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Influence of Joule Heating and Non-Linear Radiation on MHD 3D Dissipating Flow of Casson Nanofluid past a Non-Linear Stretching Sheet

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Abstract: Present analysis is to study the combined effects of viscous dissipation and Joule heating on MHD three-dimensional laminar flow of a viscous incompressible non-linear radiating Casson nanofluid past a non-linear stretching porous sheet. Present model describes that flow generated by bi-directional non-linear stretching sheet with thermophoresis and Brownian motion effects. The governing nonlinear partial differential equations are transformed into a system of nonlinear coupled ordinary differential equations by similarity transformations and then solved by employing shooting method. The effects of the flow parameters on the velocity, temperature and concentration as well as the skin friction coefficient, Nusselt number and Sherwood number near the wall are computed for various values of the fluid properties. This study reveals that the temperature of Casson nanofluid increases with combination of viscous dissipation and Joule heating. Increasing thermophoresis parameter increases the species concentration of the nanoflow. The comparison of present results have been made with the published work and the results are found to be very good agreement.

Keywords: Thermal radiation, Magnetic field, Rheological fluids, Casson Nano fluid,Viscous dissipation, Joule heating, non-linear stretching sheet, three dimensional flow

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1 Introduction

Most of the rheological fluids with their unique behaviour are utilised in different engineering and industrial processes. Examples, molten plastic material production, polymeric liquids, lubricants performance, stuff food materials, paints production, preventing coating, slurries so on. As these fluids doesn't obey the simple theory of Navier-Stoke's, so to describe it in a specific way various fluid models are developed. A few among them are Casson, Jeffery, Maxwell, power-law, Eyring-Powel models etc. In the class of non-Newtonian fluids, Casson fluid has unique behaviour, which has wide application in food processing, in metallurgy, drilling operation and bio-engineering operations, etc. Defining Casson fluid as a shear thinning fluid, which possess infinite viscosity at zero rate of shear. Casson's constitutive equations are found to describe accurately the flow curves of suspensions of pigments in lithographic varnishes used for preparation of printing inks and silicon suspensions [1]. Various experiments performed on blood with varying haematocrits, anticoagulants, temperatures, and so forth strongly suggest the behavior of blood as a Casson fluid [2–4]. In particular, Casson fluid model describes the flow characteristics of blood more accurately at low shear rates and when it flows through small vessels [5]. For increasing thermal conductivity of Casson fluid, suspension of nanoparticles into Casson fluid is considered into an account. Sreenivasulu *et al.* [6] examined Casson dissipating flow past nonlinear stretching sheet. Poornima *et al.* [7] studied the radiation effect on Casson rheological fluid.

Basing on its fascinating application in the field of industrial and engineering processes, the concept of linear and non-linear stretching sheets revolutionised more now-a-days. Particularly in the process of polymer extrusion, when an object present in a saturated porous medium passed between two closely placed vertical solid blocks gets stretched. This gives a unidirectional elongation to the extradite and thereby refines its mechanical properties. The liquid is meant to cool the stretching sheet whose property depends greatly on the rate at which it is cooled and stretched in porous medium. The fluid mechanical properties desired for the outcome of such a process depends mainly on the rate of cooling and the stretching rate. For the desired output, it is important that proper cooing fluid is chosen and flow of the cooling liquid caused due to the stretching sheet can be controlled Pioneer in this work was Crane, who gave the exact solution for the flow due to stretching of flat surface [8]. Later development was done by Wang [9] and Donald Ariel [10], who used the stretching sheets for three dimensional flows. Analytical solution was given by Hayat and Javed [11]. On generalized three dimensional MHD flow past porous stretching sheet was analysed by Donald Ariel [12], investigated the generalized three dimensional flow due to a stretching sheet. Most cases, the stretching may not be linear; it may be power-law, exponential, non-linear and so on. Hence, the flow and heat transfer caused by a non-linearly stretching sheet is of great interest because of some physical situations rise in which the need of a nonlinearly stretching sheet should be faced. It was first identified by Gupta and Gupta [13]. Vajravelu [14] presented the viscous flow over a nonlinear stretching sheet. Nadeem *et al.* [15], studied the MHD flow of a Casson fluid over an exponentially shrinking sheet. Mukhopadhyay [16] investigated the study of heat transfer in a viscous fluid over a non-linearly stretching sheet with Casson fluid flow and heat transfer over a nonlinearly stretching surface. Three dimensional flow of nanofluid over a non-liner stretching sheet was reported by Khan *et al.* [17].

A major field of science, Magnetohydrodynamics which involves the electrically conducting physiological fluid flow under the magnetic field effects, whose main application is in the field of bio-medical science. In this field, bio-magnetic fluids are used working fluids. Magnetic drug targeting, magnetic devices for cell separation, transporting complex bio-waste fluids, cancer tumour treatment, magnetic endoscopy, and cell death by hyperthermia created by an alternating magnetic field are some examples. Modelling of such fluids was characterized by Maxwell's equation. Inducing electric currents into the moving conducting fluids causes transverse magnetic field. The interaction between these induced currents and the exerted magnetic field exerts a type of resistive type force called Lorentz force. Various applications of magnetic fields arise in energy equipment devices [18], intelligent aerospace productions [19] and industrial processing devices [20]. Poornima *et al.* [21] studied the MHD Mixed Convective and Radiative Flow with chemical reaction effects. Uddin *et al.* [22] solved the problem of MHD convective nanofluid over a stretching sheet using scaling group transformation method. Mustafa and Khan [23] described comprehensive solutions for MHD Casson nanofluid over a non-linearly extending non-isothermal sheet. Uddin *et al.* [24] studied the impact of MHD on convective slip flow using Lie group analysis.

Major application of radiation is in industry and research, next comes the public health care and last is the veterinary. In public health care, radiation is used to examine and treat patients, especially large dosage of radiation is given for destroying diseased tissue, in treating cancer. Industrially, for control of quality of materials, consistency of paper. Practical applications for non-ionising radiation are lasers, microwave ovens, solariums, mobile telephones, MRI devices in the medical field, and industrial heaters. Uddin *et al.* [25] discussed the radiation influence on convective nanofluid. A new numerical approach for MHD laminar boundary layer flow and heat transfer of nanofluids over a moving surface in the presence of radiation have been analysed by Shateyi and Prkash [26]. Radiation effect on MHD free convection of a Casson fluid flow from a horizontal circular cylinder with partial slip in non-Darcian porous medium with viscous dissipation have been presented by Makanda *et. al.* [27]. Uddin *et al.* [28] studied the conversion of energy under magneto-convection,non-linear radiation past a stretching sheet. Gireesha *et al.* [29] examined the non-linear radiative heat transfer of a Casson nanofluid over a non-linear stretching sheet.Khan *et al.* [30] studied the influence of non-linear thermal radiation on Magneto-Burgers nanofluid.

In electromagnetic application, impedance boundary conditions, for heat transfer problems, convective boundary conditions are utilised. The convective boundary condition is an energy balance at a sheet-fluid interface. One side is conduction in the sheet and the other is describing convection in the fluid. The boundary conditions for the top and bottom of the sheet are not exactly the same because of spatial orientation. The difference is in the negative sign in the conduction term. It indicates that a positive temperature gradient in the solid corresponds to a negative heat flow at the surface. Nayak *et al.* [31] studied the thermal radiation impact on three dimensional MHD convective slip flow past a stretching sheet. Akbar *et al.* [32] examined MHD stagnation point flow of nanofluid towards a stretching surface with convective boundary condition. Bhaskar Reddy *et al.* [33] studied the influence of variable thermal conductivity on MHD boundary layer slip flow of Ethylene-Glycol based Cu-nanofluid over a stretching sheet with convective boundary condition. Bilal Ashraf *et al.* [34] analysed the mixed convection radiative flow of 3D Maxwell fluid past an inclined stretching sheet in the presence of thermophoresis and convective condition. Recently, Sulochana *et al.* [35] investigated 3D Casson nanofluid flow over a stretching sheet with convective boundary conditions.

The internal heat friction present inside the fluid is the dissipation. When the flow field are of extreme size or high gravitational field or in the case of plate heating or cooling the values of viscous dissipation and Joule heating are not deniable. There are numerous applications like power generator systems, liquid metal fluids, cooling of nuclear reactors, cooling of metallic sheets or electronic chips etc. Viscous dissipation and radiation effects on MHD boundary layer flow of a nanofluid past a rotating stretching sheet has been reported by Wahiduzzaman *et al.* [36]. Sreenivasulu *et al.* [37] presented the effect of radiation on MHD thermo-solutal Marangonic convection boundary layer flow with Joule heating and viscous dissipation. Jaber [38] investigated the Joule heating and viscous dissipation on effects on MHD flow over a stretching porous sheet subjected to power law heat flux in presence of heat source. Unsteady MHD three dimensional flow with viscous dissipation and Joule heating has been presented by Hayat *et al.* [39]. Farooq *et al.* [40] examined the non-linear thermal radiation effects on MHD stagnation point flow. Muhammad Ramzan [41] studied the Influence of Newtonian heating on three dimensional flow of couple stress nanofluid with viscous dissipation and Joule heating. Hussain *et al.* [42] investigated the combined effects of viscous dissipation and Joule heating on MHD Sisko nanofluid over a stretching cylinder.

The aim of the present paper is to investigate the influence of Joule heating and viscous dissipation effects on MHD 3D Casson nanofluid fluid past a non-linear stretching sheet imbedded in Darcian porous medium with radiation and convective boundary condition.

2 Mathematical analysis

Consider a steady, three dimensional boundary layer flow of a viscous incompressible electrically conducting and radiating Casson nanofluid past a non-linear stretching porous sheet with convective surface boundary condition. It is assumed that the sheet is stretched along the *xy*-plane, while the fluid is placed along the Z- axis. The sheet moves with the non-uniform velocities $u = u_w = a(x+y)^n$, $v = v_w = b(x+y)^n$ where a, b, n > 0, along x and y directions respectively as shown in Fig. 1. A non-uniform magnetic field of strength B is applied in the direction perpendicular to the fluid flow. The transversely applied magnetic field and magnetic Reynolds number are very small and hence the induced magnetic field is negligible. Addition effects of Brownian motion, Thermophoresis, Viscous dissipation and Joule heating are taken into account.

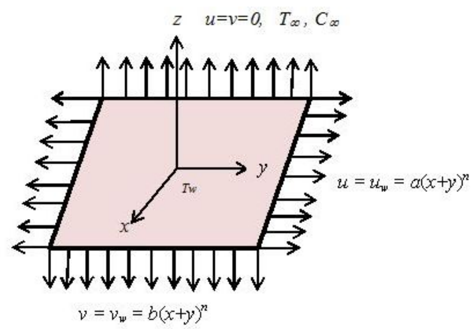

Fig. 1: Physical model and coordinate system

We also assume the rheological equation of Casson fluid reported by Mukopadyaya [16]

$$\tau_{ij} = \begin{cases} 2\left(\mu_B + p_y/\sqrt{2\pi}\right) e_{ij}, & \pi > \pi_c \\ 2\left(\mu_B + p_y/\sqrt{2\pi_c}\right) e_{ij}, & \pi < \pi_c \end{cases}$$

Where π is the product of the component of deformation rate with itself, $\pi = e_{ij}e_{ij}$, e_{ij} - (i, j) component of the deformation rate, π_{c-} critical value of this product based on the non-Newtonian model, μ_B - the plastic dynamic viscosity of the non-Newtonian fluid and P_y - the yield stress of the fluid.

The governing equations for three dimensional flow with heat and mass transfer can be expressed under the usual boundary conditions are as follows Khan *et al.* [17]

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = \nu_f\left(1 + \frac{1}{\beta}\right)\frac{\partial^2 u}{\partial z^2} - \frac{\sigma B^2}{\rho_f}u \tag{2}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = \nu_f\left(1 + \frac{1}{\beta}\right)\frac{\partial^2 v}{\partial z^2} - \frac{\sigma B^2}{\rho_f}v \tag{3}$$

$$\begin{aligned} u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z} &= \alpha_f\frac{\partial^2 T}{\partial z^2} - \frac{1}{\rho Cp}\frac{\partial q_r}{\partial z} \\ &+ \tau\left[D_B\frac{\partial T}{\partial z}\frac{\partial C}{\partial z} + \frac{D_T}{T_\infty}\left(\frac{\partial T}{\partial z}\right)^2\right] \\ &+ \frac{2\mu_f}{\rho_f Cp_f}\left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2\right] \\ &+ \frac{\sigma B^2}{\rho_f cp_f}(u^2 + v^2) \end{aligned} \tag{4}$$

$$u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} + w\frac{\partial C}{\partial z} = D_B\frac{\partial^2 C}{\partial z^2} + \frac{D_T}{T_\infty}\frac{\partial^2 T}{\partial z^2} \tag{5}$$

The appropriate boundary conditions for the velocity, temperature and concentration fields are

$$\left.\begin{array}{l} u = u_W = a(x+y)^n, \;\; v = v_W = b(x+y)^n, \\ w = 0, \;\; -k_f\left(\frac{\partial T}{\partial z}\right) = h_f(T_f - T), \;\; C = C_W(x) \end{array}\right\} \text{ at } Z = 0$$
$$u \to 0, \;\; v \to 0, \;\; T \to T_\infty, \;\; C \to C_\infty \quad \text{as} \;\; z \to \infty \tag{6}$$

where u, v and w are the velocity components in the x, y and z directions, respectively, T is the temperature of the fluid, T_W is the wall temperature of the fluid, T_∞ is the temperature of the fluid far away from the plate, C is the concentration of the fluid, C_W is the wall concentration of the solute, C_∞ is the concentration of the solute far away from the sheet. q_r is the radiative heat flux.μ_f, p_f, ν_f, ρ_f, and α_f are the dynamic viscosity, the pressure, kinematic viscosity, density of the fluid respectively. Furthermore σ, B and Cp_f are the fluid electric conductivity, magnetic induction and the specific heat at constant pressure, respectively. k_1 is the porosity of the porous medium, D_B is the effective diffusion coefficient, k_f is the thermal conductivity of the fluid, h_f is the convective heat transfer coefficient and T_f is the convective fluid temperature below the moving sheet.

The radiative heat flux q_r is described by Rosseland approximation is such that

$$q_r = -\frac{4}{3}\frac{\sigma_s}{k_e}\frac{\partial T^4}{\partial y} = -\frac{4}{3}\frac{\sigma_s}{k_e}T^3\frac{\partial T}{\partial y} \tag{7}$$

where σ_s is the Stefan-Boltzmann constant and k_e is the mean absorption coefficient.

It should be noted that by Rosseland approximation, we limit our analysis to optically thick fluids. If the temperature differences within in the flow are sufficiently small, then equation (7) can be linearized by expanding T^4 into the Taylor series about T_∞ and neglecting higher order terms to take the form:

$$T^4 \cong 4T_\infty^3 T - 3T_\infty^4 \tag{8}$$

In view of equations (7) and (8), equation (4) becomes

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z} = \frac{k_f}{\rho_f Cp_f}\left(1 + \frac{16\sigma_s T_\infty^3}{3k_e k_f}\right)\frac{\partial^2 T}{\partial z^2} + \tau\left[D_B\frac{\partial T}{\partial z}\frac{\partial C}{\partial z} + \frac{D_T}{T_\infty}\left(\frac{\partial T}{\partial z}\right)^2\right] + \frac{2\mu_f}{\rho_f Cp_f}\left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2\right] + \frac{\sigma B^2}{\rho_f Cp_f}\left(u^2 + v^2\right) \tag{9}$$

However, the energy equation with nonlinear thermal radiation will take the following form [43]

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z} = \frac{k_f}{\rho_f Cp_f}\left(1 + \frac{16\sigma_s}{3k_e k_f}T^3\right)\frac{\partial^2 T}{\partial z^2} + \tau\left[D_B\frac{\partial T}{\partial z}\frac{\partial C}{\partial z} + \frac{D_T}{T_\infty}\left(\frac{\partial T}{\partial z}\right)^2\right] + \frac{2\mu_f}{\rho_f Cp_f}\left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2\right] + \frac{\sigma B^2}{\rho_f Cp_f}\left(u^2 + v^2\right) \tag{10}$$

To obtain similarity solutions, it is assumed that the variable magnetic field B is of the form (Devi and thiyagarajan [44] and Habibi Matin *et al.* [45])

$$B(x) = \frac{B_0}{(x+y)^{\frac{1-n}{2}}} \tag{11}$$

where B_0 is the constant magnetic field.

Introducing the following similarity transformations,

$$\begin{array}{l} u = a(x+y)^n f'(\eta), \;\; v = b(x+y)^n g'(\eta), \\ w = -\sqrt{a\nu_f}\,(x+y)^{\frac{n-1}{2}}\left(\frac{n+1}{2}\left(f(\eta) + cg(\eta)\right) + \frac{n-1}{2}\eta\left(f'(\eta) + cg'(\eta)\right)\right), \\ \eta = \sqrt{\frac{a}{\nu_f}}(x+y)^{\frac{n-1}{2}}z, \;\; \theta(\eta) = \frac{T-T_\infty}{T_f-T_\infty}, \quad \phi(\eta) = \frac{C-C_\infty}{C_w-C_\infty}, \\ M = \frac{\sigma B_0^2}{\rho_f a}, \;\; c = \frac{b}{a}, \Pr = \frac{\nu_f}{\alpha_f}, \;\; R = \frac{16\sigma_s}{3k_e k_f}, \; Nb = \frac{\tau D_B(C_w - C_\infty)}{\alpha_f}, \\ Nt = \frac{\tau D_T(T_f - T_\infty)}{T_\infty\,\alpha_f}, \;\; Ec_x = \frac{u_w^2}{Cp_f(T_f - T_\infty)}, \;\; Ec_y = \frac{v_w^2}{Cp_f(T_f - T_\infty)}, \\ Le = \frac{\alpha_f}{D_B}, \;\; Bi = \frac{h_f}{k_f}\sqrt{\frac{\nu_f}{a}}, \mathrm{Re}_x = \frac{u_w(x+y)}{\nu_f}, \; \mathrm{Re}_y = \frac{v_w(x+y)}{\nu_f} \end{array} \tag{12}$$

In view of equations (9)–(11), the governing equations (2), (3), (5) and (9) reduce to the dimensionless form:

$$\left(1 + \frac{1}{\beta}\right)f''' + \left(\frac{n+1}{2}\right)(f + cg)f'' - n\left(f' + g'\right)f' - Mf' = 0 \tag{13}$$

$$\left(1 + \frac{1}{\beta}\right)g''' + \left(\frac{n+1}{2}\right)(f + cg)g'' - n\left(f' + g'\right)g' - Mg' = 0 \tag{14}$$

$$\frac{1}{\Pr}\left(1 + R(1 + (\theta_m - 1)\,\theta)^3\theta'\right)' + \left(\frac{n+1}{2}\right)(f + cg)\theta' + Nb\theta'\phi' + Nt\theta'^2 + 2\left[Ec_x f''^2 + Ec_y g''^2\right] + M\left[Ec_x f'^2 + Ec_y g'^2\right] = 0 \tag{15}$$

$$\phi'' + \left(\frac{n+1}{2}\right)Le(f + cg)\phi' + \frac{Nt}{Nb}\theta'' = 0 \tag{16}$$

The corresponding boundary conditions are

$$\begin{array}{l} f(0) = 0, \;\; f'(0) = 1, \quad g(0) = 0, \;\; g'(0) = c, \\ \theta'(0) = -Bi(1 - \theta(0)), \;\; \phi(0) = 1 \\ f'(\infty) \to 0, \quad g'(\infty) \to 0, \;\; \theta(\infty) \to 0, \;\; \phi(\infty) \to 0 \end{array} \tag{17}$$

where prime denote the differentiation with respect to η, n is the non-linear stretching parameter, M- the magnetic parameter, K- the porosity parameter, c- the ratio of stretching rate along y-direction to the stretching rate along x- direction, Pr- the Prandtl number, R-the radiation parameter, Nb- the Brownian motion parameter, Nt- the thermophoresis parameter, Ec_x- the Eckert number along x- direction, Ec_y- the Eckert number along y- direction , Le- the Lewis number and Bi- the Biot number.

From the technological point of view, the skin-friction, rate of heat transfer and the rate of mass transfer are important physical quantities for this type of boundary layer flow.

Knowing the velocity, the skin-friction at the plate can be obtained, which in non-dimensional form is given by

$$\begin{aligned} &\tau_{zx} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)_{z=0}, \quad \tau_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)_{z=0} \\ &C_{fx} = \frac{\tau_{zx}}{\rho_f u_w^2}, \quad C_{fy} = \frac{\tau_{zy}}{\rho_f u_w^2}, \\ &\Rightarrow C_{fx}\mathrm{Re}_x^{1/2} = \left(1 + \frac{1}{\beta}\right) f''(0), \\ &C_{fy} c^{3/2}\mathrm{Re}_y^{1/2} = \left(1 + \frac{1}{\beta}\right) g''(0) \end{aligned} \tag{18}$$

Knowing the temperature field in the boundary layer, we can calculate the heat transfer coefficient at the porous plate, which in the non-dimensional form, in terms of the Nusselt number, is given by

$$\begin{aligned} &Nu_x = \frac{q_w(x+y)}{k(T_f - T_\infty)}, \quad where \;\; q_W = -k\left(\frac{\partial T}{\partial z}\right)_{z=0} \\ &Nu_x \mathrm{Re}_x^{-1/2} = -\theta'(0) \end{aligned} \tag{19}$$

Knowing the concentration field in the boundary layer, we can calculate the mass transfer coefficient at the porous plate, which in the non-dimensional form, in terms of the Sherwood number, is given by

$$\begin{aligned} &Sh_x = \frac{j_w x}{D_m(C_w - C_\infty)}, \quad where \;\; j_W = -D_m\left(\frac{\partial C}{\partial z}\right)_{z=0} \\ &Sh_x \mathrm{Re}_x^{-1/2} = -\phi'(0) \end{aligned} \tag{20}$$

where Re are the local Reynolds numbers along x- and y-directions respectively.

3 Results and discussion

The system of dimensional partial differential equations is transformed to dimensionless equations using similarity variables. Then the resultant system of non-linear coupled ordinary differential equations is solved employing shooting technique with Runge-Kutta fifth order integration scheme. In this section, we have discussed the effects of governing thermo-physical parameters on the velocity profiles $f'(\eta)$ and $g'(\eta)$, temperature and concentration as well as the skin friction coefficients along x and y- directions, Nusselt number and Sherwood number are numerically computed and portrayed through graphs 2-23 and tables. The comparison of present results have been made with the published work and the results are found to be very good agreement which is shown in Tables 1 and 2. The default physical parameter values are $\beta = 0.1$, $c = 0.1$, $M = 1$, $Pr = 3$, $R = 3$, $\theta_w = 1.3$, $Nt = 0.3$, $Nb = 0.3$, $y = 0.1$, $Ec_x = 0.3$, $Ec_y = 0.3$, $Le = 10$.

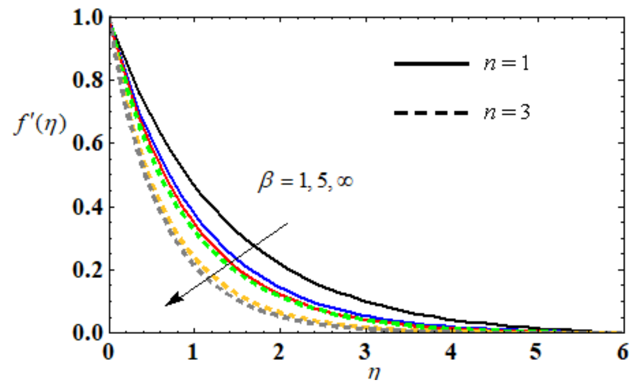

Fig. 2: Influence of β on $f'(\eta)$

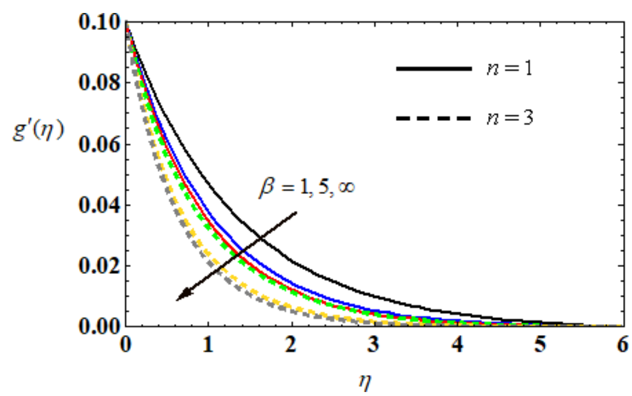

Fig. 3: Influence of β on $g'(\eta)$

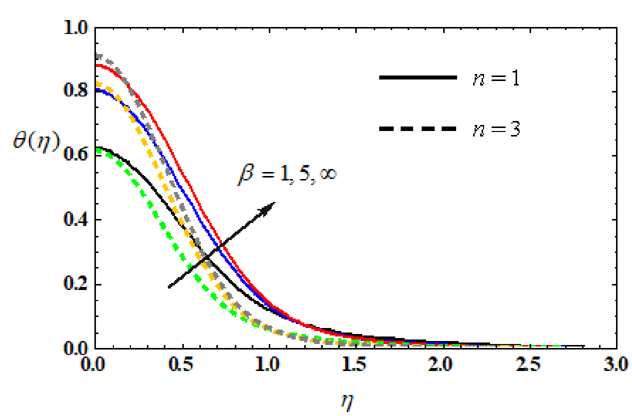

Fig. 4: Influence of β on $\theta(\eta)$

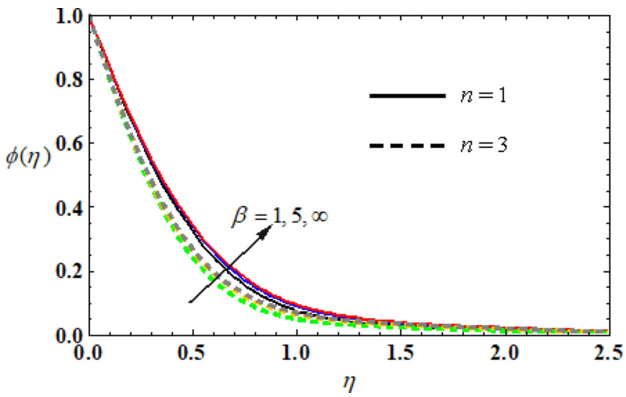

Fig. 5: Influence of β on $\phi(\eta)$

Figures 2–5 depict the variation of Casson parameter (β) on the non-dimensional velocity, temperature and concentration, respectively. It is observed from the figure that fluid flow decreases with an increase in Casson fluid parameter, while the temperature as well as concentration of the fluid decreases with an increase in Casson parameter β. The influence of the magnetic field on the velocity profiles, temperature and concentration is present respectively in Figures 6–9. The magnetic parameter is found to retard the velocity at all points of the flow field. The boundary layer thickness decreases with an increase in the magnetic parameter. It is obvious that the effect of increasing values of M results in a decreasing velocity profiles across the boundary layer. The temperature and concentration of the fluid increases due to rise in the values of magnetic parameter M.

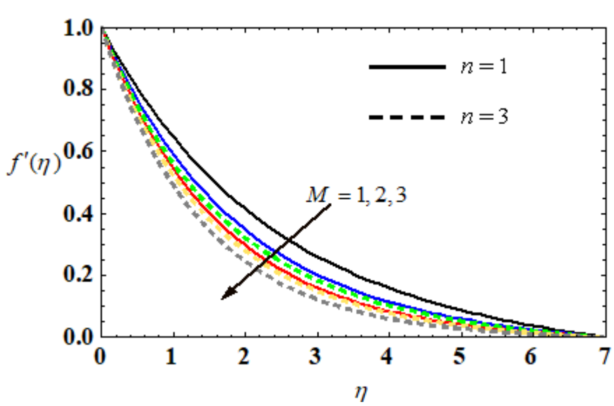

Fig. 6: Influence of M on $f'(\eta)$

It is evident from Figure 10 that increasing stretching ratio parameter reduces the velocity in x- direction as well as both the temperature and concentration, while the velocity in y-direction exhibits the opposite behaviour (Figure 11). Generally increasing the stretching parameter

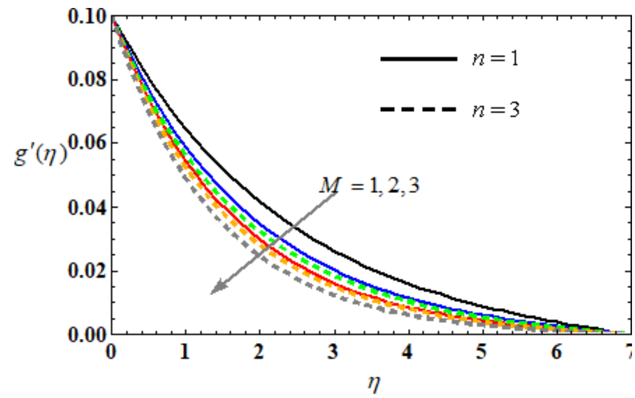

Fig. 7: Influence of M on $g'(\eta)$

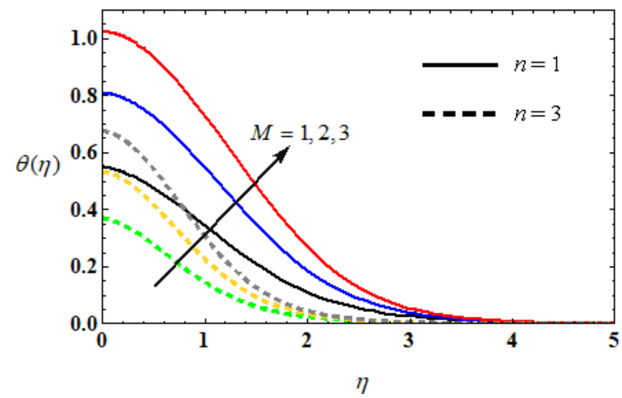

Fig. 8: Influence of M on $\theta(\eta)$

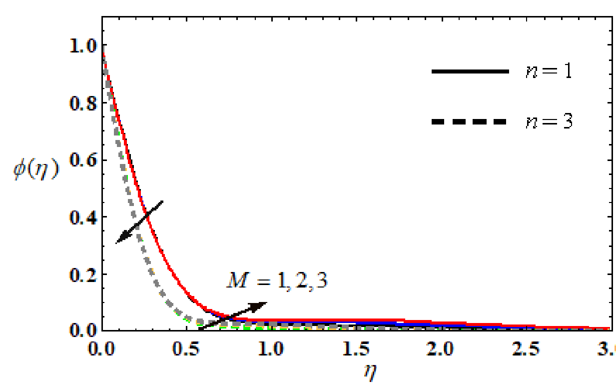

Fig. 9: Influence of M on $\theta(\eta)$

causes to increase the pressure on the fluid flow, due to this both the temperature and concentration decreases (Figures 12–13). It is obvious to see that increase in the radiation parameter results in decreasing the temperature of the fluid near the surface and increasing the temperature (Figure 14) for away from the surface within the boundary layer.

It is clear from these Figures 15 and 16 that both the temperature and concentration of the fluid increases with

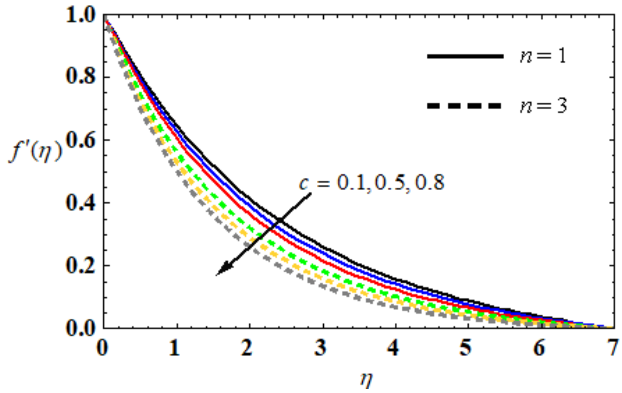

Fig. 10: Influence of c on $f'(\eta)$

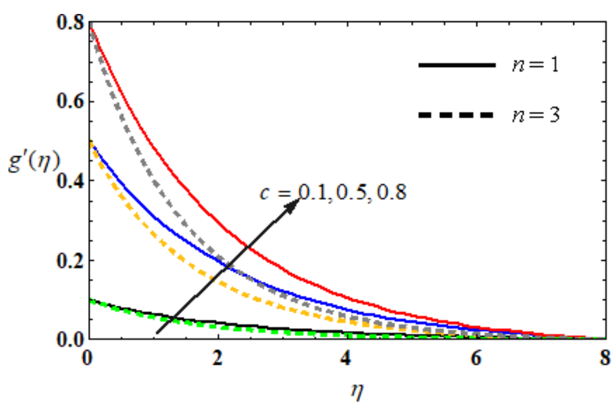

Fig. 11: Influence of c on $g'(\eta)$

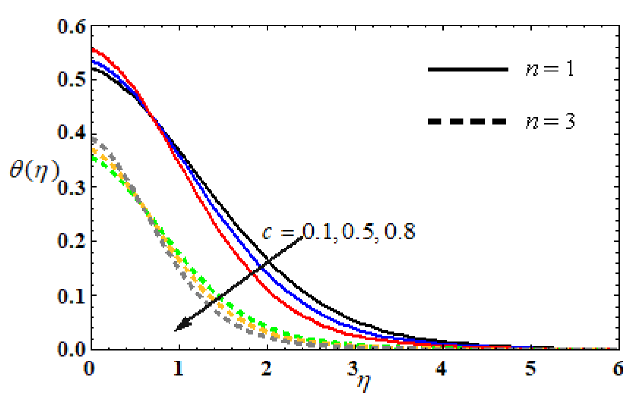

Fig. 12: Influence of c on $\theta(\eta)$

an increase in the value of Nt. This is because, if there is a temperature gradient in the flow region of the system, small particles tend to disperse faster in hotter region than in colder region. The collective effect of the differential dispersion of the particles is their apparent migration from hotter to colder regions. The result of the migration is accumulation of particles and higher particle concentrations in the colder regions of the particulate mixture.

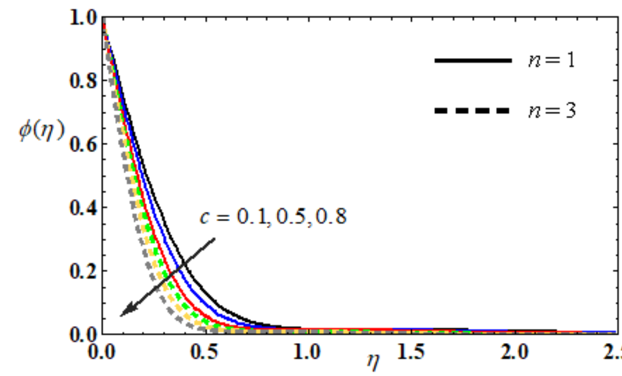

Fig. 13: Influence of c on $\phi(\eta)$

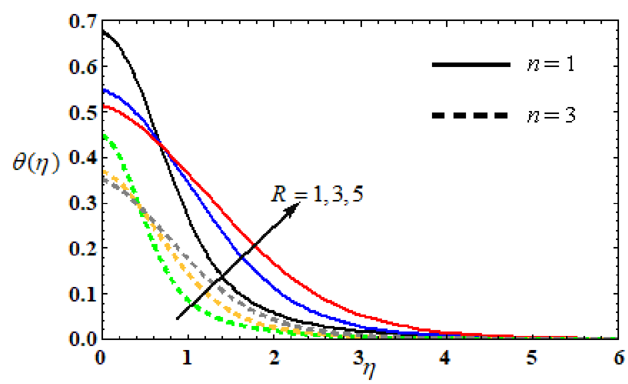

Fig. 14: Influence of R on $\theta(\eta)$

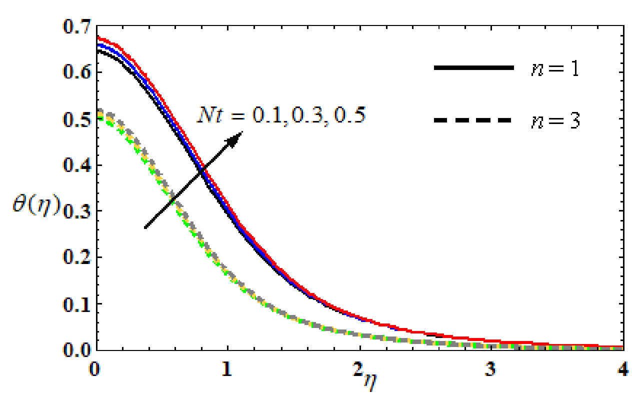

Fig. 15: Influence of Nt on $\theta(\eta)$

The effects of the Brownian motion parameter Nb on the temperature and nanoparticle volume fraction are shown in Figures 17–18, respectively. It is observed that the temperature of the fluid increases, while the concentration of the fluid decreases with an increase in the values of Nb. Physically, the Brownian motion of particles is simply the result of all the impulses of the fluid molecules on the surface of the particles. The fluid molecules have significantly high velocities and these velocities depend on the temperature of the fluid. The molecular velocities de-

Table 1: Comparison of $f''(0)$ and $g''(0)$ for different values of n, when $\beta \to \infty$, $c = 1$, $M = K = Ec_x = Ec_y = R = 0$.

n	$f''(0)$		$g''(0)$	
	Khan [17]	Present study	Khan [17]	Present study
1	-1.414214	-1.414215	-1.414214	-1.414215
3	-2.297186	-2.297184	-2.297186	-2.297184

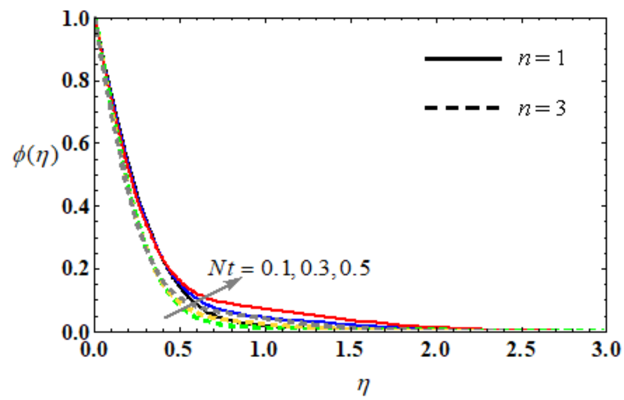

Fig. 16: Influence of Nt on $\phi(\eta)$

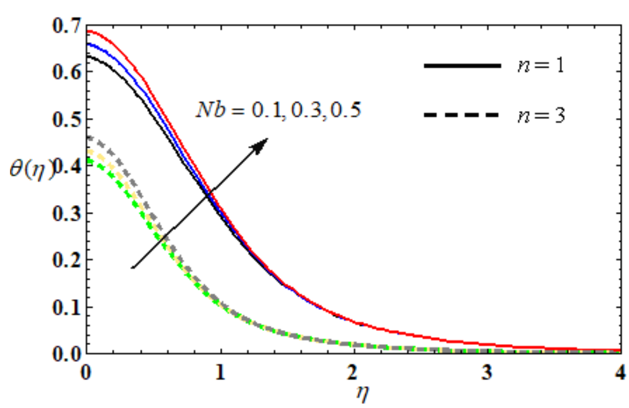

Fig. 17: Influence of Nb on $\theta(\eta)$

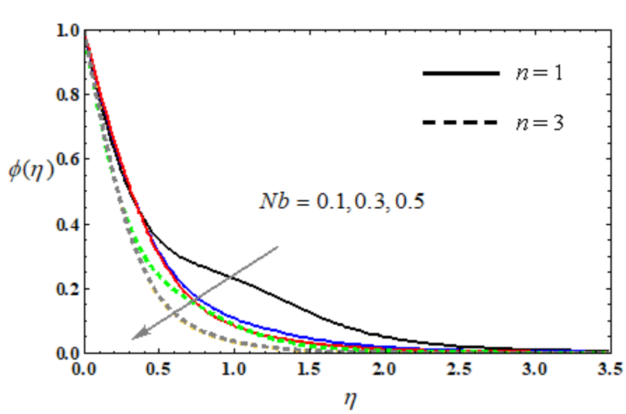

Fig. 18: Influence of Nb on $\phi(\eta)$

fine the temperature of a homogeneous fluid. The Brownian movement of particles is more intense at high temperatures. Molecular collision with particles are almost random and take place at the molecular time scales, which are of the order of femtoseconds and much shorter than the time scales of the particles.

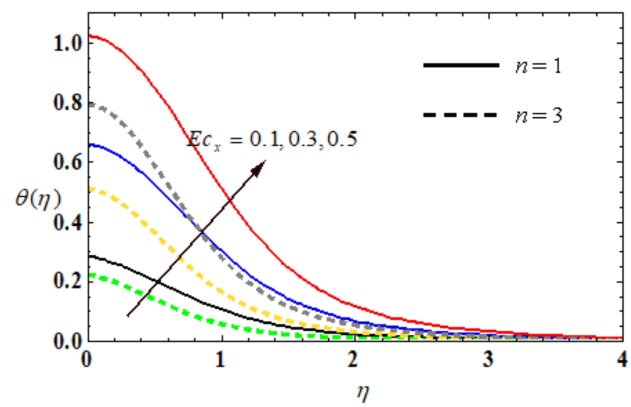

Fig. 19: Influence of Ec_x on $\theta(\eta)$

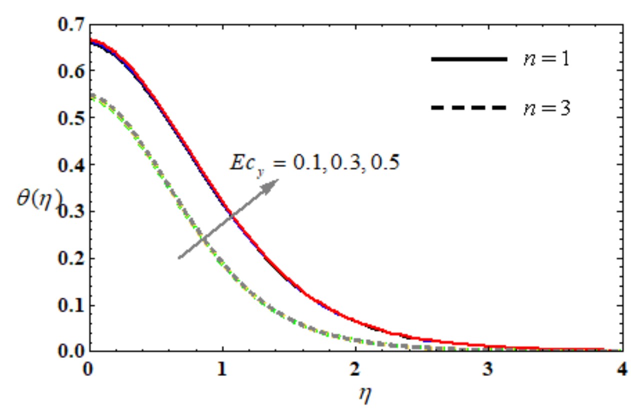

Fig. 20: Influence of Ec_y on $\theta(\eta)$

Figures 19 and 20 show the effect of Eckert number along x and y-direction on temperature profiles respectively. It is clear that the temperature of the fluid increases with an increase in Ec along both x and y directions. From Figure 21, it is evident that the concentration of the nanoparticle volume fraction decreases with an increase

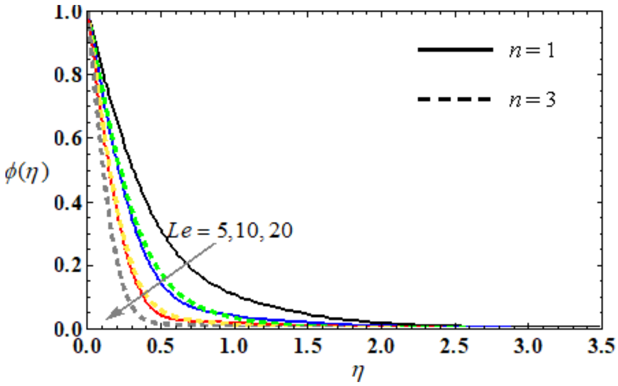

Fig. 21: Influence of Le on $\phi(\eta)$

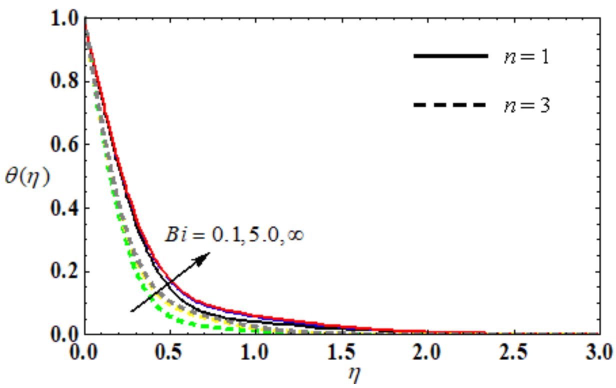

Fig. 22: Influence of Bi on $\theta(\eta)$

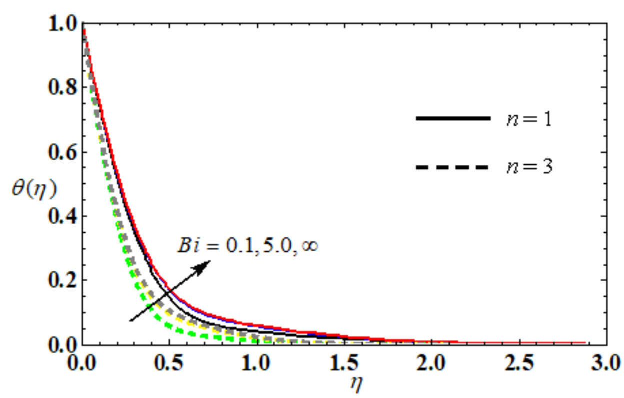

Fig. 23: Influence of Bi on $\theta(\eta)$

in Lewis number Le. This is probably due to the fact that mass transfer rate increases as Lewis number increases.

Biot number is defined as the convection at the surface of the body to the conduction with in the surface of the body. Temperature and concentration profiles versus Biot number Bi is depicted in Figures 22 and 23 respectively. It is evident that both the temperature as well as concentration of the fluid increases as increase in Biot number. This is fact that, Generally $Bi < 1$ represents the temperature field inside the surface is uniform while in case of $Bi >$ 1, the temperature field inside the surface is non uniform. The Biot number tends to infinity corresponds to the case of constant wall temperature. From Figures 2–23, a fascinating point to be noted that the velocity profiles along x-direction $f'(\eta)$, along y- direction $g'(\eta)$, temperature and concentration profiles having higher values for linear stretching sheet than the non-linear stretching sheet. This outcome is consistent with the results obtained by Khan *et al.* [17].

The variations of surface skin friction along x, y - axes, heat transfer rate and mass transfer rate for various physical parameters are shown in Table 3. It is observed that the friction coefficients along x,y- directions reduces with an increase in Casson parameter (β), magnetic parameter (M) and stretching ratio parameter (c). Also it is observed that, an increase in Casson parameter (β) and thermophoresis parameter (Nt) reduces in both the Nusselt number as well as Sherwood, while these predicts an opposite behaviour when increase in stretching ratio parameter. From Table 3, it is evident that the rate of heat transfer reduces while the rata of mass transfer rises with an increase in magnetic parameter (M), radiation parameter (R), Brownian motion parameter (Nb), Eckert number along x,y-directions and Lewis number (Le). Finally it is noticed that the upcoming values of Biot number (Bi) results in increase the Nusselt number while the mass transfer rate reduces.

4 Conclusions

The present study investigates the effects of Joule heating and viscous dissipation on radiating MHD three dimensional Casson nanofluid past a non-linear stretching porous sheet with convective boundary condition. Numerical results are presented with the help of graphs and tables. The conclusions are made as follows.

- Increase in the fluid parameter and porosity parameter reduces the friction factor, heat and mass transfer rates.
- Increase in the magnetic parameter reduces the friction factor and heat transfer rate while enhances the mass transfer rate.
- The rate of heat and mass transfer enhances while the friction factor reduces with an increase in stretching ratio parameter.
- An increase in thermophoresis, Brownian motion parameter and Eckert number through x,y-directions does not influence the friction factor.
- A rise in the value of radiation parameter, Brownian motion parameter, Eckert number along x,y-directions

Table 2: Comparison of $-\theta'(0)$ for different values of *Pr*, when $n = 3$, $c = 1$, $Nt = 0.5$, $Le = 10$, $M = K = Ec_x = Ec_y = R = 0$ and $\beta \to \infty$.

Pr	$-\theta'(0)$	
	Khan [17]	Present study
7	2.233474	2.233474
13	2.705553	2.705552
25	3.287790	3.287793
50	4.115189	4.115192
100	5.336667	5.336659

Table 3: Variations of $f''(0)$, $g''(0)$, $-\theta'(0)$ and $-\phi'(0)$ with different values of various parameters when $Pr = 3$, $\theta_w = 1.3$ and $n = 3$.

β	*M*	*C*	*R*	*Nt*	*Nb*	Ec_x	Ec_y	*Le*	*Bi*	$f''(0)$	$g''(0)$	$-\theta'(0)$	$-\phi'(0)$
0.1	1	0.1	3	0.3	0.3	0.3	0.3	10	0.1	(0.594977	(0.0594977	0.0582366	3.59657
5										(1.79131	(0.179131	0.0107873	3.50525
∞	1									(1.95254	(0.195254	0.0022835	3.48494
	2									(0.66596	(0.0665069	0.0385044	3.65990
	3	0.1								(0730605	(0.0730605	0.0212727	3.70800
		0.5								(0.662813	(0.331407	0.0555671	4.01710
		0.8	1							(0.718558	(0.574847	0.0521805	4.62901
			2							(0.594977	(0.0594977	0.0628559	3.53682
			3							(0.594977	(0.0594977	0.0644431	3.51336
			1	0.1						(0.594977	(0.0594977	0.063612	3.46625
				0.3						(0.594977	(0.0594977	0.0633829	3.47928
				0.5						(0.594977	(0.0594977	0.0631506	3.49253
				0.3	0.1					(0.594977	(0.0594977	0.0641781	3.51222
					0.3					(0.594977	(0.0594977	0.0633829	3.47928
					0.5					(0.594977	(0.0594977	0.625747	3.47270
					0.3	0.1				(0.594977	(0.0594977	0.0749489	3.46075
						0.3				(0.594977	(0.0594977	0.0633829	3.47928
						0.5				(0.594977	(0.0594977	0.0524293	3.49534
						0.3	0.1			(0.594977	(0.0594977	0.0634954	3.47911
							0.3			(0.594977	(0.0594977	0.0633829	3.47928
							0.5	5		(0.594977	(0.0594977	0.0632705	3.47946
								10		(0.594977	(0.0594977	0.0633829	3.47928
								20	0.1	(0.594977	(0.0594977	0.0634253	4.96431
									1	(0.594977	(0.0594977	0.189668	4.95582
									2	(0.594977	(0.0594977	0.207464	4.95513

and Lewis number reduces the heat transfer rate and enhances the mass transfer rate.

- An increase in Biot number increases the rate of heat transfer and decreases the mass transfer rate.

Conflict of Interest : We assure this paper has no conflict of interest

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