IOP Conference Series: Materials Science and Engineering

i-v-f β-Ideals of β-Algebras

To cite this article: P Hemavathi *et al* 2017 *IOP Conf. Ser.: Mater. Sci. Eng.* **263** 042111

IOP Conf. Series: Materials Science and Engineering **263** (2017) 042111 doi:10.1088/1757-899X/263/4/042111

i-v-f $\boldsymbol{\beta}$-Ideals of $\boldsymbol{\beta}$-Algebras

P Hemavathi [1] , P Muralikrishna [2] and K Palanivel [3]
[1]Department of Mathematics, SVCET, Thirupachur-631203, India
[2]PG and Research Department of Mathematics, Muthurangam Government Arts College,Vellore-632002 India
[3]Department of Mathematics, School of Advanced Sciences, VIT University, Vellore-632014, India

E-mail: drkpalanivel@gmail.com

Abstract. The notions of the interval valued fuzzy set were first introduced by Zadeh as a generalization of fuzzy sets. Using interval valued fuzzy set, various algebraic structures and related topics were discussed. This paper deals the notion of Interval valued Fuzzy β-ideal of a β-algebra and some related results.

1. Introduction

In 1965 Zadah [15, 16] introduced a notion of fuzzy sets. In [11] Neggers et.al introduced a new class of algebra namely $\boldsymbol{\beta}$-algebra. Jun et.al [13] also dealt some related topics on $\boldsymbol{\beta}$-Subalgebra. In 2013 Ansari et.al [1, 2] introduced fuzzy $\boldsymbol{\beta}$-Subalgebras of $\boldsymbol{\beta}$-algebras and also they initiated fuzzy $\boldsymbol{\beta}$ −ideals of $\boldsymbol{\beta}$ −algebras. In [3] Biswas described Interval valued fuzzy subgroups (ie. i-v fuzzy subgroups) and examined some properties. Moreover the authors of [12, 14] applied the notion of i-v fuzzy set in BCI and BCK-algebras.

In [4, 10], the methods and models of interval valued games and Linear programming technique for determining interval-valued restraint matrix games have been discussed. There was an enormous contribution for the fuzzy graph by the authors in [5, 6, 7, 8] and which is enforced in the field of graph theory.

Recently interval valued fuzzy $\boldsymbol{\beta}$-Subalgebra of a $\boldsymbol{\beta}$-algebra introduced in [9]. With all these ideas in this paper the conception of interval valued fuzzy $\boldsymbol{\beta}$-ideals of $\boldsymbol{\beta}$-algebra to be introduced and deal some related results

2. Preliminaries

In this part, some primary definitions and outcomes are related which is essential, in the sequel.

Definition: 2.1[11]

A β-algebra is a non-empty set X with a constant 0 and dual operations $+$ and $-$ satisfying the subsequent axioms:

$(i) x - 0 = x$

$(ii) (0 - x) + x = 0$

$(iii) (x - y) - z = x - (z + y) \quad \forall\, x, y, z \in X$

Example: 2.2
Let $X = \{0,1,2,3\}$ be a set with constant 0 and dual operations + and – are defined on X by the following cayley's table

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

-	0	1	2	3
0	0	3	2	1
1	1	0	3	2
2	2	1	0	3
3	3	2	1	0

Therefore, $(X, +, -, 0)$ is a β-algebra.

Definition: 2.3
A non empty subset A of a β-algebra $(X, +, -, 0)$ is said to be a β-subalgebra of X, if
$(i) x + y \in A$
$(ii) x - y \in A \quad \forall\, x, y \in A$

Example: 2.4
In the above illustration, the subset $A = \{0,2\}$ is a β-sub algebra of X.

Definition: 2.5
A non empty subset I of a β-algebra $(X, +, -, 0)$ is said to be a β-ideals of X , if
$(i) 0 \in I$
$(ii) x + y \in I \quad \forall\, x, y \in I$
$(iii) if\ x - y\ \&\ y \in I\ implies\ \ x \in I \quad \forall\, x, y \in X$

Example: 2.6
Consider the β-algebra X in example 2.2. Then the subset $I_1 = \{0,1\}$ is a β-ideals of X. But $I_2 = \{0,1,3\}$ is not a β-ideal of X, (since $1 + 3 = 2 \notin I_2$)

Definition: 2.7
Let$(X, +, -, 0)$ and $(Y, +, -, 0)$ be two β-algebras. A mapping $f: X \to Y$ is called a $\beta - homomorphism$ if $\forall\, x, y \in X$
$(i) f(x + y) = f(x) + f(y)$
$(ii) f(x + y) = f(x) + f(y)$

Definition: 2.8 [12]
An interval valued fuzzy set (briefly i-v fuzzy set) A represented on X is known as
$A = \{(x, [\sigma_A^L(x), \sigma_A^U(x)])\} \quad \forall\, x \in X$ (briefly expressed as $A = [\sigma_A^L, \sigma_A^U]$), where σ_A^L and σ_A^U are two fuzzy sets in X such that $\sigma_A^L(x) \le \sigma_A^U(x) \quad \forall x \in X$.
Let $\bar{\sigma}_A(x) = [\sigma_A^L(x), \sigma_A^U(x)] \quad \forall\, x \in X$ and let $D[0,1]$ denotes the relations of all closed sub intervals of $[0,1]$. If $\sigma_A^L(x) = \sigma_A^U(x) = c$, say, where $0 \le c \le 1$, then we have $\bar{\sigma}_A(x) = [c, c]$ which we also guess, for the sake of accessibility, to belong to $D[0,1]$.
Thus $\bar{\sigma}_A(x) \in D[0,1] \quad \forall x \in X$, and hence the i-v fuzzy set A is given by
$A = \{(x, \bar{\sigma}_A(x))\}\ \forall\, x \in X$, where $\bar{\sigma}_A: X \to D[0,1]$.
Now let us illustrate what is identified as $refined\ minimum$ (briefly $rmin$) of two elements in $D[0,1]$. We also characterized the symbols "$\ge$", " $\le$ " and " $=$ " in case of two elements in $D[0,1]$.
Suppose two elements
$D_1 = [a_1, b_1]$ and $D_2 = [a_2, b_2] \in D[0,1]$.

Therefore, we have

$$rmin(D_1, D_2) = [\min\{a_1, a_2\}, \min\{b_1, b_2\}];$$

$D_1 \geq D_2$ if and only if $a_1 \geq a_2$, $b_1 \geq b_2$;
likewise, we may have $D_1 \leq D_2$ and $D_1 = D_2$.

Definition: 2.9
Let $\bar{\sigma}$ be an i-v fuzzy set of X. $\bar{\sigma}$ is assumed to have the r-supremum property if for any subset A of X, there exist a $a_0 \in A$ such that $\bar{\sigma}(a_0) = \mathrm{rsup}_{a \in A}\, \bar{\sigma}(a)$

Definition: 2.10 [9]
Let$(X, +, -, 0)$ be a β-algebra. Then the interval valued fuzzy subset $A = \{\langle x, \bar{\sigma}_A(x)\rangle : x \in X\}$ is known as an interval valued fuzzy (i-v fuzzy) β-subalgebra of X if
$(i)\bar{\sigma}_A(x + y) \geq rmin\{\bar{\sigma}_A(x), \bar{\sigma}_A(y)\}$
$(ii)\bar{\sigma}_A(x - y) \geq rmin\,\{\bar{\sigma}_A(x), \bar{\sigma}_A(y)\} \quad \forall x, y \in X$

3. i-v-f $\boldsymbol{\beta}$-Ideals of $\boldsymbol{\beta}$-algebra

This segment introduces the notion of interval valued fuzzy(i-v-f) $\boldsymbol{\beta}$-ideals of $\boldsymbol{\beta}$-algebras and deals various simple results.

Definition: 3.1[2]
Let σ be a fuzzy set in a β-algebra of X. Then σ is said to be a fuzzy β-ideal of X, if $\forall\, x, y \in X$
$(i)\ \sigma(0) \geq \sigma(x)$
$(ii)\sigma(x + y) \geq \min\{\sigma(x), \sigma(y)\}$
$(iii)\sigma(x) \geq \min\{\sigma(x - y), \sigma(y)\}$

Definition: 3.2
Let $A = \{\langle x, \bar{\sigma}_A(x)\rangle : x \in X\}$ be an interval valued fuzzy set in a β-algebra X. Then A is known as an interval valued fuzzy(i-v-f) β-ideal of X, if $\forall\, x, y \in X$

$(i)\ \bar{\sigma}_A(0) \geq \bar{\sigma}_A(x)$
$(ii)\bar{\sigma}_A(x + y) \geq rmin\{\bar{\sigma}_A(x), \bar{\sigma}_A(y)\}$
$(iii)\bar{\sigma}_A(x) \geq rmin\,\{\bar{\sigma}_A(x - y), \bar{\sigma}_A(y)\}$

Example: 3.3
The i-v fuzzy set defined in the β-algebra X in the example 2.2 as ,

$\bar{\sigma}_A : X \to D[0,1]$ such that $\bar{\sigma}_A(x) = \begin{cases} [0.3,0.7] & x = 0 \\ [0.1,0.5] & x = 1,3 \\ [0.2,0.6] & x = 2 \end{cases}$ is a i-v fuzzy β-ideal of X.

Proposition: 3.4
The intersection of any two i-v fuzzy β-ideals of a β-algebra is also an i-v fuzzy β-ideal.
Proof:
Let $\bar{\sigma}_1$and $\bar{\sigma}_2$ be two i-v fuzzy β-ideals of a β-algebra X.Now

$$\begin{aligned}(i)\ (\bar{\sigma}_1 \cap \bar{\sigma}_2)(0) &\geq rmin\{\bar{\sigma}_1(0), \bar{\sigma}_2(0)\} \\ &= rmin\{\bar{\sigma}_1(x), \bar{\sigma}_2(x)\} \\ &= (\bar{\sigma}_1 \cap \bar{\sigma}_2)(x)\end{aligned}$$

$$\begin{aligned}(ii)(\bar{\sigma}_1 \cap \bar{\sigma}_2)(x + y) &\geq rmin\{\bar{\sigma}_1(x + y), \bar{\sigma}_2(x + y)\} \\ &= rmin\big\{rmin\{\bar{\sigma}_1(x), \bar{\sigma}_1(y)\}, rmin\{\bar{\sigma}_2(x), \bar{\sigma}_2(y)\}\big\} \\ &= rmin\big\{rmin\{\bar{\sigma}_1(x), \bar{\sigma}_2(x)\}, rmin\{\bar{\sigma}_1(y), \bar{\sigma}_2(y)\}\big\} \\ &= rmin\{(\bar{\sigma}_1 \cap \bar{\sigma}_2)(x), (\bar{\sigma}_1 \cap \bar{\sigma}_2)(y)\}\end{aligned}$$

IOP Conf. Series: Materials Science and Engineering **263** (2017) 042111 doi:10.1088/1757-899X/263/4/042111

$(iii)(\bar{\sigma}_1 \cap \bar{\sigma}_2)(x) \geq rmin\{\bar{\sigma}_1(x), \bar{\sigma}_2(x)\}$
$= rmin\{rmin\{\bar{\sigma}_1(x-y), \bar{\sigma}_1(y)\}, rmin\{\bar{\sigma}_2(x-y), \bar{\sigma}_2(y)\}\}$
$= rmin\{rmin\{\bar{\sigma}_1(x-y), \bar{\sigma}_2(x-y)\}, rmin\{\bar{\sigma}_1(y), \bar{\sigma}_2(y)\}\}$
$= rmin\{(\bar{\sigma}_1 \cap \bar{\sigma}_2)(x-y), (\bar{\sigma}_1 \cap \bar{\sigma}_2)(y)\}$

Hence $(\bar{\sigma}_1 \cap \bar{\sigma}_2)$ is a i-v fuzzy β-ideal of X.

The exceeding theorem can be generalized as

Proposition: 3.5
The intersection of any set of i-v fuzzy β-ideals of a β-algebra is also a i-v fuzzy β-ideal.

Proposition: 3.6
Let $A = \{\langle x, \bar{\sigma}_A(x)\rangle : x \in X\}$ be an i-v fuzzy β-ideal of a β-algebra X. If $x \leq y$ then $\bar{\sigma}_A(x) \geq \bar{\sigma}_A(y)$.
Proof:
For any $x, y \in X$, $x \leq y \Rightarrow x - y = 0$
$\Rightarrow \bar{\sigma}_A(x) \geq rmin\{\bar{\sigma}_A(x-y), \bar{\sigma}_A(y)\}$
$= rmin\{\bar{\sigma}_A(0), \bar{\sigma}_A(y)\}$
$= \bar{\sigma}_A(y)$

Proposition: 3.7
Let $A = \{\langle x, \bar{\sigma}_A(x)\rangle : x \in X\}$ be an i-v fuzzy β-ideal of a β-algebra X. Whenever $x \leq z + y$ then $\bar{\sigma}_A(x) \geq rmin\{\bar{\sigma}_A(z), \bar{\sigma}_A(y)\}$
Proof:
For $x, y, z \in X$
$\bar{\sigma}_A(x) \geq rmin\{\bar{\sigma}_A(x-y), \bar{\sigma}_A(y)\}$
$= rmin\{rmin\{\bar{\sigma}_A((x-y)-z), \bar{\sigma}_A(z)\}, \bar{\sigma}_A(y)\}$
$= rmin\{rmin\{\bar{\sigma}_A(x-(z+y)), \bar{\sigma}_A(z)\}, \bar{\sigma}_A(y)\}$
$= rmin\{rmin\{\bar{\sigma}_A(0), \bar{\sigma}_A(z)\}, \bar{\sigma}_A(y)\}$
$= rmin\{\bar{\sigma}_A(z), \bar{\sigma}_A(y)\}$

Proposition: 3.8
An i-v fuzzy set $A = [\sigma_A^L, \sigma_A^U]$ in X is an i-v fuzzy β-ideal of X if and only if σ_A^L and σ_A^U are fuzzy β-ideals of X.
Proof:
Suppose that σ_A^L as well as σ_A^U are fuzzy β-ideal of X.
$\therefore \sigma_A^L(0) \geq \sigma_A^L(x)$ and $\sigma_A^U(0) \geq \sigma_A^U(x)$
$\Rightarrow \quad \bar{\sigma}_A(0) \geq \bar{\sigma}_A(x)$
Let $x, y, z \in X$. Then
$\bar{\sigma}_A(x+y) = [\sigma_A^L(x+y), \sigma_A^U(x+y)]$
$\geq [min\{\sigma_A^L(x), \sigma_A^L(y)\}, min\{\sigma_A^U(x), \sigma_A^U(y)\}]$
$= rmin\{[\sigma_A^L(x), \sigma_A^L(y)], [\sigma_A^U(x), \sigma_A^U(y)]\}$
$= rmin\{\bar{\sigma}_A(x), \bar{\sigma}_A(y)\}$

$\bar{\sigma}_A(x) = [\sigma_A^L(x), \sigma_A^U(x)]$
$\geq [min\{\sigma_A^L(x-y), \sigma_A^L(y)\}, min\{\mu_A^U(x-y), \mu_A^U(y)\}]$
$= rmin\{[\sigma_A^L(x-y), \sigma_A^U(x-y)], [\sigma_A^L(y), \sigma_A^U(y)]\}$
$= rmin\{\bar{\sigma}_A(x-y), \bar{\sigma}_A(y)\}$
Thus A is an i-v fuzzy β-ideal of X.

Conversely,
Let A be an i-v fuzzy β-ideal of X
Then for each $x, y \in X$, we have

$$[\sigma_A^L(x+y), \sigma_A^U(x+y)] = \bar{\sigma}_A(x+y)$$
$$\geq rmin\{\bar{\sigma}_A(x), \bar{\sigma}_A(y)\}$$
$$= rmin\{[\sigma_A^L(x), \sigma_A^U(x)], [\sigma_A^L(y), \sigma_A^U(y)]\}$$
$$= [min\{\sigma_A^L(x), \sigma_A^L(y)\},\ min\{\sigma_A^U(x), \sigma_A^U(y)\}]$$

It follows that

$\sigma_A^L(x+y) \geq min\{\sigma_A^L(x), \sigma_A^L(y)\}$ and $\sigma_A^U(x+y) \geq min\{\sigma_A^U(x), \sigma_A^U(y)\}$

$$[\sigma_A^L(x), \sigma_A^U(x)] = \bar{\sigma}_A(x)$$
$$\geq rmin\{\bar{\sigma}_A(x-y), \bar{\sigma}_A(y)\}$$
$$= rmin\{[\sigma_A^L(x-y), \sigma_A^U(x-y)], [\sigma_A^L(y), \sigma_A^U(y)]\}$$
$$= [min\{\mu_A^L(x-y), \mu_A^L(y)\},\ min\{\mu_A^U(x-y), \mu_A^U(y)\}]$$

$\therefore$ $\sigma_A^L(x) \geq min\{\sigma_A^L(x-y), \sigma_A^L(y)\}$ and $\sigma_A^U(x) \geq min\{\sigma_A^U(x-y), \sigma_A^U(y)\}$

Therefore σ_A^L and σ_A^U are fuzzy β-ideals of X.

Proposition: 3.9
Suppose A is subset of X. Describe an i-v fuzzy set $\bar{\sigma}_A: X \to D[0,1]$such that

$$\bar{\sigma}_A(x) = \begin{cases} [t_0, t_1] & if\ x \in A \\ [s_0, s_1] & if x \notin A \end{cases}$$ where $[t_0, t_1]$ and $[s_0, s_1] \in D[0,1]$ with $[t_0, t_1] \geq [s_0, s_1]$.

Then $\bar{\sigma}$ is an i-v fuzzy β-ideal of X, iff A is β-ideal of X.

Proof:
Consider $\bar{\sigma}_A$ is an i-v fuzzy β-ideal of X.

(i) We have $\bar{\sigma}_A(0) \geq \bar{\sigma}(x)\ \ \forall\, x \in X \Rightarrow\ \bar{\sigma}_A(0) = [t_0, t_1]\ \Rightarrow 0 \in A$

(ii) For any $x, y \in A \Rightarrow\ \bar{\sigma}_A(x) = [t_0, t_1] = \bar{\sigma}_A(y)$.
Then $\bar{\sigma}_A(x+y) \geq rmin\{\ \bar{\sigma}_A(x),\ \bar{\sigma}_A(y)\} = rmin\{\ [t_0, t_1], [t_0, t_1]\ \} = [t_0, t_1]$
$\therefore$ $\bar{\sigma}_A(x+y) = [t_0, t_1]\ \Rightarrow\ x+y \in A$

(iii) For any $x, y \in X$, if $x-y$ and $y \in A\ \Rightarrow\ \bar{\sigma}_A(x+y) = [t_0, t_1] = \bar{\sigma}_A(y)$
Now $\bar{\sigma}_A(x)\ \geq rmin\{\ \bar{\sigma}_A(x-y),\ \bar{\sigma}_A(y)\} = rmin\{[t_0, t_1], [t_0, t_1]\} = [t_0, t_1]$
$\Rightarrow\ \bar{\sigma}_A(x) = [t_0, t_1]\ \Rightarrow\ x \in A$
Therefore A is a β-ideal of X.

Conversely, if A is a β-ideal of X.

(i) If $0 \in A\ \Rightarrow\ \bar{\sigma}_A(0) = [t_0, t_1]$.
As well as $\forall\, x \in X, Im(\bar{\sigma}) = [[t_0, t_1], [s_0, s_1]]$ and $[t_0, t_1] > [s_0, s_1]$
$\Rightarrow\ \bar{\bar{\sigma}}_A(0) \geq\ \bar{\sigma}_A(x)\ \ \forall\, x \in X$

(ii) For $x, y \in A\ \Rightarrow$ if $x+y \in A \Rightarrow\ \bar{\sigma}_A(x) = \bar{\sigma}_A(y) = \bar{\sigma}_A(x+y) = [t_0, t_1] = rmin\{\ \bar{\sigma}_A(x),\ \bar{\sigma}_A(y)\}$
Hence $\bar{\sigma}_A(x+y) \geq\ rmin\{\ \bar{\sigma}_A(x),\ \bar{\sigma}_A(y)\}$

(iii) For $x, y \in A$ if $x-y \in A$ and $y \in A\ \Rightarrow\ x \in A$
$\Rightarrow\ \bar{\sigma}_A(x) = [t_0, t_1] = rmin\{[t_0, t_1], [t_0, t_1]\} = rmin\{\ \bar{\sigma}_A(x-y),\ \bar{\sigma}_A(y)\}$
$\therefore$ $\bar{\sigma}_A$ is an i-v fuzzy β-ideal of X.

Corollary: 3.10
Let $A = \{\langle x, \bar{\sigma}_A(x)\rangle : x \in X\}$ be an i-v fuzzy β-ideal of X, then the set $X_{\bar{\sigma}_A} = \{x \in X : \bar{\sigma}_A(x) = \bar{\sigma}_A(0)\}$ is a β-ideal of X.
Proof:
Since $\bar{\sigma}_A(x) = \bar{\sigma}_A(0) \Rightarrow 0 \in X_{\bar{\sigma}_A}$
If $x - y,\ y \in X_{\bar{\sigma}_A} \Rightarrow \bar{\sigma}_A(x - y) = \bar{\sigma}_A(0)$,
$\bar{\sigma}_A(y) = \bar{\sigma}_A(0)$
And so,
$$\begin{aligned} \bar{\sigma}_A(x) &\geq rmin\,\{\bar{\sigma}_A(x-y), \bar{\sigma}_A(y)\} \\ &= rmin\,\{\bar{\sigma}_A(0), \bar{\sigma}_A(0)\} \\ &= \bar{\sigma}_A(0) \\ \bar{\sigma}_A(x) &\geq \bar{\sigma}_A(0). \end{aligned}$$
But $\bar{\sigma}_A(x) \leq \bar{\sigma}_A(0) \Rightarrow \bar{\sigma}_A(x) = \bar{\sigma}_A(0) \Rightarrow x \in X_{\bar{\sigma}_A}$
$i.e,\quad x - y, y \in X_{\bar{\sigma}_A} \Rightarrow x \in X_{\bar{\sigma}_A}$
$\therefore\ X_{\bar{\sigma}_A}$ is an β-ideal of X.

Proposition: 3.11
Let $f: X \to Y$ be an onto homomorphism of β-algebras. Suppose A is an i-v fuzzy β-ideal of Y, then the preimage of $f^{-1}(A)$ is an i-v fuzzy β-ideal of X.
Proof:
Suppose A be an i-v fuzzy β-ideal of Y.
For any $x \in X$,
$f^{-1}(\bar{\sigma}_A(0)) = \bar{\sigma}_A(f(0)) = \bar{\sigma}_A(0) \geq \bar{\sigma}_A(x)$

For some $x, y \in X$,

$$\begin{aligned} f^{-1}(\bar{\sigma}_A)(x+y) &= \bar{\sigma}_A(f(x+y)) \\ &= \bar{\sigma}_A(f(x) + f(y)) \\ &\geq rmin\,\{\bar{\sigma}_A(f(x)), \bar{\sigma}_A(f(y))\} \\ &= rmin\{f^{-1}(\bar{\sigma}_A(x)), f^{-1}(\bar{\sigma}_A(y))\}. \end{aligned}$$

$$\begin{aligned} f^{-1}(\bar{\sigma}_A)(x) &= \bar{\sigma}_A(f(x)) \\ &\geq rmin\,\{\bar{\sigma}_A(f(x) - f(y)), \bar{\sigma}_A(f(y))\} \\ &= rmin\,\{\bar{\sigma}_A(f(x-y)), \bar{\sigma}_A(f(y))\} \\ &= \mathrm{r}min\{f^{-1}(\bar{\sigma}_A(x-y)), f^{-1}(\bar{\sigma}_A(y))\} \end{aligned}$$

$\therefore\ f^{-1}(\bar{\sigma}_A)$ is an i-v fuzzy β-ideal of X.

Hence $f^{-1}(A)$ is an i-v fuzzy β-ideal of X.

Proposition: 3.12
Let $f: X \to Y$ be an onto homomorphism of β-algebras. If $\bar{\sigma}_A$ is an i-v fuzzy β-ideal of X, with supremum property and $\ker(f) \subseteq X_{\bar{\sigma}_A}$ then by the image of $\bar{\sigma}_A$, f($\bar{\sigma}_A$) is an i-v fuzzy β-ideal of Y.
Proof:
Now,
$$f(\bar{\sigma}_A)(0) = r \operatorname*{rsup}_{x \in f^{-1}(0)} \{\bar{\sigma}_A(x)\} = \bar{\sigma}_A(0) \geq \bar{\sigma}_A(x), \forall\, x \in X$$
Hence,

$f(\bar{\sigma}_A)(0) = \underset{x\in f^{-1}(0)}{\text{rsup}} \{\bar{\sigma}_A(x)\} = f(\bar{\sigma}_A)(y)\ , \forall\ y\ \in\ Y$

Let $y_1, y_2 \in Y$. Then there exist $x_1, x_2 \in X$ such that

$f(x_1) = y_1, f(x_2) = y_2$

$f(\bar{\sigma}_A)(y_1 + y_2) = rsup\{\bar{\sigma}_A(x): x\ \in\ f^{-1}(y_1 + y_2)\}$
$\geq rsup\{\bar{\sigma}_A(x_1 + x_2): x_1 \in f^{-1}(\ y_1)\&\ x_2 \in f^{-1}(\ y_2)\}$
$= rsup\ \{\ rmin\{\ \bar{\sigma}_A(x_1), \bar{\sigma}_A(x_2)\ \}\ , x_1 \in f^{-1}(\ y_1)\&\ x_2 \in f^{-1}(\ y_2)\}$
$= rmin\ \{\ rsup\ \{\ \bar{\sigma}_A(\ x_1): x_1 \in f^{-1}(\ y_1)\ \},\ \ rsup\ \{\ \bar{\sigma}_A(\ x_2): x_2 \in f^{-1}(\ y_2)\ \}\ \}$
$= rmin\{\ \underset{x_1\in f^{-1}(y_1)}{\text{rsup}}\ \{\bar{\sigma}_A(x_1)\},\ \underset{x_2\in f^{-1}(y_2)}{\text{rsup}}\ \{\bar{\sigma}_A(x_2)\}$
$= rmin\{\ f(\bar{\sigma}_A)(\ y_1)\ , f(\bar{\mu}_A)(\ y_2)\ \}$

Suppose that for some $y_1, y_2 \in Y$.

Then $f(\bar{\sigma}_A)(\ y_1) \leq rmin\{\ f(\bar{\sigma}_A)(\ y_1 - y_2), f(\bar{\sigma}_A)(\ y_2)\}$

Since f is onto there exist $x_1, x_2 \in X$ such that

$f(x_1) = y_1$ and $f(x_2) = y_2$

$f(\bar{\sigma}_A)(f(\ x_1)) < rmin\{f(\bar{\sigma}_A)\big(f(x_1) - f(x_2)\big), f(\bar{\sigma}_A)f(\ x_2)\}$
$= rmin\{\ f(\bar{\sigma}_A)f(\ x_1 - x_2), f(\bar{\sigma}_A)f(\ x_2)\}$

$\Rightarrow f(\bar{\sigma}_A)(f(\ x_1)) <\ rmin\{\ f^{-1}(f(\bar{\sigma}_A))(\ x_1 - x_2),\ f^{-1}(f(\bar{\sigma}_A))(\ x_2)\}$

$\Rightarrow \bar{\sigma}_A(\ x_1) < rmin\ \{\ \bar{\sigma}_A(\ x_1 - x_2), \bar{\sigma}_A(\ x_2)\}$

Hence $f(\bar{\sigma}_A)$ is an i-v fuzzy β-ideal of Y.

Proposition: 3.13

Let $f: X \to Y$ be an on homomorphism of β-algebras. If $\bar{\sigma}_A$ is an i-v fuzzy β-ideal of X, with $\ker(f) \subseteq X_{\bar{\sigma}_A}$ then the pre image of $f^{-1}\big(f(\bar{\sigma}_A)\big) = \bar{\sigma}_A$.

Proof:

Let $x\ \in X$ and $f(x) = y$

Hence

$f^{-1}f(\bar{\sigma}_A)(x) = f(\bar{\sigma}_A)\big(f(x)\big)$
$= f(\bar{\sigma}_A)$
$= \text{rsup}_{x\in f^{-1}(y)}\{\bar{\sigma}_A(x)\}$

For any $x' \in X, x' \in f^{-1}(y) \Rightarrow f(x') = y$

$\Rightarrow f(x') = f(x) \Rightarrow f(x') - f(x) = 0$

$f(x' - x) = 0 \Rightarrow x' - x \in \ker(f)$

$x' - x \in\ X_{\bar{\mu}_A}$

$\Rightarrow \bar{\sigma}_A(x' - x) = \bar{\sigma}_A(0)$

$\therefore\quad \bar{\sigma}_A(x')\ \geq rmin\{\ \bar{\sigma}_A(x' - x), \bar{\sigma}_A(x)\}$
$= rmin\{\ \bar{\sigma}_A(0), \bar{\sigma}_A(x)\}$
$=\ \bar{\sigma}_A(x)$

We can also prove $\bar{\mu}_A(x)\ \geq \bar{\sigma}_A(x')$

Hence $\bar{\sigma}_A(x') = \bar{\sigma}_A(x)$

$\therefore\ \ f^{-1}\big(f(\bar{\sigma}_A)\big)(x) = \underset{x'\in f^{-1}(y)}{\text{rsup}}\ \{\bar{\sigma}_A(x')\} = \bar{\sigma}_A(x)$

$\Rightarrow\ f^{-1}\big(f(\bar{\sigma}_A)\big)(x) =\ \bar{\sigma}_A(x)$

Proposition: 3.14
Let $f: X \to X$ be an endomorphism on X. Let $\bar{\sigma}$ be an i-v fuzzy β-ideal of X. Then $\bar{\sigma}_f: X \to D[0,1]$ defined by $\bar{\sigma}_f(x) = \bar{\sigma}(f(x)$, $\forall\, x \in X$, is an i-v fuzzy β-ideal of X.

Proof:
Suppose $\bar{\sigma}$ be an i-v fuzzy β-ideal of X
For some $x \in X$,
$\bar{\sigma}_f(0) = \bar{\sigma}(f(0) = \bar{\sigma}(0) \geq \bar{\sigma}(x), \forall x, y \in X$
$\bar{\sigma}_f(x+y) = \bar{\sigma}(f(x+y))$
$= \bar{\sigma}(f(x) + f(y))$
$\geq rmin\{\bar{\sigma}(f(x)), \bar{\sigma}(f(y))\}$
$= rmin\{\bar{\sigma}_f(x), \bar{\sigma}_f(y)\}$
Also, $\bar{\sigma}_f(x) = \bar{\sigma}(f(x))$
$\geq rmin\{\bar{\sigma}(f(x) - f(y)), \bar{\sigma}(f(y))\}$
$= rmin\{\bar{\sigma}(f(x-y)), \bar{\sigma}(f(y))\}$
$= rmin\{\bar{\sigma}_f(x-y), \bar{\sigma}_f(y)\}$
Thus, $\bar{\sigma}_f$ is an i-v fuzzy β-ideal of X.

4. Product on i-v fuzzy $\boldsymbol{\beta}$-ideal of $\boldsymbol{\beta}$-algebra
In this segment, we talk about the product on interval valued fuzzy $\boldsymbol{\beta}$-ideals of $\boldsymbol{\beta}$-algebras and various related results.

Definition:4.1
Let $(X, +, -, 0)$ and $(Y, +, -, 0)$ be two β-algebras.
Let $A = \{\langle x, \bar{\sigma}_A(x)\rangle : x \in X\}$ and $B = \{\langle y, \bar{\sigma}_B(y)\rangle : y \in Y\}$ be i-v fuzzy subsets in X and Y respectively. If $A \times B$ is the Cartesian product of A and B which is defined to be the set
$A \times B = \{\langle (x,y), \bar{\sigma}_{A\times B}(x, y)\rangle : (x, y) \in X \times Y\}$.

Proposition: 4.2
Let A and B be two i-v fuzzy β-ideals of X and Y correspondingly. Then $A \times B$ is also an i-v fuzzy β-ideal of $X \times Y$.

Proof:
Let $A = \{\langle x, \bar{\sigma}_A(x)\rangle : x \in X\}$ and $B = \{\langle y, \bar{\sigma}_B(y)\rangle : y \in Y\}$ be i-v fuzzy β-ideal in X and Y.
Take $(x, y) \in X \times Y$
$\bar{\sigma}_{A\times B}(0,0) \geq rmin\{\bar{\sigma}_{A\times B}(0), \bar{\sigma}_{A\times B}(0)\}$
$= rmin\{\bar{\sigma}_{A\times B}(x), \bar{\sigma}_{A\times B}(y)\}$
$= \bar{\sigma}_{A\times B}(x, y)$
Take $(a, b) \in X \times Y$, where $a = (x_1, y_1)$ and $b = (x_2, y_2)$.
Clearly, $\bar{\sigma}_{A\times B}(a+b) \geq rmin\{\bar{\sigma}_{A\times B}(a), \bar{\sigma}_{A\times B}(b)\}$
and
$\bar{\sigma}_{A\times B}(a) = \bar{\sigma}_{A\times B}(x_1, y_1)$
$= rmin\{\bar{\sigma}_{A\times B}(x_1), \bar{\sigma}_{A\times B}(y_1)\}$
$\geq rmin\{rmin\{\bar{\sigma}_A(x_1 - x_2), \bar{\sigma}_A(x_2)\}, \; rmin\{\bar{\sigma}_B(y_1 - y_2), \bar{\sigma}_B(y_2)\}\}$
$= rmin\{rmin\{\bar{\sigma}_A(x_1 - x_2), \bar{\sigma}_B(y_1 - y_2)\}, \; rmin\{\bar{\sigma}_A(x_2), \bar{\sigma}_B(y_2)\}\}$
$= rmin\{\bar{\sigma}_{A\times B}((x_1, y_1) - (x_2, y_2)), \bar{\sigma}_{A\times B}(x_2, y_2)\}$,
$= rmin\{\bar{\sigma}_{A\times B}(a-b), \bar{\sigma}_{A\times B}(b)\}$
Hence $A \times B$ is also an i-v fuzzy β-ideal of $X \times Y$.

Lemma:4.3
Let A and B be two i-v fuzzy subsets of X and Y. If $A \times B$ is an i-v-fuzzy β-ideal of $X \times Y$ in that case $\bar{\sigma}_A(0) \geq \bar{\sigma}_B(y)$ and $\bar{\sigma}_B(0) \geq \bar{\sigma}_A(x)$
Proof:
Let A and B be two i-v fuzzy subsets of X and Y
Assume $\bar{\sigma}_B(y) \geq \bar{\sigma}_A(0)$ and $\bar{\mu}_A(x) \geq \bar{\mu}_B(0)$ for any $x \in X, y \in Y$. Then

$$\bar{\sigma}_{A\times B}(x,y) \geq rmin\{\bar{\sigma}_A(x), \bar{\sigma}_B(y)\}$$
$$= rmin\{\bar{\sigma}_B(0), \bar{\sigma}_A(0)\}$$
$$= \bar{\sigma}_{A\times B}(0,0)$$

which shows a contradiction. Hence the given result is proved.

Proposition: 4.4
Let A and B be two i-v fuzzy subsets of X and Y such that $A \times B$ is an i-v fuzzy β-ideals of $X \times Y$. Then either A is an i-v fuzzy β-ideal of X or B is an i-v fuzzy β-ideal of Y.
Proof:
From lemma 4.3, if we take $\bar{\sigma}_A(0) \geq \bar{\sigma}_B(y)$ then

$$\bar{\mu}_{A\times B}(0,y) = rmin\{\bar{\sigma}_A(0), \bar{\sigma}_B(y)\} \quad (1)$$

Since $A \times B$ is an i-v fuzzy β-ideals of $X \times Y$,

$$\bar{\sigma}_{A\times B}\big((x_1,y_1),(x_2,y_2)\big) \geq rmin\{\bar{\sigma}_{A\times B}((x_1,y_1)-(x_2,y_2)), \bar{\sigma}_{A\times B}(x_2,y_2)\}$$

and since

$$\bar{\sigma}_{A\times B}\big((x_1,y_1)-(x_2,y_2)\big) \geq rmin\{\bar{\sigma}_{A\times B}(x_1,y_1), \bar{\sigma}_{A\times B}(x_2,y_2)\}$$

We have $\bar{\mu}_{A\times B}(x_1,y_1) \geq rmin\{\bar{\sigma}_{A\times B}\big((x_1-x_2),(y_1-y_2)\big), \bar{\sigma}_{A\times B}(x_2,y_2)\}$

$$\bar{\sigma}_{A\times B}\big((x_1-x_2),(y_1-y_2)\big) \geq rmin\{\bar{\sigma}_{A\times B}(x_1,y_1), \bar{\sigma}_{A\times B}(x_2,y_2)\} \quad (2)$$

Putting $x_1 = x_2 = 0$ in (2) we get

$\bar{\sigma}_{A\times B}(0,y_1) \geq rmin\{\bar{\sigma}_{A\times B}\big(0,(y_1-y_2)\big), \bar{\sigma}_{A\times B}(0,y_2)\}$ and

$$\bar{\sigma}_{A\times B}\big(0,(y_1-y_2)\big) \geq rmin\{\bar{\sigma}_{A\times B}(0,y_1), \bar{\sigma}_{A\times B}(0,y_2)\} \quad (3)$$

Using equations (1) in (3) we have
$\bar{\sigma}_B(y_1) \geq rmin\{\bar{\sigma}_B(y_1-y_2), \bar{\sigma}_B(y_2)\}$ and $\bar{\sigma}_B(y_1-y_2) \geq rmin\{\bar{\sigma}_B(y_1), \bar{\sigma}_B(y_2)\}$
Hence B is an i-v fuzzy β-ideal of Y.

Conclusion
This paper presents interval valued fuzzy $\boldsymbol{\beta}$ −ideals of $\boldsymbol{\beta}$ −algebras. In the future work, it is planned to extend the above ideas into the concept of intuitionistic interval valued fuzzy $\boldsymbol{\beta}$ −ideals of $\boldsymbol{\beta}$ −algebras and other substructures of $\boldsymbol{\beta}$ −algebras.

References
[1] Aub Ayub Ansari M and Chandramouleeswaran M 2013 *Int. J. Math. Sci. Eng. App.* **5(7)** 239-49
[2] Aub Ayub Ansari M and Chandramouleeswaran M 2014 *Int. J. Math. Sci. Eng. App.* **8(1)** 1-10
[3] Biswas R and Rosenfeld's 1994 *fuzzy Sets and systems* **63(1)** 87-90
[4] Deng-Feng Li. 2014 *Springer, Cham, Switzerland*
[5] Dey A and Pal A 2013 *Annals of Pure and Applied Mathematics* **3(2)** 169-78
[6] Dey A, Pradhan R, Pal A, and Pal A *Springer* 805-13
[7] Dey A and Pal A 2012 *Int. J. Management, IT and Engineering*, **2(8)** 340-52
[8] Dey A and Pal A **12(3)** 419–30
[9] Hemavathi P, Muralikrishna P and Palanivel K 2015 *Global J. Pure and Applied Mathematics* **11(4)** 2553-60

[10] Jiang-Xia Nan and Deng-Feng Li. 2014 *J. Industrial and Management Optimization* **10 (4)** 1059-70
[11] Neggers and Kim H.S 2002 *Math.Slovaca* **52(5)** 517-30
[12] Sung Min Houng, Young Bae Jun, SeonJeong Kim, and GwangIl Kim *Math Japonica* **40(2)** 199-202
[13] Young Bae, Jun and Kim 2012 *Commun.Korean.Math.Soc* **27(2)** 243-55
[14] Young Bae, Jun 2000 *Soc. Math*.**3** 435-44
[15] Zadeh L.A 1965 *Inform.Control* **8(3)** 338-53
[16] Zadeh L.A 1975 *Information Sci*. **8** 199-249