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Inequalities for the Taylor coefficients of spiralike functions involving q-differential operator

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Abstract. Making use of q-analogue of the well-known differential operator, we provide a formal extension of a bi-univalent spiralike and bi-univalent strongly spiralike functions. We obtain the inequalities for the Maclaurin-Taylor coefficients of the functions belonging to the defined subclasses. Further we have provided some applications of our main results.

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Key Words and Phrases: Starlike Functions, Spiralike Functions, Bi-Univalent Functions, Coefficient Inequalities, q-Calculus Operator

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1. Introduction of Quantum Calculus in dual with Univalent Functions

Quantum calculus popularly called as q-calculus is based on the idea of finite difference rescaling. The difference of quantum differentials from the ordinary ones is that notion of limit is removed in q-calculus, that is q-derivative is merely a ratio which is given by

$$D_q f(z) = \frac{f(qz) - f(z)}{(q-1)z}.$$

Notice that as limit $q \to 1^-$, $D_q f(z) = f'(z)$. q-calculus has numerous applications in variety of disciplines such as theory of special functions, operator theory, quantum-mechanics, relativity etc. Notations and symbols play an very important role in the study of q-calculus. Throughout this paper, we let

$$[n]_q = \sum_{k=1}^{n} q^{k-1}, \quad [0]_q = 0, \quad (q \in \mathbb{C})$$

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and the q-shifted factorial by

$$(a;q)_n = \begin{cases} 1, & n=0 \\ (1-a)(1-aq)\dots\left(1-aq^{n-1}\right), & n=1,2,\dots. \end{cases}$$

The q-hypergeometric series was developed by Heine as a generalization of the hypergeometric series

$$ {}_2F_1[a,b;\,c|q,z] = \sum_{n=0}^{\infty} \frac{(a;q)_n(b;q)_n}{(q;q)_n(c;q)_n} z^n. \tag{1}$$

Generalizing the Heine's series, we define ${}_r\phi_s$ the basic hypergeometric series by

$$\begin{aligned} &{}_r\phi_s\left(a_1,\, a_2,\, \dots,\, a_r;\, b_1,\, b_2,\, \dots,\, b_s; q,\, z\right) \\ &\qquad = \sum_{n=0}^{\infty} \frac{(a_1;q)_n(a_2;q)_n\dots(a_r;q)_n}{(q;q)_n(b_1;q)_n\dots(b_s;q)_n} \left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} z^n \end{aligned} \tag{2}$$

with $\binom{n}{2} = \frac{n(n-1)}{2}$, where $q \neq 0$ when $r > s+1$. In (1) and (2), it is assumed that the parameters $b_1,\, b_2,\, \dots,\, b_s$ are such that the denominator factors in the terms of the series are never zero.

Let $\mathcal{A}$ denote the class of all functions having a Taylor series expansion of the form

$$f(z) = z + \sum_{n=2}^{\infty} k_n z^n, \quad (z \in \mathcal{U}). \tag{3}$$

For complex parameters $a_1,\, \dots,\, a_r$ and $b_1, \dots,\, b_s$ ($\beta_j \in \mathbb{C}\backslash\mathbb{Z}_0^-$; $\mathbb{Z}_0^- = 0, -1, -2, \dots$; $j = 1,\, \dots,\, s$), we define the generalized q-hypergeometric function ${}_r\Psi_s(a_1,\, \dots,\, a_q;\, b_1,\, \dots,\, b_s;\, q, z)$ by

$$ {}_r\Psi_s(a_1,\, a_2,\, \dots,\, a_q;\, b_1,\, b_2,\, \dots,\, b_s;\, q, z) = \sum_{n=0}^{\infty} \frac{(a_1;q)_n(a_2;q)_n\dots(a_r;q)_n}{(q;q)_n(b_1;q)_n\dots(b_s;q)_n} z^n \tag{4}$$

$$(r = s+1;\, r,\, s \in N_0 = N \cup \{0\};\, z \in \mathcal{U}),$$

where N denotes the set of positive integers. By using the ratio test, we should note that, if $|q| < 1$, the series (4) converges absolutely for $|z| < 1$ and $r = s+1$. For more mathematical background of these functions, one may refer to [2].

Corresponding to a function $\mathcal{G}_{r,\,s}(a_i,\, b_j;\, q,\, z)$ ($a_i,\, b_j$ are real; $i = 1,\, 2,\, \dots,\, r$; $j = 1,\, 2,\, \dots,\, s$) defined by

$$\mathcal{G}_{r,\,s}(a_i,\, b_j;\, q,\, z) := z\, {}_q\Psi_s(a_1,\, a_2,\, \dots,\, a_r;\, b_1,\, b_2,\, \dots,\, b_s;\, q, z). \tag{5}$$

We now define the following operator $\mathcal{J}_\lambda^m(a_1, b_1;\, q,\, z)f \,:\, \mathcal{U} \longrightarrow \mathcal{U}$ by

$$\mathcal{J}_\lambda^0(a_1, b_1;\, q,\, z)f(z) = f(z) * \mathcal{G}_{r,\,s}(a_i,\, b_j;\, q,\, z)$$

$$\mathcal{J}_\lambda^1(a_1, b_1;\, q,\, z)f(z) = (1-\lambda)(f(z)*\mathcal{G}_{r,\,s}(a_i,\, b_j;\, q,\, z)) + \lambda\, zD_q(f(z)*\mathcal{G}_{r,\,s}(a_i,\, b_j;\, q,\, z)) \tag{6}$$

$$\mathcal{J}_{\lambda}^{m}(a_1, b_1;\, q,\, z)f(z) = \mathcal{J}_{\lambda}^{1}(\mathcal{J}_{\lambda}^{m-1}(a_1, b_1;\, q,\, z)f(z)). \tag{7}$$

If $f \in \mathcal{A}$, then from (6) and (7) we may easily deduce that

$$\mathcal{J}_{\lambda}^{m}(a_1, b_1;\, q,\, z)f = z + \sum_{n=2}^{\infty} [1 - \lambda + [n]_q \lambda]^m \ \Upsilon_n k_n z^n, \tag{8}$$

$$(m \in N_0 = N \cup \{0\} \text{ and } \lambda \geq 0)\,,$$

where

$$\Upsilon_n = \frac{(a_1; q)_{n-1}(a_2; q)_{n-1} \dots (a_r; q)_{n-1}}{(q; q)_{n-1}(b_1; q)_{n-1} \dots (b_s; q)_{n-1}}, \quad (|q| < 1)\,.$$

Remark 1. *We note that the linear operator (8) is q-analogue of the operator defined by Selvaraj and Karthikeyan [5]. Here we list some special cases of the operator* $\mathcal{J}_{\lambda}^{m}(a_1, b_1;\, q,\, z)f$*.*

1. *For a choice of the parameter* $m = 0$*, the operator* $\mathcal{J}_{\lambda}^{0}(\alpha_1,\, \beta_1)f(z)$ *reduces to the q-analogue of Dziok- Srivastava operator [1].*
2. *For* $a_i = q^{\alpha_i}$, $b_j = q^{\beta_j}$, $\alpha_i,\, \beta_j \in \mathbb{C}, \beta_j \neq 0, , (i = 1,\, \dots,\, r,\, j = 1,\, \dots,\, s)$ *and* $q \to 1^-$*, we get the operator defined by Selvaraj and Karthikeyan [5].*
3. *For* $r = 2,\, s = 1;\, a_1 = b_1,\, a_2 = q,$ *and* $\lambda = 1$*, we get the q- analogue of the well known Sălăgean operator (see [4]).*

Also many (well known and new) integral and differential operators can be obtained by specializing the parameters.

We let $\mathcal{S}^*$, $\mathcal{C}$ and $\mathcal{K}$ to denote the well known classes of starlike,convex and close to convex function respectively. We refer Goodman[3] which provides the study of various subclasses of univalent functions in slow motion. Another very important class in the study of various subclasses of univalent functions is the class of functions with positive real part. We denote by $P(\rho)$ the class of functions with $p(0) = 1$ which satisfies $\mathcal{R}\{p(z)\} > \rho$. It is well known that $p(z) = 1 + c_1 z + c_2 z^2 + \cdots \in P(\rho)$ implies $|\ p_n\ | \leq 2(1-\rho)$ for all $n \geq 1$.

The coefficients for the inverse of a function $f(z)$ of the form (3) is given by

$$g(w) = f^{-1}(z) = w - k_2 w^2 + (2k_2^2 - k_3)w^3 - (5k_2^3 - 5k_2 k_3 + k_4)w^4 + \cdots\,, \tag{9}$$

for details on the coefficients of the inverse of a function, we refer to chapter 5 in [3].

The area of a closed disc of radius r of a function $f \in \mathcal{S}$, provided us with an inequality which in turn was used to prove several central theorems in the field of univalent functions. In the class $\mathcal{S}$, the upper bound on a_2 was very useful in establishing the growth, distortion and radius problems of univalent functions. So finding the initial coefficients of various subclasses of analytic functions has always been a very attractive topic in the study of univalent function theory. The main purpose of this paper is to obtain the initial coefficients of the two classes of spiralike functions namely $\alpha - \mathcal{SP}^*(\beta, a, b; q, z)$ and $\alpha - \mathcal{SP}(\rho, a, b; q, z)$.

Now we begin with the following definitions.

Definition 1. *The function $f(z)$, given by (3), is said to be a member of $\alpha - \mathcal{SP}^*(\beta, a, b; q, z)$, if each of the following conditions are satisfied.*

$$\left|\arg\left(e^{i\alpha}\frac{z\left[D_q(J_\lambda^m(a_1, b_1; q, z)f)\right]}{(J_\lambda^m(a_1, b_1; q, z)f)}\right)\right| < \beta\frac{\pi}{2}, \quad (z \in \mathcal{U};\ \mid \alpha \mid \leq \pi/2,\ 0 \leq \beta < 1)$$

and

$$\left|\arg\left(e^{i\alpha}\frac{w\left[D_q(J_\lambda^m(a_1, b_1; q, w)f)\right]}{(J_\lambda^m(a_1, b_1; q, w)f)}\right)\right| < \beta\frac{\pi}{2}, \quad (w \in \mathcal{U};\ \mid \alpha \mid \leq \pi/2,\ 0 \leq \beta < 1).$$

Definition 2. *The function $f(z)$ given by (3), is said to be a member of $\alpha - \mathcal{SP}(\rho, a, b; q, z)$, if each of the following conditions are satisfied.*

$$\mathcal{R}\left(e^{i\alpha}\frac{z\left[D_q(J_\lambda^m(a_1, b_1; q, z)f)\right]}{(J_\lambda^m(a_1, b_1; q, z)f)}\right) > \rho cos(\alpha), \quad (z \in \mathcal{U};\ \mid \alpha \mid \leq \pi/2,\ 0 \leq \rho < 1)$$

and

$$\mathcal{R}\left(e^{i\alpha}\frac{w\left[D_q(J_\lambda^m(a_1, b_1; q, w)f)\right]}{(J_\lambda^m(a_1, b_1; q, w)f)}\right) > \rho cos(\alpha), \quad (w \in \mathcal{U};\ \mid \alpha \mid \leq \pi/2,\ 0 \leq \rho < 1).$$

The classes of $\alpha - \mathcal{SP}^*(\beta, a, b; q, z)$ and $\alpha - \mathcal{SP}(\rho, a, b; q, z)$ were motivated by [6]. If we let $m = 0, r = 2,\ s = 1;\ a_1 = b_1,\ a_2 = q$ and by taking limit $q \to 1^-$ in $\alpha - \mathcal{SP}^*(\beta, a, b; q, z)$ and $\alpha - \mathcal{SP}(\rho, a, b; q, z)$, we get the classes introduced by M. M. Soren and A. K. Misra [6].

2. Main Results

Theorem 1. *Let $f(z)$ given by (3), be in the class $\alpha - \mathcal{SP}(\rho, a, b; q, z)$, $(\mid \alpha \mid \leq \frac{\pi}{2}, 0 \leq \rho < 1)$. Then*

$$\mid k_2 \mid \leq \frac{\sqrt{2\cos\alpha(1-\rho)}}{\sqrt{q\left[1-\lambda+(1+q)\lambda\right]^{2m}\gamma_2^2 + \left[1-\lambda+[3]_q\lambda\right]^m([3]_q - 1)\gamma_3}}$$

,

$$\mid k_3 \mid \leq (1-\rho)\cos\alpha\left(\frac{2}{q\left[1-\lambda+(1+q)\lambda\right]^{2m}\gamma_2^2 + \left[1-\lambda+[3]_q\lambda\right]^m([3]_q - 1)\gamma_3}\right)$$

and

$$| k_4 |\leq \frac{2(1-\rho)\cos\alpha}{[1-\lambda+[4]_q\lambda]^m\,([4]_q-1)\gamma_4}+ \frac{10\sqrt{2}[(1-\rho)\cos\alpha]^{\frac{3}{2}}}{[[1-\lambda+(1+q)\lambda]^{2m}\,(q)\gamma_2^2+[1-\lambda+[3]_q\lambda]^m\,([3]_q-1)\gamma_3]^{\frac{3}{2}}}+ \frac{2\sqrt{2}[(1-\rho)\cos\alpha]^{\frac{3}{2}}}{[[1-\lambda+(1+q)\lambda]^{2m}\,(q)\gamma_2^2+[1-\lambda+[3]_q\lambda]^m\,([3]_q-1)\gamma_3]^{\frac{3}{2}}} \left[\frac{2\,[1-\lambda+(1+q)\lambda]^m\,[1-\lambda+[3]_q\lambda]^m\,\gamma_2\gamma_3(1+q+[3]_q)}{[1-\lambda+[4]_q\lambda]^m\,([4]_q-1)\gamma_4}+5\right].$$

Proof.

Let $f\in\alpha-\mathcal{SP}(\rho,a,b;q,z)$. Then the inequalities in Definition 2 can be equivalently rewritten as,

$$\left(e^{i\alpha}\frac{z\,[D_q(J_\lambda^m(a_1,b_1;q,z)f)]}{(J_\lambda^m(a_1,b_1;q,z)f)}\right)=P_1(z)\cos\alpha+i\sin\alpha \tag{10}$$

and

$$\left(e^{i\alpha}\frac{w\,[D_q(J_\lambda^m(a_1,b_1;q,w)f)]}{(J_\lambda^m(a_1,b_1;q,w)f)}\right)=Q_1(w)\cos\alpha+i\sin\alpha \tag{11}$$

respectively, where $\mathcal{R}(P_1(z))>\rho$ and $\mathcal{R}(Q_1(z))>\rho$,

$$P_1(z)=1+c_1z+c_2z^2+\cdots \qquad (z\in\mathcal{U})$$

and

$$Q_1(w)=1+l_1w+l_2w^2+\cdots \qquad (w\in\mathcal{U}).$$

By comparing the coefficients in (10), we have

$$e^{i\alpha}\,[1-\lambda+(1+q)\lambda]^m\,q\gamma_2k_2=c_1\cos\alpha \tag{12}$$

$$e^{i\alpha}\left[[1-\lambda+(1+q)\lambda]^{2m}\,(q)\gamma_2^2k_2^2+[1-\lambda+[3]_q\lambda]^m\,([3]_q-1)\,\gamma_3k_3\right]=c_2\cos\alpha \tag{13}$$

$$e^{i\alpha}\Bigg[[1-\lambda+[4]_q\lambda]^m\,([4]_q-1)\gamma_4k_4-(1+q+[3]_q)\,[1-\lambda+(1+q)\lambda]^m\,[1-\lambda+[3]_q\lambda]^m\,\gamma_2\gamma_3k_2k_3+ (1+q)\,[1-\lambda+(1+q)\lambda]^{2m}\,\gamma_2^2k_2^2\Bigg]=c_3\cos\alpha. \tag{14}$$

Similarly by equating the coefficients in (11), we get

$$-e^{i\alpha}\,[1-\lambda+(1+q)\lambda]^m\,(q)\,\gamma_2k_2=l_1\cos\alpha \tag{15}$$

$$e^{i\alpha}\left[[1-\lambda+(1+q)\lambda]^{2m}\,(q)\gamma_2^2k_2^2+[1-\lambda+[3]_q\lambda]^m\,([3]_q-1)\,\gamma_3(2k_2^2-k_3)\right]=l_2\cos\alpha. \tag{16}$$

$$e^{i\alpha}\Bigg[[1-\lambda+[4]_q\lambda]^m\left([4]_q-1\right)\gamma_4(5k_2k_3-5k_2^3-k_4) - \left(1+q+[3]_q\right)[1-\lambda+(1+q)\lambda]^m[1-\lambda+[3]_q\lambda]^m\gamma_2\gamma_3k_2k_3 + (1+q)[1-\lambda+(1+q)\lambda]^{2m}\gamma_2^2k_2^2\Bigg] = l_3\cos\alpha. \tag{17}$$

From (12) and (15), we get $l_1 = -c_1$. Before computing $\mid a_2 \mid$ and $\mid a_3 \mid$, we will obtain a refined estimate of $\mid c_1 \mid$. For this purpose, we first add (13) and (16)

$$2k_2^2 = (c_2+l_2)\frac{\cos\alpha}{e^{i\alpha}[[1-\lambda+(1+q)\lambda]^{2m}(q)\gamma_2^2+[1-\lambda+[3]_q\lambda]^m(1-[3]_q)\gamma_3]}.$$

Using (12) in the above equation in conjunction with the known result that $|c_n| \le 2(1-\rho)$ and $|l_n| \le 2(1-\rho)$, we have

$$\begin{aligned}
\mid c_1^2 \mid &= \left|(c_2+l_2)\frac{e^{i\alpha}[1-\lambda+(1+q)\lambda]^{2m}(q)^2\gamma_2^2}{2\cos\alpha[[1-\lambda+(1+q)\lambda]^{2m}(q)\gamma_2^2+[1-\lambda+[3]_q\lambda]^m(1-[3]_q)\gamma_3]}\right| \\
&\le \frac{|c_2|+|l_2|}{2}\frac{1\,[1-\lambda+(1+q)\lambda]^{2m}(q)^2\gamma_2^2}{\cos\alpha[[1-\lambda+(1+q)\lambda]^{2m}(q)\gamma_2^2+[1-\lambda+[3]_q\lambda]^m([3]_q-1)\gamma_3]} \\
&\le \frac{2(1-\rho)[1-\lambda+(1+q)\lambda]^{2m}(q)^2\gamma_2^2}{\cos\alpha[[1-\lambda+(1+q)\lambda]^{2m}(q)\gamma_2^2+[1-\lambda+[3]_q\lambda]^m([3]_q-1)\gamma_3]}
\end{aligned}$$

$$\mid c_1 \mid \le \frac{\sqrt{2(1-\rho)}\,[1-\lambda+(1+q)\lambda]^m(q)\gamma_2}{\sqrt{\cos\alpha[[1-\lambda+(1+q)\lambda]^{2m}(q)\gamma_2^2+[1-\lambda+[3]_q\lambda]^m([3]_q-1)\gamma_3]}} \tag{18}$$

and

$$\begin{aligned}
\mid k_2 \mid &\le \frac{\mid c_1 \mid \cos\alpha}{[1-\lambda+(1+q)\lambda]^m([2]_q-1)\gamma_2} \\
&= \frac{\sqrt{2(1-\rho)\cos\alpha}}{\sqrt{[[1-\lambda+(1+q)\lambda]^{2m}(q)\gamma_2^2+[1-\lambda+[3]_q\lambda]^m([3]_q-1)\gamma_3]}}
\end{aligned}$$

which proves the assertion.
We next find the upper bound on a_3. For this we subtract (13) and (16) and using $c_1 = -l_1$, we get

$$k_3 = \frac{(c_2-l_2)\cos\alpha}{2e^{i\alpha}[1-\lambda+[3]_q\lambda]^m([3]_q-1)\gamma_3} + k_2^2. \tag{19}$$

On simplification, we get

$$k_3 = \frac{(c_2 - l_2)\cos\alpha}{2e^{i\alpha}\left[1-\lambda+[3]_q\lambda\right]^m([3]_q-1)\gamma_3} + \frac{c_1^2\cos^2\alpha}{e^{2i\alpha}\left[1-\lambda+(1+q)\lambda\right]^{2m}(q)^2}$$

$$= \frac{(c_2 - l_2)\cos\alpha}{2e^{i\alpha}\left[1-\lambda+[3]_q\lambda\right]^m([3]_q-1)\gamma_3} + \frac{(c_2 + l_2)\cos\alpha}{2e^{i\alpha}[\left[1-\lambda+(1+q)\lambda\right]^{2m}(q)\gamma_2^2 + \left[1-\lambda+[3]_q\lambda\right]^m([3]_q-1)\gamma_3]}$$

$$= \frac{\cos\alpha}{2e^{i\alpha}}\left[\frac{c_2}{\left[1-\lambda+[3]_q\lambda\right]^m([3]_q-1)\gamma_3} + \frac{c_2}{\left[1-\lambda+(1+q)\lambda\right]^{2m}(q)\gamma_2 + \left[1-\lambda+[3]_q\lambda\right]^m([3]_q-1)\gamma_3}\right] + \frac{\cos\alpha}{2e^{i\alpha}}\left[\frac{l_2}{\left[1-\lambda+(1+q)\lambda\right]^{2m}(q)\gamma_2 + \left[1-\lambda+[3]_q\lambda\right]^m([3]_q-1)\gamma_3} - \frac{l_2}{\left[1-\lambda+[3]_q\lambda\right]^m(1-[3]_q)\gamma_3}\right].$$

On taking the modulus, we have

$$\mid k_3 \mid \leq (1-\rho)\cos\alpha\left[\frac{2}{\left[1-\lambda+(1+q)\lambda\right]^{2m}(q)\gamma_2^2 + \left[1-\lambda+[3]_q\lambda\right]^m([3]_q-1)\gamma_3}\right]. \qquad (20)$$

Hence the upper bound of k_3. Now we shall we move onto find the estimate on $\mid k_4 \mid$.
By subtracting the equations (14) and (17), we get

$$2k_4 = \frac{e^{-i\alpha}\cos\alpha(c_3 - l_3)}{\left[1-\lambda+[4]_q\lambda\right]^m([4]_q-1)\gamma_4} + \frac{5c_1\cos\alpha k_3}{e^{i\alpha}\left[1-\lambda+(1+q)\lambda\right]^m(q)\gamma_2} - \frac{c_1^3\cos^3\alpha}{e^{3i\alpha}\left[1-\lambda+(1+q)\lambda\right]^{3m}(q)^3\gamma_2^3}\left[\frac{2\left[1-\lambda+(1+q)\lambda\right]^m\left[1-\lambda+[3]_q\lambda\right]^m\gamma_2\gamma_3((1+q)+[3]_q)}{\left[1-\lambda+[4]_q\lambda\right]^m([4]_q-1)\gamma_4} + 5\right].$$

$$\mid k_4 \mid \leq \frac{2(1-\rho)\cos\alpha}{\left[1-\lambda+[4]_q\lambda\right]^m([4]_q-1)\gamma_4} + \frac{10\sqrt{2}[(1-\rho)\cos\alpha]^{\frac{3}{2}}}{[\left[1-\lambda+(1+q)\lambda\right]^{2m}(q)\gamma_2^2 + \left[1-\lambda+[3]_q\lambda\right]^m([3]_q-1)\gamma_3]^{\frac{3}{2}}} + \frac{2\sqrt{2}[(1-\rho)\cos\alpha]^{\frac{3}{2}}}{[\left[1-\lambda+(1+q)\lambda\right]^{2m}(q)\gamma_2^2 + \left[1-\lambda+[3]_q\lambda\right]^m([3]_q-1)\gamma_3]^{\frac{3}{2}}}\left[\frac{2\left[1-\lambda+(1+q)\lambda\right]^m\left[1-\lambda+[3]_q\lambda\right]^m\gamma_2\gamma_3(1+q+[3]_q)}{\left[1-\lambda+[4]_q\lambda\right]^m([4]_q-1)\gamma_4} + 5\right].$$

This completes the proof of the Theorem 1.

Theorem 2. *Let $f(z)$, given by (3), be in the class $\alpha - \mathcal{SP}^*(\beta, a, b; q, z)$ ($| \alpha | \leq \frac{\pi}{2}, 0 \leq \beta < 1$). Then*

$$| k_2 | \leq \frac{\beta\sqrt{2}\cos(\frac{\alpha}{\beta})}{q\left[1-\lambda+(1+q)\lambda\right]^m | \gamma_2 | \sqrt{\left[1-\lambda+[3]_q\lambda\right]^m ([3]_q-1)\gamma_3\cos(\frac{\alpha}{\beta})\delta}}$$

and

$$| k_3 | \leq \frac{2\beta\cos(\frac{\alpha}{\beta})}{\left[1-\lambda+[3]_q\lambda\right]^m ([3]_q-1)\gamma_3}\left[\frac{\beta}{q^2\left[1-\lambda+(1+q)\lambda\right]^{2m}\gamma_2^2\delta}-1\right]$$

where

$$\delta = \frac{\beta}{q^2\left[1-\lambda+(1+q)\lambda\right]^{2m}\gamma_2^2} - \frac{\left[1+\frac{\beta-1}{2}(q+2)\right]}{q\left[1-\lambda+[3]_q\lambda\right]^m ([3]_q-1)\gamma_3}.$$

Proof. From Definition 1, we have

$$D_q(J_\lambda^m(a_1, b_1; q, z)f) = \frac{J_\lambda^m(a_1, b_1; q, z)f}{z}e^{-i\alpha}h(z) \tag{21}$$

where $h(z)$ is analytic in $\mathcal{U}$ and satisfies

$$h(0) = e^{i\alpha} \quad \text{and} \quad | \arg h(z) | < \beta\pi/2 \quad (z \in \mathcal{U}).$$

It can be checked that the function $q(z)$ defined by

$$h(z)^{\frac{1}{\beta}} = \cos\left(\frac{\alpha}{\beta}\right)q(z) + i\sin\left(\frac{\alpha}{\beta}\right) \quad (z \in \mathcal{U})$$

is a member of the class $\mathcal{P}$. Suppose that

$$q(z) = 1 + c_1z + c_2z^2 + \ldots. \quad (z \in \mathcal{U}).$$

By comparing coefficients in (21), we have

$$k_2 = \frac{\beta c_1 e^{-i(\frac{\alpha}{\beta})}\cos(\frac{\alpha}{\beta})}{q\left[1-\lambda+(1+q)\lambda\right]^m\gamma_2} \tag{22}$$

and

$$k_3 = \frac{\beta c_2 e^{-i(\frac{\alpha}{\beta})}\cos(\frac{\alpha}{\beta}) + \frac{\beta}{q}c_1^2 e^{-2i(\frac{\alpha}{\beta})}\cos^2(\frac{\alpha}{\beta})\left[1+\frac{\beta-1}{2}(q+2)\right]}{\left[1-\lambda+[3]_q\lambda\right]^m([3]_q-1)\gamma_3}. \tag{23}$$

Similarly, we take

$$D_q(J_\lambda^m(a_1, b_1; q, w)f) = \frac{J_\lambda^m(a_1, b_1; q, w)f}{w}e^{-i\alpha}h(w), \tag{24}$$

where $h(w)$ is analytic in $\mathcal{U}$ and satisfies

$$\mathcal{H}(0) = e^{i\alpha} \quad \text{and} \quad |\arg h(w)| < \beta\pi/2 \quad (w \in \mathcal{U}).$$

It can be checked that the function $p(w)$ defined by:

$$h(w)^{\frac{1}{\beta}} = \cos\left(\frac{\alpha}{\beta}\right) p(w) + i\sin\left(\frac{\alpha}{\beta}\right) \quad (w \in \mathcal{U})$$

is a member of the class $\mathcal{P}$. If

$$p(w) = 1 + l_1 w + l_2 w^2 + \dots \quad (w \in \mathcal{U}),$$

then again by comparing the coefficients in (24), we have the following

$$-k_2 = \frac{\beta l_1 e^{-i(\frac{\alpha}{\beta})}\cos(\frac{\alpha}{\beta})}{q\left[1-\lambda+(1+q)\lambda\right]^m \gamma_2} \tag{25}$$

and

$$2k_2^2 - k_3 = \frac{\beta l_2 e^{-i(\frac{\alpha}{\beta})}\cos(\frac{\alpha}{\beta}) + \frac{\beta}{q} l_1^2 e^{-2i(\frac{\alpha}{\beta})}\cos^2(\frac{\alpha}{\beta})\left[1+\frac{\beta-1}{2}(q+2)\right]}{\left[1-\lambda+[3]_q\lambda\right]^m([3]_q-1)\gamma_3}. \tag{26}$$

It is obvious from (22) and (25) that $l_1 = -c_1$. From (23) and (26), we get

$$c_1^2 = \frac{(c_2+l_2)}{2\left[1-\lambda+[3]_q\lambda\right]^m([3]_q-1)\gamma_3 e^{-i(\frac{\alpha}{\beta})}\cos(\frac{\alpha}{\beta})\delta} \tag{27}$$

where

$$\delta = \frac{\beta}{q^2\left[1-\lambda+(1+q)\lambda\right]^{2m}\gamma_2^2} - \frac{\left[1+\frac{\beta-1}{2}(q+2)\right]}{q\left[1-\lambda+[3]_q\lambda\right]^m([3]_q-1)\gamma_3}.$$

By applying the familiar inequalities $|c_2| \leq 2$ and $|l_2| \leq 2$, we get

$$|c_1| \leq \frac{\sqrt{2}}{\sqrt{\left[1-\lambda+[3]_q\lambda\right]^m([3]_q-1)\gamma_3\cos(\frac{\alpha}{\beta})\delta}} \tag{28}$$

and

$$|k_2| = \frac{\beta\,|c_1|\cos(\frac{\alpha}{\beta})}{q\left[1-\lambda+(1+q)\lambda\right]^m|\gamma_2|} \tag{29}$$

$$\leq \frac{\beta\sqrt{2}\cos(\frac{\alpha}{\beta})}{q\left[1-\lambda+(1+q)\lambda\right]^m|\gamma_2|\sqrt{\left[1-\lambda+[3]_q\lambda\right]^m([3]_q-1)\gamma_3\cos(\frac{\alpha}{\beta})\delta}}. \tag{30}$$

We next find a upper bound on $|a_3|$. For this we subtract (26) from (23) and get

$$2k_3 = 2k_2^2 - \frac{\beta l_2 e^{-i(\frac{\alpha}{\beta})}\cos(\frac{\alpha}{\beta}) + \frac{\beta}{q} l_1^2 e^{-2i(\frac{\alpha}{\beta})}\cos^2(\frac{\alpha}{\beta})\left[1+\frac{\beta-1}{2}(q+2)\right]}{\left[1-\lambda+[3]_q\lambda\right]^m([3]_q-1)\gamma_3} +$$

$$\frac{\beta c_2 e^{-i(\frac{\alpha}{\beta})}\cos(\frac{\alpha}{\beta}) + \frac{\beta}{q}c_1^2 e^{-2i(\frac{\alpha}{\beta})}\cos^2(\frac{\alpha}{\beta})\left[1+\frac{\beta-1}{2}(q+2)\right]}{\left[1-\lambda+[3]_q\lambda\right]^m([3]_q-1)\gamma_3}.$$

Now putting that $c_1^2 = l_1^2$ and k_2 values in above equation, we obtain

$$2k_3 = \frac{\beta e^{-i(\frac{\alpha}{\beta})}\cos(\frac{\alpha}{\beta})}{\left[1-\lambda+[3]_q\lambda\right]^m([3]_q-1)\gamma_3}\left[\frac{\beta(c_2+l_2)}{q^2\left[1-\lambda+(1+q)\lambda\right]^{2m}\gamma_2^2\delta} - (c_2-l_2)\right]. \qquad (31)$$

By applying the familiar inequalities $\mid c_2 \mid \leq 2$ and $\mid l_2 \mid \leq 2$ we get

$$\mid k_3 \mid \leq \frac{2\beta\cos(\frac{\alpha}{\beta})}{\left[1-\lambda+[3]_q\lambda\right]^m([3]_q-1)\gamma_3}\left[\frac{\beta}{q^2\left[1-\lambda+(1+q)\lambda\right]^{2m}\gamma_2^2\delta} - 1\right]. \qquad (32)$$

The proof of Theorem 2 is thus completed.

3. Concluding Remarks

Remark 2. *For the choice of the parameters, $m = 0, r = 2$, $s = 1$; $a_1 = b_1$, $a_2 = q$, and by taking limit $q \to 1^-$ in Theorem 1 and Theorem 2, we get the results obtained in [6].*

Remark 3. *For appropriate choice of the parameter in Theorem 1, we get the following inequalities for a class of functions bi-starlike of order ρ $(0 \leq \rho < 1)$.*

$$|k_2| \leq \sqrt{2(1-\rho)} \qquad \text{and} \qquad |k_3| \leq 2(1-\rho).$$

Remark 4. *Similarly for the appropriate choice of the parameter in Theorem 1, we get the following inequalities for a class of functions which are bi-convex of order ρ $(0 \leq \rho < 1)$.*

$$|k_2| \leq \sqrt{(1-\rho)} \qquad \text{and} \qquad |k_3| \leq (1-\rho).$$

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