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Independent Dominator Sequence Number of a Graph

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Abstract

Let $G = (V, E)$ be a connected graph. A dominator sequence in G is a sequence of vertices $S = (v_1, v_2, \ldots, v_k)$ such that for each i with $2 \le i \le k$, the vertex v_i dominates at least one vertex which is not dominated by $v_1, v_2, \ldots, v_{i-1}$. If further the set of vertices in S is an independent set, then S is called an *independent dominator sequence* (IDS) in G. The maximum length of an IDS in G is called the independent dominator sequence number of G and is denoted by $l_\iota(G)$. In this paper we initiate a study of this parameter.

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1. Introduction

By a graph $G = (V, E)$ we mean a finite, undirected and connected graph with neither loops nor multiple edges. The order $|V|$ and the size $|E|$ are denoted by n and m respectively. For graph theoretic terminology we refer to Chartrand and Lesniak [2]. Motivated by an application in optimal networks, the concept of dominator sequences in bipartite graphs was introduced in Jayaram et al. [4].

Definition 1. *Let G be a bipartite graph with bipartition X, Y. A Y-dominator sequence is a sequence of vertices $(x_1, x_2, \ldots, x_k)$ in X such that for each $i, 2 \le i \le k$, there exists $y_i \in Y$ such that y_i is dominated by x_i and y_i is not dominated by any of the vertices $x_1, x_2, \ldots, x_{i-1}$, or equivalently, $y_i \in N(x_i) - (N(x_1) \cup N(x_2) \cup \cdots \cup N(x_{i-1}))$. The maximum length of a Y-dominator sequence of G is called the Y-dominator sequence number of G and is denoted by $l_Y(G)$. Similarly a X-dominator sequence is a sequence of vertices $(y_1, y_2, \ldots, y_r)$ in Y such that for each $i, 2 \le i \le r$, there exists $x_i \in X$ such that x_i is dominated by y_i but not dominated by any of the vertices $y_1, y_2, \ldots, y_{i-1}$. The maximum length of a X-dominator sequence of G is called the X-dominator sequence number of G and is denoted by $l_X(G)$.*

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It has been proved in Jayaram et al.[4] that $l_X(G) = l_Y(G)$. Let $l(G) = l_X(G) = l_Y(G)$. Then $l(G)$ is called the *dominator sequence number* of G.

In Jayaram et al.[4] we confined ourselves to bipartite graphs since the graph involved in the application of this concept in the context of optical networks is a bipartite graph. Obviously any dominator sequence in a bipartite graph G is an independent set and the set of vertices of a dominator sequence need not be a dominating set of G. In a recent paper Brešar et al.[1] introduced the concept of dominating sequence and the dominating sequence number, which they called Grundy domination number in any graph G. The dominating sequence considered in their paper is not necessarily independent and is always a dominating set of G.

In this paper we introduce the concept of independent dominator sequence in arbitrary graphs.

We need the following theorems given in Jayaram et al.[4].

Theorem 2. *Let C_n be an even cycle. Then $l(C_n) = \frac{n}{2} - 1$.*

Theorem 3. *Let $G = C_n \square K_2$, where $C_n \square K_2$ denotes the Cartesian product of C_n and K_2. Then $l(G) = n - 2$.*

Theorem 4. *For any path P_n, $l(P_n \square K_2) = n - 1$.*

2. Main Results

Definition 5. *Let $G = (V, E)$ be a connected graph. A dominator sequence in G is a sequence of vertices $S = (v_1, v_2, \ldots, v_k)$ such that for each i with $2 \le i \le k$, the vertex v_i dominates at least one vertex which is not dominated by $v_1, v_2, \ldots, v_{i-1}$. If further the set of vertices in S is an independent set, then S is called an independent dominator sequence (IDS) in G. The maximum length of an IDS in G is called the independent dominator sequence number of G and is denoted by $l_\iota(G)$.*

It follows from the definition that

Observation 6. *$l_\iota(G) \le \beta_0(G)$, where $\beta_0(G)$ is the independence number of G, and the bound is tight.*

Theorem 8 shows that the bound is tight for odd cycles.

We start with the following basic result which asserts that for a bipartite graph G the dominator sequence number $l(G)$ and the independent dominator sequence number $l_\iota(G)$ are equal.

Theorem 7. *Let G be a bipartite graph. Then $l(G) = l_\iota(G)$.*

Proof. Since any Y-dominator sequence of G is an independent dominator sequence, it follows that $l_\iota(G) \ge l(G)$. Now, let $S = (v_1, v_2, \ldots, v_{l_\iota(G)})$ be an independent dominator sequence in G. For each $v_j \in S \cap Y$, choose a vertex w_j in X such that w_j is dominated by v_j and is not dominated by $v_1, v_2, \ldots, v_{j-1}$. Replace v_j by w_j in S. Clearly w_j dominates v_j and v_j is not dominated by $v_1, v_2, \ldots, v_{j-1}$. This gives an X-dominator sequence S_1 in G with $|S_1| = |S| = l_\iota(G)$. Hence it follows that $l(G) \ge l_\iota(G)$ and thus $l(G) = l_\iota(G)$. □

We now proceed to determine $l_\iota(G)$ for some standard graphs.

Theorem 8.

(i) *For the complete graph K_n, $l_\iota(K_n) = 1$.*
(ii) *For the path P_n, $l_\iota(P_n) = \lfloor \frac{n}{2} \rfloor$.*
(iii) *For the Cartesian product $G = K_n \square K_2$, $l_\iota(G) = 2$.*
(iv) *For the cycle $C_n = (v_1, v_2, \ldots, v_n, v_1)$, $l_\iota(C_n) = \begin{cases} \frac{n}{2} - 1 & \text{if } n \text{ is even} \\ \frac{n-1}{2} & \text{if } n \text{ is odd.} \end{cases}$*
(v) *If G is any connected graph and $H = G + K_1$, then $l_\iota(H) = l_\iota(G)$. In particular for the wheel $W_n = C_{n-1} + K_1$, we have $l_\iota(W_n) = l_\iota(C_{n-1})$.*

Proof. Since $l_\iota(K_n) = \beta_0(K_n) = 1$, (i) follows. Since $l(P_n) = \lfloor \frac{n}{2} \rfloor$, it follows from Theorem 7 that $l_\iota(P_n) = \lfloor \frac{n}{2} \rfloor$. Now for the Cartesian product $G = K_n \square K_2$, one vertex u in the first copy of K_n and one vertex v which is nonadjacent to u in the second copy of K_n in G is an independent dominator sequence of G. Hence $l_\iota(K_n \Box K_2) \geq 2$. Further $l_\iota(K_n \Box K_2) \leq \beta_0(K_n \Box K_2) = 2$ and hence $l_\iota(K_n \Box K_2) = 2$.

Now consider the cycle $C_n = (v_1, v_2, \ldots, v_n, v_1)$. If n is even, then the result follows from Theorem 7 and Theorem 2. Now, suppose n is odd. Then $S = \{v_i : 1 \leq i < n$ and i is odd $\}$ is an independent dominator sequence of C_n and hence $l_\iota(C_n) \geq |S| = \frac{n-1}{2}$. Further the first vertex in the dominator sequence dominates two vertices and all the remaining vertices dominate at least one vertex which is not dominated by earlier vertices in the sequence. Hence any independent dominator sequence of length k in C_n dominates at least $k+1$ vertices and hence $2k+1 \leq n$. Thus $k \leq \frac{n-1}{2}$, so that $l_\iota(C_n) = \frac{n-1}{2}$.

Now consider $H = G + K_1$ where G is any connected graph. Since any independent dominator sequence of G is also a dominator sequence of H, we have $l_\iota(H) \geq l_\iota(G)$. Further any independent dominator sequence of H of maximum length cannot contain the vertex of the copy K_1. Hence $l_\iota(H) = l_\iota(G)$. □

Theorem 9. *Let C_n be any cycle. Then $l_\iota(C_n \Box K_2) = 2l_\iota(C_n)$.*

Proof. Let $G = C_n \Box K_2$. Let $C_n^0 = (x_1, x_2, \ldots, x_n, x_1)$ and $C_n^1 = (y_1, y_2, \ldots, y_n, y_1)$ be the two copies of C_n in G with $x_i y_i \in E(G)$. If n is even, the result follows from Theorem 7 and Theorem 2. Suppose n is odd. Let $S = (x_1, y_2, x_3, y_4, \ldots, x_{n-2}, y_{n-1})$. Then S is an independent dominator sequence in G and $|S| = n-1$. Hence $l_\iota(G) \geq n-1$. Further the first vertex in the dominator sequence dominates three vertices and all the remaining vertices dominate at least one vertex which is not dominated by earlier vertices in the sequence. Hence any independent dominator sequence of length k dominates at least $k+2$ vertices, so that $k+k+2 \leq 2n$. Thus $l_\iota(G) \leq k \leq n-1$. Hence $l_\iota(G) = n-1 = 2l_\iota(C_n)$. □

Lemma 10. *Let P_n be an even path and let $G = P_n + e$ where $e = v_i v_j, i < j$. If $l_\iota(G) < l_\iota(P_n)$, then the unique cycle in G is an even cycle.*

Proof. Let $P_n = (v_1, v_2, \ldots, v_n)$. The sequences $S_1 = (v_1, v_3, v_5, \ldots, v_{n-1})$ and $S_2 = (v_2, v_4, v_6, \ldots, v_{n-2}, v_n)$ are both independent dominator sequences of P_n having maximum length. Hence $l_\iota(P_n) = |S_1| = |S_2| = \frac{n}{2}$. Now, suppose $l_\iota(G) < l_\iota(P_n)$. If both i and j are even, then S_1 is an independent dominator sequence of G. If both i and j are odd, then S_2 is an independent dominator sequence of G. Thus $l_\iota(G) \geq l_\iota(P_n)$, which is a contradiction. Hence i and j are of opposite parity, so that the unique cycle in G is even. □

A *chord* of a cycle C in a graph is an edge that joins two nonconsecutive vertices of C. A graph G is called a *chordal graph* if every cycle of length 4 or more in G has a chord. A *clique* of a graph G is a complete subgraph of G. If a clique is not contained in a larger clique in G, then it is called a *maximal clique*. A *simplicial vertex* of a graph G is a vertex v such that the subgraph induced by its closed neighborhood $N[v]$ forms a clique in G.

The following theorem is given in Fulkerson and Gross[3].

Theorem 11. *A chordal graph G of order n has at most n maximal cliques, with equality if and only if $G = \overline{K_n}$.*

We now present a few basic results on the independent dominator sequence number of chordal graphs. It follows from Theorem 11 that a connected chordal graph of order n has at most k maximal cliques where $k < n$.

Theorem 12. *Let G be a chordal graph and let k denote the number of maximal cliques in G. Then $l_\iota(G) \leq k$.*

Proof. Let $l_\iota(G) = r$ and let $S = (v_1, v_2, \ldots, v_r)$ be an independent dominator sequence in G. Let C_i be the maximal clique in G such that $v_i \in C_i$. Since the set of vertices in S is an independent set, it follows that the cliques $C_1, C_2, \ldots, C_r$ are distinct. Hence $l_\iota(G) = r \leq k$. □

The following theorem gives a condition under which equality holds in Theorem 12.

Theorem 13. *Let G be a chordal graph with k maximal cliques $C_1, C_2, \ldots, C_k$ such that each C_i has a simplicial vertex v_i. Then $l_\iota(G) = k$.*

Proof. Let $S = (v_1, v_2, \ldots, v_k)$. Since v_i is a simplicial vertex of G, it follows that $\langle N[v_i] \rangle$ is a complete subgraph of G and $N[v_i] \supseteq C_i$. Since C_i is a maximal clique $N[v_i] = C_i$. Thus the vertices in C_i are not dominated by $v_1, v_2, \ldots, v_{i-1}$ for each $i \geq 2$. Hence S is an independent dominator sequence of G and $l_\iota(G) \geq k$. Thus by Theorem 12, we have $l_\iota(G) = k$. □

3. Conclusion and Scope

In this paper we have initiated a study of independent dominator sequence number of a graph which has been motivated by the corresponding concept in bipartite graphs given in Jayaram et al.[4]. The following are interesting problems for further investigation.

Problem 14. *Characterize the class of graphs G for which* $l_\iota(G) = \beta_0(G)$.

Problem 15. *Determine the value of* $l_\iota(G)$ *for various graph products.*

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