u,\alpha) := (1-(2N+1)\alpha u)(1+(N-1)\alpha u)(1+(N+2)\alpha u). \tag{40}$$

For $N = 0, 1$ and 2 in Equation (16).

$$C_0 := \frac{1}{(1-2\alpha u) + 2u\frac{B_2}{C_0}}. \tag{41}$$

$$C_1 := B_0 \times \frac{u}{(1-4\alpha u) + 3u\frac{B_3}{C_1}}. \tag{42}$$

$$C_2 := B_1 \times \frac{2u}{(1-6\alpha u) + 4u\frac{B_4}{C_2}}. \tag{43}$$

Hence, Equation (31) is obtained by iterating Equations (38) and (39) starting with Equation (41). Equation (32) is obtained from Equation (42) with B_0 given by Equation (28) iterating both respectively with Equations (17), (18) and Equations (38), (39). Following the same way, Equation (33) is obtained from Equation (43). Equations (34)–(37) are derived by continuous iteration of Equations (38) and (39), giving the results. □

4. Hankel Determinants of the Dixon Elliptic Functions ($\alpha \neq 0$)

By assuming the denominator of the quasi C fractions as one, Lemma 1 is applied to the coefficients of the quasi C fractions for deriving the Hankel determinants of non-zero modulus. In Lemma 1, $H^{(1)}_{(.)}([c_v])$ represents the Hankel determinants of Dixon elliptic functions with $\alpha = 0$, which are calculated from the associated continued fractions. As quasi C fractions are derived in this work, the quasi associated continued fractions are considered, which were discussed in detail by the authors in [23]. Therefore, $H^{(1)}_{(.)}([c_v])$ of Lemma 1 is taken from [23], which are Hankel determinants of the non-zero modulus Dixon elliptic functions.

Theorem 4. *Hankel determinants of the Dixon elliptic function* $sm^N(x,\alpha)$ *are given by the following equations:*

1.

$$H_m^{(2)}\left([sm(x,\alpha)]_{3v+1}\right) := \begin{cases} -4(1-5\alpha u)\ ;\ m=1. & (44)\\ 2400(1-5\alpha u)^2 E_4\ ;\ m=2. & (45)\\ (-1)^m 6^{(m-1)}(1-5\alpha u)^m \prod_{j=0}^{m-1} E_{3j+4}^{(m-j-1)} \prod_{j=1}^{m-2} H_{3j+1}^{(m-j-1)} & \\ \times \prod_{j=1}^{m}(3j-2)(3j-1)^2\ ;\ m\geq 3. & (46)\end{cases}$$

2.

$$H_m^{(2)}\left(\left[sm^2(x,\alpha)\right]_{3v+2}\right) := \begin{cases} -36(1-7\alpha u)\ ;\ m=1. & (47)\\ (-1)^m 2^m (1-7\alpha u)^m \prod_{j=1}^{m-1} E_{3j+2}^{(m-j)} H_{3j-1}^{(m-j)} \prod_{j=1}^{m}(3j-1)(3j)^2\ ;\ m\geq 2. & (48)\end{cases}$$

3. *For* $N = 3, 6, 9, 12, \cdots$.

$$H_m^{(2)}\left(\left[sm^N(x,\alpha)\right]_{3v+N}\right) := \begin{cases} -(3i)(3i+1)^2 E_{3i}\ ;\ m=1. & (49)\\ (-1)^m E_{3i}^m \prod_{j=1}^{m-1} E_{3j+3i}^{(m-j)} H_{3j+3i-3}^{(m-j)} \prod_{j=1}^{m}(3j+3i-3)(3j+3i-2)^2\ ;\ m\geq 2. & (50)\end{cases}$$

where $i = 1, 2, 3, \cdots, \frac{N}{3}$.

4. *For* $N = 4, 7, 10, 13, \cdots$.

$$H_m^{(2)}\left(\left[sm^N(x,\alpha)\right]_{3v+N}\right) := \begin{cases} -(3i+1)(3i+2)^2 E_{3i+1}\ ;\ m=1. & (51)\\ (-1)^m E_{3i+1}^m \prod_{j=1}^{m-1} E_{3j+3i+1}^{(m-j)} H_{3j+3i-2}^{(m-j)} \prod_{j=1}^{m}(3j+3i-2)(3j+3i-1)^2\ ;\ m\geq 2. & (52)\end{cases}$$

where $i = 1, 2, 3, \cdots, \frac{N-1}{3}$.

5. *For* $N = 5, 8, 11, 14, \cdots$.

$$H_m^{(2)}\left(\left[sm^N(x,\alpha)\right]_{3v+N}\right) := \begin{cases} -(3i+2)(3i+3)^2 E_{3i}\ ;\ m=1. & (53)\\ (-1)^m E_{3i+2}^m \prod_{j=1}^{m-1} E_{3j+3i+2}^{(m-j)} H_{3j+3i-1}^{(m-j)} \prod_{j=1}^{m}(3j+3i-1)(3j+3i)^2\ ;\ m\geq 2. & (54)\end{cases}$$

where $i = 1, 2, 3, \cdots, \frac{N-2}{3}$.

Here, $E_{(.)}$ *and* $H_{(.)}$ *are given by the following polynomials for* $N \geq 3$ *from [23].*

$$E_N := (N-2)(N-1)N(1-(2N+3)\alpha u).$$
$$H_N := N(N+1)(N+2)(1-(2N-3)\alpha u).$$

Proof. Let $\left[sm^N(x,\alpha)\right]_{3v+N}$ denote the coefficients in the Maclaurin series of $sm^N(x,\alpha)$.

$$sm^N(x,\alpha) := \sum_{v=0}^{\infty} \frac{[sm^N(x,\alpha)]_{3v+N} x^{3v+N}}{(3v+N)!} \ ; \ N = 1,2,3,\cdots .$$

Assuming the denominators of Theorem 1 as unity and applying Equation (8) of Lemma 1 to the coefficients of Equations (9)–(13) (where $H^{(1)}_{(.)}(.)$ are the Hankel determinants of the quasi associated continued fraction given in [23]), iterating and simplifying complete the proof. □

Theorem 5. *Hankel determinants of the Dixon elliptic function $sm^N(x,\alpha)\, cm(x,\alpha)$ are given by the following equations:*

1.

$$H^{(2)}_m\left([cm(x,\alpha)]_{3v}\right) := \begin{cases} -2P^*_0 \ ; \ m = 1. & (55)\\ 960(P^*_0)^2 P_3 \ ; \ m = 2. & (56)\\ (-1)^m 6^{m-1} (P^*_0)^m P_3^{m-1} \prod_{j=1}^{m-2} S_{3j}^{(m-j-1)} P_{3j+3}^{(m-j-1)} \prod_{j=1}^{m} (3j-2)^2(3j-1) \ ; \ m \geq 3. & (57) \end{cases}$$

2.

$$H^{(2)}_m\left([sm(x,\alpha)cm(x,\alpha)]_{3v+1}\right) := \begin{cases} -12P^*_1 \ ; \ m = 1. & (58)\\ (-1)^m (P^*_1)^m \prod_{j=1}^{m-1} S_{3j-2}^{(m-j)} P_{3j+1}^{(m-j)} \prod_{j=1}^{m} (3j-1)^2(3j) \ ; \ m \geq 2. & (59) \end{cases}$$

3.

$$H^{(2)}_m\left(\left[sm^2(x,\alpha)cm(x,\alpha)\right]_{3v+2}\right) := \begin{cases} -72P^*_2 \ ; \ m = 1. & (60)\\ (-1)^m 2^m (P^*_2)^m \prod_{j=1}^{m-1} S_{3j-1}^{(m-j)} P_{3j+2}^{(m-j)} \prod_{j=1}^{m} (3j)^2(3j+1) \ ; \ m \geq 2. & (61) \end{cases}$$

4. *For $N = 3,6,9,12,\cdots$.*

$$H^{(2)}_m\left(\left[sm^N(x,\alpha)cm(x,\alpha)\right]_{3v+N}\right) := \begin{cases} -(3i+1)^2(3i+2)P_{3i} \ ; \ m = 1. & (62)\\ (-1)^m (P_{3i})^m \prod_{j=1}^{m-1} S_{3j+3i-3}^{(m-j)} P_{3j+3i}^{(m-j)} & \\ \times \prod_{j=1}^{m} (3j+3i-2)^2(3j+3i-1) \ ; \ m \geq 2, & (63) \end{cases}$$

where $i = 1,2,3,\cdots \frac{N}{3}$.

5. *For $N = 4,7,10,13,\cdots$.*

$$H^{(2)}_m\left(\left[sm^N(x,\alpha)cm(x,\alpha)\right]_{3v+N}\right) := \begin{cases} -(3i+2)^2(3i+3)P_{3i+1} \ ; \ m = 1. & (64)\\ (-1)^m (P_{3i+1})^m \prod_{j=1}^{m-1} S_{3j+3i-2}^{(m-j)} P_{3j+3i+1}^{(m-j)} & \\ \times \prod_{j=1}^{m} (3j+3i-1)^2(3j+3i) \ ; \ m \geq 2, & (65) \end{cases}$$

where $i = 1, 2, 3, \cdots \frac{N-1}{3}$.

6. *For* $N = 5, 8, 11, 14, \cdots$.

$$H_m^{(2)}\left(\left[sm^N(x,\alpha)cm(x,\alpha)\right]_{3v+N}\right) := \begin{cases} -(3i+3)^2(3i+4)P_{3i+2}\ ;\ m=1. & (66)\\ (-1)^m(P_{3i+2})^m \prod_{j=1}^{m-1} S_{3j+3i-1}^{(m-j)} P_{3j+3i+2}^{(m-j)} & \\ \times \prod_{j=1}^{m}(3j+3i)^2(3j+3i+1)\ ;\ m \geq 2, & (67) \end{cases}$$

where $i = 1, 2, 3, \cdots \frac{N-2}{3}$.

Here, $P^*_{(.)}$ *and* $S_{(.)}$ *are given by the following polynomials in [23].*

$$\begin{aligned} P_0^*(u,\alpha) :=& (1-4\alpha u)(1+2\alpha u).\\ P_1^*(u,\alpha) :=& (1-6\alpha u)(1+3\alpha u).\\ P_2^*(u,\alpha) :=& (1-8\alpha u)(1+4\alpha u).\\ P_N(u,\alpha) :=& (N-2)(N-1)N(1-(2N+4)\alpha u)(1+(N+2)\alpha u)\ ;\ (N \geq 3).\\ S_N(u,\alpha) :=& (N+1)(N+2)(N+3)(1-(2N-2)\alpha u)(1+(N-1)\alpha u)\ ;\ (N \geq 1). \end{aligned}$$

Proof. Let $\left[sm^N(x,\alpha)cm(x,\alpha)\right]_{3v+N}$ denote the coefficients of series for $sm^N(x,\alpha)cm(x,\alpha)$.

$$sm^N(x,\alpha)cm(x,\alpha) := \sum_{v=0}^{\infty} \frac{[sm^N(x,\alpha)cm(x,\alpha)]_{3v+N}\, x^{3v+N}}{(3v+N)!}\ ;\ N = 0, 1, 2, \cdots .$$

Assuming the denominator of Theorem 2 as unity and applying the coefficients of Theorem 2 in Equation (8) of Lemma 1 (where $H_{(.)}^{(1)}(.)$ are the Hankel determinants of the quasi associated continued fraction given in [23]), iterating and simplifying complete the proof. □

Theorem 6. *Hankel determinants of the Dixon elliptic function* $sm^N(x,\alpha)\, cm^2(x,\alpha)$ *are given by the following equations:*

1.

$$H_m^{(2)}\left(\left[cm^2(x,\alpha)\right]_{3v}\right) := \begin{cases} -4T_0^*(1+4\alpha u)\ ;\ m=1. & (68)\\ 400(T_0^*)^2 X_0^* T_3(1+4\alpha u)(1+7\alpha u)\ ;\ m=2. & (69)\\ (-1)^m(T_0^*)^m(X_0^*)^{m-1}T_3^{m-1}\prod_{j=1}^{m-2} X_{3j}^{(m-j-1)}T_{3j+3}^{(m-j-1)} & \\ \times \prod_{j=1}^{m}(3j-2)(3j-1)^2(1+(3j+1)\alpha u)\ ;\ m \geq 3. & (70) \end{cases}$$

2.

$$H_m^{(2)}\left(\left[sm(x,\alpha)cm^2(x,\alpha)\right]_{3v+1}\right) := \begin{cases} -18T_1^*(1+5\alpha u)\ ;\ m=1. & (71) \\ 3240(T_1^*)^2X_1^*T_4(1+5\alpha u)(1+8\alpha u)\ ;\ m=2. & (72) \\ (-1)^m(T_1^*)^m(X_1^*)^{m-1}T_4^{m-1}\prod_{j=1}^{m-2}X_{3j+1}^{(m-j-1)}T_{3j+4}^{(m-j-1)} & \\ \times\prod_{j=1}^{m}(3j-1)(3j)^2(1+(3j+2)\alpha u)\ ;\ m\geq 3. & (73) \end{cases}$$

3.

$$H_m^{(2)}\left(\left[sm^2(x,\alpha)cm^2(x,\alpha)\right]_{3v+1}\right) := \begin{cases} -48T_2^*(1+6\alpha u)\ ;\ m=1. & (74) \\ (-1)^m(T_2^*)^m\prod_{j=1}^{m-1}X_{3j-1}^{(m-j)}T_{3j+2}^{(m-j)} & \\ \times\prod_{j=1}^{m}(3j)(3j+1)^2(1+(3j+3)\alpha u)\ ;\ m\geq 2. & (75) \end{cases}$$

4.

$$H_m^{(2)}\left(\left[sm^3(x,\alpha)cm^2(x,\alpha)\right]_{3v+3}\right) := \begin{cases} -100T_3(1+7\alpha u)\ ;\ m=1. & (76) \\ (-1)^mT_3^m\prod_{j=1}^{m-1}X_{3j}^{(m-j)}T_{3j+3}^{(m-j)} & \\ \times\prod_{j=1}^{m}(3j+1)(3j+2)^2(1+(3j+4)\alpha u)\ ;\ m\geq 2. & (77) \end{cases}$$

5. *For* $N=4,7,10,13,\cdots$.

$$H_m^{(2)}\left(\left[sm^N(x,\alpha)cm^2(x,\alpha)\right]_{3v+N}\right) := \begin{cases} -(3i+2)(3i+3)^2(1+(3i+5)\alpha u)T_{3i+1}\ ;\ m=1. & (78) \\ (-1)^mT_{3i+1}^m\prod_{j=1}^{m-1}X_{3j+3i-2}^{(m-j)}T_{3j+3i+1}^{(m-j)} & \\ \times\prod_{j=1}^{m}(3j+3i-1)(3j+3i)^2(1+(3j+3i+2)\alpha u)\ ;\ m\geq 2, & (79) \end{cases}$$

where $i=1,2,3,\cdots\frac{N-1}{3}$.

6. *For* $N=5,8,11,14,\cdots$.

$$H_m^{(2)}\left(\left[sm^N(x,\alpha)cm^2(x,\alpha)\right]_{3v+N}\right) := \begin{cases} -(3i+3)(3i+4)^2(1+(3i+6)\alpha u)T_{3i+2}\ ;\ m=1. & (80) \\ (-1)^mT_{3i+2}^m\prod_{j=1}^{m-1}X_{3j+3i-1}^{(m-j)}T_{3j+3i+2}^{(m-j)} & \\ \times\prod_{j=1}^{m}(3j+3i)(3j+3i+1)^2(1+(3j+3i+3)\alpha u)\ ;\ m\geq 2, & (81) \end{cases}$$

where $i=1,2,3,\cdots\frac{N-2}{3}$.

7. *For* $N = 6, 9, 12, 15, \cdots$.

$$H_m^{(2)}\left(\left[sm^N(x,\alpha)cm^2(x,\alpha)\right]_{3v+N}\right) := \begin{cases} -(3i+4)(3i+5)^2(1+(3i+7)\alpha u)T_{3i+3}\ ;\ m=1. & (82)\\ (-1)^m T_{3i+3}^m \prod_{j=1}^{m-1} X_{3j+3i}^{(m-j)} T_{3j+3i+3}^{(m-j)} & \\ \times \prod_{j=1}^{m}(3j+3i+1)(3j+3i+2)^2(1+(3j+3i+4)\alpha u)\ ;\ m\geq 2, & (83) \end{cases}$$

where $i = 1, 2, 3, \cdots \frac{N-3}{3}$.

Here, $T^*_{(.)}$, $X^*_{(.)}$, $T_{(.)}$ *and* $X_{(.)}$ *are given by the following polynomials in [23].*

$$\begin{aligned}
T_0^*(u,\alpha) &:= (1+\alpha u)(1+4\alpha u)(1-5\alpha u).\\
X_0^*(u,\alpha) &:= 24(1+\alpha u).\\
T_1^*(u,\alpha) &:= (1+2\alpha u)(1-7\alpha u)(1+5\alpha u).\\
X_1^*(u,\alpha) &:= 60(1-\alpha u)(1+2\alpha u).\\
T_2^*(u,\alpha) &:= 2(1+3\alpha u)(1-9\alpha u)(1+6\alpha u).\\
T_N(u,\alpha) &:= (N-2)(N-1)N(1+(N+1)\alpha u)(1+(N+4)\alpha u)(1-(2N+5)\alpha u)\ ;\ (N\geq 3).\\
X_N(u,\alpha) &:= (N+2)(N+3)(N+4)(1+(N+1)\alpha u)(1+(N-2)\alpha u)(1-(2N-1)\alpha u)\ ;\ (N\geq 1).
\end{aligned}$$

Proof. Let $\left[sm^N(x,\alpha)cm^2(x,\alpha)\right]_{3v+N}$ denote the coefficients in the series of $sm^N(x,\alpha)cm^2(x,\alpha)$.

$$sm^N(x,\alpha)cm^2(x,\alpha) := \sum_{v=0}^{\infty} \frac{[sm^N(x,\alpha)cm^2(x,\alpha)]_{3v+N}x^{3v+N}}{(3v+N)!}\ ;\ N = 0, 1, 2, \cdots .$$

Assuming the denominator of Theorem 3 as unity and iterating by applying the coefficients of Equations (31)–(37) in Equation (8) of Lemma 1 (where $H_{(.)}^{(1)}(.)$ are the Hankel determinants of the quasi associated continued fraction given in [23]) give the Hankel determinants. □

5. Applications and Numerical Examples

The Hankel determinants of the Dixon elliptic functions with zero modulus are obtained by substituting the modulus $\alpha = 0$ in Theorems 4–6. When $\alpha = 0$ in Equations (44)–(46), this gives the $H_m^{(2)}(.)$ of $sm\,(x,0)$, which are given in Table 1 for the different values of m. When $\alpha = 0$ in Equations (47) and (48), this gives the $H_m^{(2)}(.)$ of $sm^2\,(x,0)$, which are given in Table 1 for the different values of m.

Table 1. Hankel determinants $H_m^{(2)}(.)$ of the Dixon elliptic functions $sm\,(x,0)$ and $sm^2\,(x,0)$ for m from 1–10.

m	$H_m^{(2)}\,(sm\,(x,0))$	$H_m^{(2)}\,(sm^2\,(x,0))$
1	-2^2	$-2^2\,3^2$
2	$2^8\,3^2\,5^2$	$2^{10}\,3^6\,5^2$
3	$-2^{22}\,3^6\,5^4\,7^2$	$-2^{24}\,3^{14}\,5^4\,7^2$
4	$2^{38}\,3^{14}\,5^8\,7^4\,11^2$	$2^{44}\,3^{24}\,5^8\,7^4\,11^2$
5	$-2^{60}\,3^{24}\,5^{12}\,7^8\,11^4\,13^2$	$-2^{66}\,3^{36}\,5^{14}\,7^8\,11^4\,13^2$
6	$2^{90}\,3^{36}\,5^{18}\,7^{12}\,11^6\,13^4\,17^2$	$2^{98}\,3^{52}\,5^{20}\,7^{12}\,11^6\,13^4\,17^2$
7	$-2^{126}\,3^{52}\,5^{26}\,7^{16}\,11^8\,13^6\,17^4\,19^2$	$-2^{134}\,3^{70}\,5^{28}\,7^{18}\,11^8\,13^6\,17^4\,19^2$
8	$2^{164}\,3^{70}\,5^{34}\,7^{22}\,11^{12}\,13^8\,17^6\,19^4\,23^2$	$2^{178}\,3^{90}\,5^{36}\,7^{24}\,11^{12}\,13^8\,17^6\,19^4\,23^2$
9	$-2^{210}\,3^{90}\,5^{46}\,7^{28}\,11^{16}\,13^{12}\,17^8\,19^6\,23^4$	$-2^{224}\,3^{116}\,5^{48}\,7^{30}\,11^{16}\,13^{12}\,17^8\,19^6\,23^4$
10	$2^{260}\,3^{116}\,5^{58}\,7^{36}\,11^{20}\,13^{16}\,17^{10}\,19^8\,23^6\,29^2$	$2^{276}\,3^{144}\,5^{62}\,7^{38}\,11^{20}\,13^{16}\,17^{10}\,19^8\,23^6\,29^2$

When $\alpha = 0$ in Equations (49) and (50) and $i = 1$, this gives the $H_m^{(2)}(.)$ of $sm^3\,(x,0)$, which are given in Table 2 for the different values of m. Similarly, when $i = 2$ in Equations (49) and (50), this gives the $H_m^{(2)}(.)$ of $sm^6\,(x,\alpha)$, and when $i = 3$ in Equations (49) and (50), this gives the $H_m^{(2)}(.)$ of $sm^9\,(x,\alpha)$. Substituting $\alpha = 0$ in Equations (51) and (52) and when $i = 2$, this gives the $H_m^{(2)}(.)$ of $sm^4\,(x,0)$, which are given in Table 2 for the different values of m. In the same way, when $i = 2$, $i = 3$ in Equations (51) and (52), this gives the $H_m^{(2)}(.)$ of the respective $sm^7\,(x,\alpha)$ and $sm^{10}\,(x,\alpha)$.

Table 2. Hankel determinants $H_m^{(2)}(.)$ of the Dixon elliptic functions $sm^3\,(x,0)$ and $sm^4\,(x,0)$ for m from 110.

m	$H_m^{(2)}\,(sm^3\,(x,0))$	$H_m^{(2)}\,(sm^4\,(x,0))$
1	$-2^5\,3^2$	$-3\,2^5\,5^2$
2	$2^{12}\,3^6\,5^2\,7^2$	$2^{18}\,3^4\,5^4\,7^2$
3	$-2^{27}\,3^{14}\,5^6\,7^4$	$-2^{33}\,3^{11}\,5^8\,7^4\,11^2$
4	$2^{46}\,3^{24}\,5^{10}\,7^6\,11^2\,13^2$	$2^{54}\,3^{20}\,5^{12}\,7^8\,11^4\,13^2$
5	$-2^{75}\,3^{36}\,5^{16}\,7^{10}\,11^4\,13^4$	$-2^{83}\,3^{31}\,5^{18}\,7^{12}\,11^6\,13^4\,17^2$
6	$2^{106}\,3^{52}\,5^{22}\,7^{14}\,11^6\,13^6\,17^2\,19^2$	$2^{118}\,3^{46}\,5^{26}\,7^{16}\,11^8\,13^6\,17^4\,19^2$
7	$-2^{143}\,3^{70}\,5^{30}\,7^{20}\,11^{10}\,13^8\,17^4\,19^4$	$-2^{155}\,3^{63}\,5^{34}\,7^{22}\,11^{12}\,13^8\,17^6\,19^4\,23^2$
8	$2^{186}\,3^{90}\,5^{42}\,7^{26}\,11^{14}\,13^{10}\,17^6\,19^6\,23^2$	$2^{200}\,3^{82}\,5^{46}\,7^{28}\,11^{16}\,13^{12}\,17^8\,19^6\,23^4$
9	$-2^{235}\,3^{116}\,5^{54}\,7^{34}\,11^{18}\,13^{14}\,17^8\,19^8\,23^4$	$-2^{249}\,3^{107}\,5^{58}\,7^{36}\,11^{20}\,13^{16}\,17^{10}\,19^8\,23^6\,29^2$
10	$2^{286}\,3^{144}\,5^{68}\,7^{42}\,11^{22}\,13^{18}\,17^{10}\,19^{10}\,23^6\,29^2\,31^2$	$2^{310}\,3^{134}\,5^{72}\,7^{44}\,11^{24}\,13^{20}\,17^{12}\,19^{10}\,23^8\,29^4\,31^2$

When $\alpha = 0$ in Equations (53) and (54) and $i = 1$, this gives the $H_m^{(2)}(.)$ of $sm^5\,(x,0)$, which are given in Table 3 for the different values of m. Next, when $i = 2$, $i = 3$ in Equations (53) and (54), this gives the $H_m^{(2)}(.)$ of $sm^8\,(x,\alpha)$ and $sm^{11}\,(x,\alpha)$, respectively. When $\alpha = 0$ in Equations (55)–(57), this gives the $H_m^{(2)}(.)$ of $cm\,(x,0)$, which are given in Table 3 for the different values of m.

Table 3. Hankel determinants $H_m^{(2)}(.)$ of the Dixon elliptic functions $sm^5(x,0)$ and $cm(x,0)$ for m from 1–10.

m	$H_m^{(2)}\left(sm^5(x,0)\right)$	$H_m^{(2)}\left(cm(x,0)\right)$
1	$-2^4\,3^3\,5^2$	-2
2	$2^{14}\,3^{10}\,5^4\,7^2$	$5\,2^7\,3^2$
3	$-2^{30}\,3^{19}\,5^8\,7^4\,11^2$	$-2^{18}\,3^6\,5^3\,7^2$
4	$2^{48}\,3^{30}\,5^{14}\,7^8\,11^4\,13^2$	$11\,2^{34}\,3^{14}\,5^7\,7^4$
5	$-2^{76}\,3^{45}\,5^{20}\,7^{12}\,11^6\,13^4\,17^2$	$-2^{55}\,3^{24}\,5^{11}\,7^7\,11^3\,13^2$
6	$2^{108}\,3^{62}\,5^{28}\,7^{18}\,11^8\,13^6\,17^4\,19^2$	$17\,2^{85}\,3^{36}\,5^{17}\,7^{11}\,11^5\,13^4$
7	$-2^{148}\,3^{81}\,5^{36}\,7^{24}\,11^{12}\,13^8\,17^6\,19^4\,23^2$	$-2^{119}\,3^{52}\,5^{24}\,7^{15}\,11^7\,13^6\,17^3\,19^2$
8	$2^{190}\,3^{106}\,5^{48}\,7^{30}\,11^{16}\,13^{12}\,17^8\,19^6\,23^4$	$23\,2^{157}\,3^{70}\,5^{32}\,7^{21}\,11^{11}\,13^8\,17^5\,19^4$
9	$-2^{238}\,3^{133}\,5^{62}\,7^{38}\,11^{20}\,13^{16}\,17^{10}\,19^8\,23^6\,29^2$	$-2^{202}\,3^{90}\,5^{44}\,7^{27}\,11^{15}\,13^{11}\,17^7\,19^6\,23^3$
10	$2^{296}\,3^{162}\,5^{76}\,7^{46}\,11^{26}\,13^{20}\,17^{12}\,19^{10}\,23^8\,29^4\,31^2$	$29\,2^{252}\,3^{116}\,5^{56}\,7^{35}\,11^{19}\,13^{15}\,17^9\,19^8\,23^5$

When $\alpha = 0$ in Equations (58) and (59), this gives the $H_m^{(2)}(.)$ of $sm(x,0)\,cm(x,0)$, which are given in Table 4 for the different values of m. When $\alpha = 0$ in Equations (60) and (61), this gives the $H_m^{(2)}(.)$ of $sm^2(x,0)\,cm(x,0)$, which are given in Table 4 for the different values of m.

Table 4. Hankel determinants $H_m^{(2)}(.)$ of the Dixon elliptic functions $sm(x,0)\,cm(x,0)$ and $sm^2(x,0)\,cm(x,0)$ for m from 1–10.

m	$H_m^{(2)}\left(sm(x,0)\,cm(x,0)\right)$	$H_m^{(2)}\left(sm^2(x,0)\,cm(x,0)\right)$
1	$-3\,2^2$	$-2^3\,3^2$
2	$2^9\,3^4\,5^2$	$7\,2^{10}\,3^6\,5^2$
3	$-2^{23}\,3^{10}\,5^4\,7^2$	$-2^{24}\,3^{14}\,5^5\,7^3$
4	$2^{41}\,3^{19}\,5^8\,7^4\,11^2$	$13\,2^{43}\,3^{24}\,5^9\,7^5\,11^2$
5	$-2^{63}\,3^{30}\,5^{13}\,7^8\,11^4\,13^2$	$-2^{68}\,3^{36}\,5^{15}\,7^9\,11^4\,13^3$
6	$2^{94}\,3^{44}\,5^{19}\,7^{12}\,11^6\,13^4\,17^2$	$19\,2^{99}\,3^{52}\,5^{21}\,7^{13}\,11^6\,13^5\,17^2$
7	$-2^{130}\,3^{61}\,5^{27}\,7^{17}\,11^8\,13^6\,17^4\,19^2$	$-2^{135}\,3^{70}\,5^{29}\,7^{19}\,11^9\,13^7\,17^4\,19^3$
8	$2^{171}\,3^{80}\,5^{35}\,7^{23}\,11^{12}\,13^8\,17^6\,19^4\,23^2$	$2^{178}\,3^{90}\,5^{39}\,7^{25}\,11^{13}\,13^9\,17^6\,19^5\,23^2$
9	$-2^{217}\,3^{103}\,5^{47}\,7^{29}\,11^{16}\,13^{12}\,17^8\,19^6\,23^4$	$-2^{225}\,3^{116}\,5^{51}\,7^{32}\,11^{17}\,13^{13}\,17^8\,19^7\,23^4$
10	$2^{268}\,3^{130}\,5^{60}\,7^{37}\,11^{20}\,13^{16}\,17^{10}\,19^8\,23^6\,29^2$	$31\,2^{276}\,3^{144}\,5^{65}\,7^{40}\,11^{21}\,13^{17}\,17^{10}\,19^9\,23^6\,29^2$

When $\alpha = 0$ in Equations (62) and (63) and $i = 1$, this gives the $H_m^{(2)}(.)$ of $sm^3(x,0)\,cm(x,0)$, which are given in Table 5 for the different values of m. When $i = 2$, $i = 3$ in Equations (62) and (63), this gives $H_m^{(2)}(.)$ of $sm^6(x,0)\,cm(x,\alpha)$ and $sm^9(x,0)\,cm(x,\alpha)$, respectively. Next, when $\alpha = 0$ in Equations (64) and (65) and $i = 1$, this gives the $H_m^{(2)}(.)$ of $sm^4(x,0)\,cm(x,0)$, which are given in Table 5 for the different values of m. Furthermore, when $i = 2$ and $i = 3$ in Equations (64) and (65) and $i = 1$, this gives $H_m^{(2)}(.)$ of $sm^7(x,\alpha)\,cm(x,\alpha)$ and $sm^{10}(x,\alpha)\,cm(x,\alpha)$, respectively.

Table 5. Hankel determinants $H_m^{(2)}(.)$ of the Dixon elliptic functions $sm^3(x,0)\,cm(x,0)$ and $sm^4(x,0)\,cm(x,0)$ for m from 1–10.

m	$H_m^{(2)}\left(sm^3(x,0)\,cm(x,0)\right)$	$H_m^{(2)}\left(sm^4(x,0)\,cm(x,0)\right)$
1	$-3\ 2^5\ 5$	$-2^4\ 3^2\ 5^2$
2	$2^{15}\ 3^4\ 5^3\ 7^2$	$2^{15}\ 3^7\ 5^4\ 7^2$
3	$-11\ 2^{30}\ 3^{11}\ 5^7\ 7^4$	$-2^{30}\ 3^{15}\ 5^8\ 7^4\ 11^2$
4	$2^{50}\ 3^{20}\ 5^{11}\ 7^7\ 11^3\ 13^2$	$2^{49}\ 3^{25}\ 5^{13}\ 7^8\ 11^4\ 13^2$
5	$-17\ 2^{79}\ 3^{31}\ 5^{17}\ 7^{11}\ 11^5\ 13^4$	$-2^{77}\ 3^{38}\ 5^{19}\ 7^{12}\ 11^6\ 13^4\ 17^2$
6	$2^{112}\ 3^{46}\ 5^{24}\ 7^{15}\ 11^7\ 13^6\ 17^3\ 19^2$	$2^{110}\ 3^{54}\ 5^{27}\ 7^{17}\ 11^8\ 13^6\ 17^4\ 19^2$
7	$-23\ 2^{149}\ 3^{63}\ 5^{32}\ 7^{21}\ 11^{11}\ 13^8\ 17^5\ 19^4$	$-2^{148}\ 3^{72}\ 5^{35}\ 7^{23}\ 11^{12}\ 13^8\ 17^6\ 19^4\ 23^2$
8	$2^{193}\ 3^{82}\ 5^{44}\ 7^{27}\ 11^{15}\ 13^{11}\ 17^7\ 19^6\ 23^3$	$2^{191}\ 3^{94}\ 5^{47}\ 7^{29}\ 11^{16}\ 13^{12}\ 17^8\ 19^6\ 23^4$
9	$-29\ 2^{242}\ 3^{107}\ 5^{56}\ 7^{35}\ 11^{19}\ 13^{15}\ 17^9\ 19^8\ 23^5$	$-2^{239}\ 3^{120}\ 5^{60}\ 7^{37}\ 11^{20}\ 13^{16}\ 17^{10}\ 19^8\ 23^6\ 29^2$
10	$2^{298}\ 3^{134}\ 5^{70}\ 7^{43}\ 11^{23}\ 13^{19}\ 17^{11}\ 19^{10}\ 23^7\ 29^3\ 31^2$	$2^{298}\ 3^{148}\ 5^{74}\ 7^{45}\ 11^{25}\ 13^{20}\ 17^{12}\ 19^{10}\ 23^8\ 29^4\ 31^2$

When $\alpha = 0$ in Equations (66) and (67) and $i = 1$, this gives the $H_m^{(2)}(.)$ of $sm^5(x,0)\,cm(x,0)$, which are given in Table 6 for the different values of m. In the same Equations (66) and (67), substituting $i = 2$ and $i = 3$ give the $H_m^{(2)}(.)$ of $sm^8(x,\alpha)\,cm(x,\alpha)$ and $sm^{11}(x,\alpha)\,cm(x,\alpha)$, respectively. When $\alpha = 0$ in Equations (68)–(70), this gives the $H_m^{(2)}(.)$ of $cm^2(x,0)$, which are given in Table 6 for the different values of m.

Table 6. Hankel determinants $H_m^{(2)}(.)$ of the Dixon elliptic functions $sm^5(x,0)\,cm(x,0)$ and $cm^2(x,0)$ for m from 1–10.

m	$H_m^{(2)}\left(sm^5(x,0)\,cm(x,0)\right)$	$H_m^{(2)}\left(cm^2(x,0)\right)$
1	$-5\ 2^4\ 3^3\ 7$	-2^2
2	$2^{15}\ 3^{10}\ 5^3\ 7^3$	$2^8\ 3^2\ 5^2$
3	$-13\ 2^{31}\ 3^{19}\ 5^6\ 7^5\ 11^2$	$-2^{22}\ 3^6\ 5^4\ 7^2$
4	$2^{53}\ 3^{30}\ 5^{11}\ 7^9\ 11^4\ 13^3$	$2^{38}\ 3^{14}\ 5^8\ 7^4\ 11^2$
5	$-19\ 2^{81}\ 3^{45}\ 5^{16}\ 7^{13}\ 11^6\ 13^5\ 17^2$	$-2^{60}\ 3^{24}\ 5^{12}\ 7^8\ 11^4\ 13^2$
6	$2^{114}\ 3^{62}\ 5^{23}\ 7^{19}\ 11^9\ 13^7\ 17^4\ 19^3$	$2^{90}\ 3^{36}\ 5^{18}\ 7^{12}\ 11^6\ 13^4\ 17^2$
7	$-2^{154}\ 3^{81}\ 5^{32}\ 7^{25}\ 11^{13}\ 13^9\ 17^6\ 19^5\ 23^2$	$-2^{126}\ 3^{52}\ 5^{26}\ 7^{16}\ 11^8\ 13^6\ 17^4\ 19^2$
8	$2^{198}\ 3^{106}\ 5^{43}\ 7^{32}\ 11^{17}\ 13^{13}\ 17^8\ 19^7\ 23^4$	$2^{164}\ 3^{70}\ 5^{34}\ 7^{22}\ 11^{12}\ 13^8\ 17^6\ 19^4\ 23^2$
9	$-31\ 2^{246}\ 3^{133}\ 5^{56}\ 7^{40}\ 11^{21}\ 13^{17}\ 17^{10}\ 19^9\ 23^6\ 29^2$	$-2^{210}\ 3^{90}\ 5^{46}\ 7^{28}\ 11^{16}\ 13^{12}\ 17^8\ 19^6\ 23^4$
10	$2^{305}\ 3^{162}\ 5^{69}\ 7^{48}\ 11^{27}\ 13^{21}\ 17^{13}\ 19^{11}\ 23^8\ 29^4\ 31^3$	$2^{260}\ 3^{116}\ 5^{58}\ 7^{36}\ 11^{20}\ 13^{16}\ 17^{10}\ 19^8\ 23^6\ 29^2$

When $\alpha = 0$ in Equations (71)–(73), this gives the $H_m^{(2)}(.)$ of $sm(x,0)\,cm^2(x,0)$, which are given in Table 7 for the different values of m. Next, when $\alpha = 0$ in Equations (74) and (75), this gives the $H_m^{(2)}(.)$ of $sm^2(x,0)\,cm^2(x,0)$, which are given in Table 7 for the different values of m.

Table 7. Hankel determinants $H_m^{(2)}(.)$ of the Dixon elliptic functions $sm\,(x,0)\,cm^2\,(x,0)$ and $sm^2\,(x,0)\,cm^2\,(x,0)$ for m from 1–10.

m	$H_m^{(2)}\left(sm\,(x,0)\,cm^2\,(x,0)\right)$	$H_m^{(2)}\left(sm^2\,(x,0)\,cm^2\,(x,0)\right)$
1	$-2\ 3^2$	$-3\ 2^5$
2	$2^8\ 3^6\ 5^2$	$2^{12}\ 3^4\ 5^2\ 7^2$
3	$-2^{21}\ 3^{14}\ 5^4\ 7^2$	$-2^{27}\ 3^{11}\ 5^6\ 7^4$
4	$2^{40}\ 3^{24}\ 5^8\ 7^4\ 11^2$	$2^{46}\ 3^{20}\ 5^{10}\ 7^6\ 11^2\ 13^2$
5	$-2^{61}\ 3^{36}\ 5^{14}\ 7^8\ 11^4\ 13^2$	$-2^{75}\ 3^{31}\ 5^{16}\ 7^{10}\ 11^4\ 13^4$
6	$2^{92}\ 3^{52}\ 5^{20}\ 7^{12}\ 11^6\ 13^4\ 17^2$	$2^{106}\ 3^{46}\ 5^{22}\ 7^{14}\ 11^6\ 13^6\ 17^2\ 19^2$
7	$-2^{127}\ 3^{70}\ 5^{28}\ 7^{18}\ 11^8\ 13^6\ 17^4\ 19^2$	$-2^{143}\ 3^{63}\ 5^{30}\ 7^{20}\ 11^{10}\ 13^8\ 17^4\ 19^4$
8	$2^{170}\ 3^{90}\ 5^{36}\ 7^{24}\ 11^{12}\ 13^8\ 17^6\ 19^4\ 23^2$	$2^{186}\ 3^{82}\ 5^{42}\ 7^{26}\ 11^{14}\ 13^{10}\ 17^6\ 19^6\ 23^2$
9	$-2^{215}\ 3^{116}\ 5^{48}\ 7^{30}\ 11^{16}\ 13^{12}\ 17^8\ 19^6\ 23^4$	$-2^{235}\ 3^{107}\ 5^{54}\ 7^{34}\ 11^{18}\ 13^{14}\ 17^8\ 19^8\ 23^4$
10	$2^{266}\ 3^{144}\ 5^{62}\ 7^{38}\ 11^{20}\ 13^{16}\ 17^{10}\ 19^8\ 23^6\ 29^2$	$2^{286}\ 3^{134}\ 5^{68}\ 7^{42}\ 11^{22}\ 13^{18}\ 17^{10}\ 19^{10}\ 23^6\ 29^2\ 31^2$

When $\alpha = 0$ in Equations (76) and (77), this gives the $H_m^{(2)}(.)$ of $sm^3\,(x,0)\,cm^2\,(x,0)$, which are given in Table 8 for the different values of m. When $\alpha = 0$ and $i = 1$ in Equations (78) and (79), this gives the $H_m^{(2)}(.)$ of $sm^4\,(x,0)\,cm^2\,(x,0)$, which are given in Table 8 for the different values of m. Furthermore, note that for $i = 2$ and $i = 3$ in Equations (78) and (79), this gives the $H_m^{(2)}(.)$ of $sm^7\,(x,\alpha)\,cm^2\,(x,\alpha)$ and $sm^{10}\,(x,\alpha)\,cm^2\,(x,\alpha)$, respectively.

Table 8. Hankel determinants $H_m^{(2)}(.)$ of the Dixon elliptic functions $sm^3\,(x,0)\,cm^2\,(x,0)$ and $sm^4\,(x,0)\,cm^2\,(x,0)$ for m from 1–10.

m	$H_m^{(2)}\left(sm^3\,(x,0)\,cm^2\,(x,0)\right)$	$H_m^{(2)}\left(sm^4\,(x,0)\,cm^2\,(x,0)\right)$
1	$-3\ 2^3\ 5^2$	$-5\ 2^5\ 3^3$
2	$2^{14}\ 3^4\ 5^4\ 7^2$	$2^{16}\ 3^{10}\ 5^2\ 7^2$
3	$-2^{27}\ 3^{11}\ 5^8\ 7^4\ 11^2$	$-2^{33}\ 3^{19}\ 5^5\ 7^4\ 11^2$
4	$2^{46}\ 3^{20}\ 5^{12}\ 7^8\ 11^4\ 13^2$	$2^{52}\ 3^{30}\ 5^{10}\ 7^8\ 11^4\ 13^2$
5	$-2^{73}\ 3^{31}\ 5^{18}\ 7^{12}\ 11^6\ 13^4\ 17^2$	$-2^{81}\ 3^{45}\ 5^{15}\ 7^{12}\ 11^6\ 13^4\ 17^2$
6	$2^{106}\ 3^{46}\ 5^{26}\ 7^{16}\ 11^8\ 13^6\ 17^4\ 19^2$	$2^{114}\ 3^{62}\ 5^{22}\ 7^{18}\ 11^8\ 13^6\ 17^4\ 19^2$
7	$-2^{141}\ 3^{63}\ 5^{34}\ 7^{22}\ 11^{12}\ 13^8\ 17^6\ 19^4\ 23^2$	$-2^{155}\ 3^{81}\ 5^{29}\ 7^{24}\ 11^{12}\ 13^8\ 17^6\ 19^4\ 23^2$
8	$2^{184}\ 3^{82}\ 5^{46}\ 7^{28}\ 11^{16}\ 13^{12}\ 17^8\ 19^6\ 23^4$	$2^{198}\ 3^{106}\ 5^{40}\ 7^{30}\ 11^{16}\ 13^{12}\ 17^8\ 19^6\ 23^4$
9	$-2^{231}\ 3^{107}\ 5^{58}\ 7^{36}\ 11^{20}\ 13^{16}\ 17^{10}\ 19^8\ 23^6\ 29^2$	$-2^{247}\ 3^{133}\ 5^{53}\ 7^{38}\ 11^{20}\ 13^{16}\ 17^{10}\ 19^8\ 23^6\ 29^2$
10	$2^{290}\ 3^{134}\ 5^{72}\ 7^{44}\ 11^{24}\ 13^{20}\ 17^{12}\ 19^{10}\ 23^8\ 29^4\ 31^2$	$2^{306}\ 3^{162}\ 5^{66}\ 7^{46}\ 11^{26}\ 13^{20}\ 17^{12}\ 19^{10}\ 23^8\ 29^4\ 31^2$

When $\alpha = 0$ and $i = 1$ in Equations (80) and (81), this gives the $H_m^{(2)}(.)$ of $sm^5\,(x,0)\,cm^2\,(x,0)$, which are given in Table 9 for the different values of m. In the same way, for $i = 2$ and $i = 3$ in Equations (80) and (81), this gives the $H_m^{(2)}(.)$ of the respective $sm^8\,(x,\alpha)\,cm^2\,(x,\alpha)$ and $sm^{11}\,(x,\alpha)\,cm^2\,(x,\alpha)$. When $\alpha = 0$ and $i = 1$ in Equations (82) and (83), this gives the $H_m^{(2)}(.)$ of $sm^6\,(x,0)\,cm^2\,(x,0)$, which are given in Table 9 for the different values of m. For $i = 2$ and

$i = 3$ in Equations (82) and (83), this gives the $H_m^{(2)}(.)$ of the respective $sm^9(x,\alpha)\,cm^2(x,\alpha)$ and $sm^{12}(x,\alpha)\,cm^2(x,\alpha)$.

Table 9. Hankel determinants $H_m^{(2)}(.)$ of the Dixon elliptic function $sm^5(x,0)\,cm^2(x,0)$ and $sm^6(x,0)\,cm^2(x,0)$ for m from 1–10.

m	$H_m^{(2)}\left(sm^5(x,0)\,cm^2(x,0)\right)$	$H_m^{(2)}\left(sm^6(x,0)\,cm^2(x,0)\right)$
1	$-5\;2^3\;3^2\;7^2$	$-7\;3\;2^9\;5$
2	$2^{14}\;3^8\;5^4\;7^4$	$2^{20}\;3^6\;5^4\;7^2\;11^2$
3	$-2^{29}\;3^{16}\;5^7\;7^6\;11^2\;13^2$	$-2^{37}\;3^{13}\;5^7\;7^5\;11^4\;13^2$
4	$2^{54}\;3^{26}\;5^{12}\;7^{10}\;11^4\;13^4$	$2^{62}\;3^{22}\;5^{12}\;7^8\;11^6\;13^4\;17^2$
5	$-2^{81}\;3^{40}\;5^{17}\;7^{14}\;11^6\;13^6\;17^2\;19^2$	$-2^{93}\;3^{35}\;5^{19}\;7^{11}\;11^8\;13^6\;17^4\;19^2$
6	$2^{114}\;3^{56}\;5^{24}\;7^{20}\;11^{10}\;13^8\;17^4\;19^4$	$2^{126}\;3^{50}\;5^{26}\;7^{16}\;11^{12}\;13^8\;17^6\;19^4\;23^2$
7	$-2^{153}\;3^{74}\;5^{35}\;7^{26}\;11^{14}\;13^{10}\;17^6\;19^6\;23^2$	$-2^{167}\;3^{67}\;5^{37}\;7^{21}\;11^{16}\;13^{12}\;17^8\;19^6\;23^4$
8	$2^{198}\;3^{98}\;5^{46}\;7^{34}\;11^{18}\;13^{14}\;17^8\;19^8\;23^4$	$2^{212}\;3^{90}\;5^{48}\;7^{28}\;11^{20}\;13^{16}\;17^{10}\;19^8\;23^6\;29^2$
9	$-2^{245}\;3^{124}\;5^{59}\;7^{42}\;11^{22}\;13^{18}\;17^{10}\;19^{10}\;23^6\;29^2\;31^2$	$-2^{269}\;3^{115}\;5^{61}\;7^{35}\;11^{24}\;13^{20}\;17^{12}\;19^{10}\;23^8\;29^4\;31^2$
10	$2^{304}\;3^{152}\;5^{72}\;7^{50}\;11^{28}\;13^{22}\;17^{14}\;19^{12}\;23^8\;29^4\;31^4$	$2^{328}\;3^{142}\;5^{76}\;7^{44}\;11^{30}\;13^{24}\;17^{16}\;19^{12}\;23^{10}\;29^6\;31^4$

6. Results and Discussion

Multiplying Equation (9) with u gives the Laplace transform of $sm(x,\alpha)$ in [5] (Theorem 19, page 61 [5]). Multiplication of u in Equations (22) and (23) gives the results given in [5] (Theorems 20 and 21, respectively, pp. 62–63 [5]). The remaining results in Theorems 1–3 appearing in this work are new to the literature reviewed. Letting $\alpha = 0$ in Equations (44)–(46) gives the results in [5] ($H_m^{(2)}(.)$ in Theorem 16, pp. 57–58 [5]). When $\alpha = 0$ in Equations (55)–(57) and Equations (58) and (59), this gives the results in [5] ($H_m^{(2)}(.)$ of Theorems 17 and 18, pp. 58–59 [5]), respectively. The remaining results of Theorems 4–6 are new to the literature reviewed.

7. Conclusions

In this research work, the Sumudu integral transform was applied to the non-zero modulus Dixon elliptic functions to derive general three-term recurrences. Then, from the three-term recurrences, the quasi C fractions were expanded. Next, by assuming the functions in the denominator of quasi C fractions as one, the Hankel determinants $H_m^{(2)}(.)$ of the non-zero modulus Dixon elliptic functions were obtained by using Lemma 1 in which Hankel determinants $H_m^{(1)}(.)$ were used from authors' previous work [23]. In this approach, the non-zero modulus Dixon elliptic functions need not have their Maclaurin's series expanded for the $H_m^{(2)}(.)$ calculations. In Section 5, the $H_m^{(2)}(.)$ of certain Dixon functions with $\alpha = 0$ were given, which proves that the assumptions made in the denominator were true. Section 6 also ensured the assumptions were correct and gave the previous results.

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