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Influence of predator standby capacity, harvesting and noise on a two patchy aquatic delayed eco system with migration of prey

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ABSTRACT

In this article, we analyse the dynamical behaviour of a prey predator fishery model. The model is studied and analysed on the basis of harvesting of prey species in an environment which consists of two sectors. Mathematically, we have analysed the boundedness of the solution and the local stability of positive interior equilibrium point. The time lag in terms of delay parameter corresponds to the predator gestation period. The occurrence of Hopf-bifurcation of the proposed model is shown at the positive equilibrium point by considering delay as a bifurcation parameter. We observe that the system exhibits periodic oscillations due to an increase of the delay parameter. Furthermore, we examined the impact of noise on the model system using the Fourier transform technique. Finally, we verified our analytical findings by means of graphical illustrations.

1. Introduction

Now-a-days, the world population is growing substantially, and a responsibility is to provide food to various species of the natural environment. There are several food resources in the natural environment and one of them concerns fisheries. The system of fisheries comprises bionomic and social mechanisms, delivering distinguishing view points on the fishery. Numerous species have become extinct and other species are approaching this, owing to several causes such as predation, biological contamination, uncritical garnering, mishandling of normal reserves, and so on. To defend species from extinction, refining of circumstances in natural environments is necessary, thereby reducing the interactions among species with other agents. Then the growth of species and protected populations may increase without restriction. The elementary hypothetical implementation in several surveys [1–3] is the system of Lotka-Volterra equations for a prey with population density. In the truancy of the predator populace, the prey populace develops exponentially, and in the truancy of prey, the predator population decays exponentially.

Investigators have proceeded to examine the stability mediating the effects of a long list of such processes [4–8], and studied a more complex populace with special resources in a two patchy environment. They show and develop a steadiness analysis in view of local and global stable states of complex models with an induced Holling type interactive environment. A few researchers [9–18] have contributed some qualitative role plays, which inspired us to develop modified models including harvesting strategies with and without feedback controls, and similarly with and without stochasticity. Kunal Chakraborty et al. [22], studied the dynamics, bifurcation and control of the biological and economical controls of the prey predator system with time delay, but did not develop a stochastic analysis, which inspires us to extend the work based on environmental dynamics in a different dimension.

Some researchers have worked on permanence or stability of positive equilibria or positive periodic solutions under limited conditions, which are sufficient to attain the goal of simulating an appropriate bio-ecological system. In these simulations, most of the systems were built on the assumption that intrinsic growth rates are always positive, which translates biologically to every organism developing under a suitable environment without any disturbances.

Whereas, looking into an actual real scenario, more organisms are under the influence of environmental disturbances. Environmental factors which can disturb may be manmade - human activities and waste from industries which can pollute the environment including the atmosphere, rivers, soil and water. The natural resources and environment are polluted by many means, leading to dispersals and resulting in a breakup of the cultivating and fishing area into patches. In some of these patches, sometimes even in every patch, species will become extinct without contributions from other patches. In the present paper, our interest is to study the species population densities by considering a two patchy environment with a time delay in gestation.

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Thus, by applying a strategy of gradual growth in the harvesting coefficient of the prey population, it does not affect the death rate of the predator populace extensively when developing simulative strategic modelling. This sort simulation can be achieved with the help of a strong elemental constraint out of several regulatory mechanisms and methodologies, like the two patchy environment, stochastic theory, feedback control theory etc. These are the most popular, and are possible regulatory mechanisms and techniques studied and developed by many authors [19–27] for fish populations under a dynamic environment. Considering some creative strategies like marine reserve zones, marine patchy zones and a two patchy environment are treated as significant techniques, which inspires us to simulate and develop a model for organisms under a two patchy environment with some strategies like harvesting, time delay and noise introduction.

However, as far as bio-ecological simulations are concerned in view of predator-prey developments and interactions, some research studies have treated population with an enhanced harvesting or delay effort and were developed using a trendy and suitable harvesting, spatial dynamics and time delay. In the model with harvesting, some contributions focused on population to solve economic problems and meet the expectations of the system study. An induced time delayed model and model without time delay in population are considered by some creative studies [12–17,20–22], and inspires us to consider the current model in a two patchy environment with a time delay in gestation.

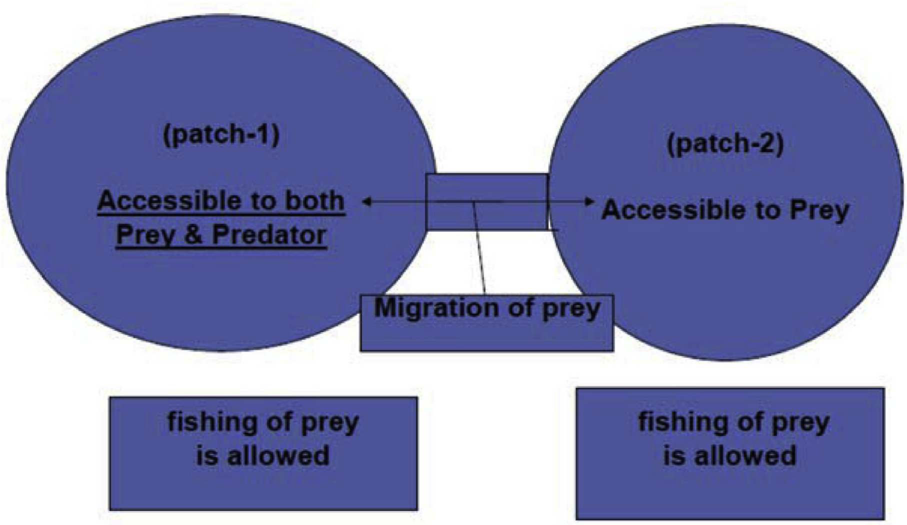
The above analysis suggests that a considerable effort may be needed to simulate a model under the influence of environmental factors and noise. Many bio-ecological models have been simulated and implemented by introducing new strategies with and without stochastic theories developed and verified with appropriate results, which provides the inspiration for many researchers [9–11,19–22,26]. Manju Agarwal et al. [23,24]. considered and developed a two patchy environment without a noise factor. They also checked the effect of non-selective harvesting on the prey-predator system in an unreserved zone, and stability analysis in terms of both local and global factors. But in view of the authors' knowledge and a literature search, there is no prior study which considers an environmental factor noise on a two patchy environment with a selective timing strategy.

Keeping these in mind, we considered a prey predator model with migration of prey, delay and noise, in which we analysed the effect of predator standby capacity, harvesting and noise on a two patchy aquatic delayed eco system. The current article is different from other work in view of the environmental factor dynamics and bifurcational dynamics. Therefore, the proposed model is significantly different from other models using differential algebraic equations, and requires separate investigation.

Keeping these in mind, the current model is proposed and developed with strategies such as migration induced prey, delay and noise, in which we analysed the effect of predator standby capacity, harvesting and noise on a two patchy aquatic delayed ecosystem. The current article is also different other work in view of environmental factor dynamics and bifurcation dynamics. Thus the proposed model is potentially interesting and attractive to researchers as a new combinational and updated model using a stochastic approach is strategically defined. The total segments of the paper are framed as seg1, seg2, seg3 … seg8. In seg 2, we formulate a mathematical model with assumption. In seg 3 boundedness of solution of a deterministic model is discussed. Seg 4 deals with the existence of equilibrium points with feasible condition. In seg 5, local stability analysis of positive interior equilibrium points is discussed. Seg 6 deals with delay analysis of positive interior equilibrium point. In seg 7, we computed the population intensity of fluctuation due to incorporation of noise which leads to chaos. Numerical simulation of the proposed model is presented in seg 8. The discussion and conclusion are presented in the last seg.

2. Mathematical model and its qualitative role play

In seg 2, we simulate a structure with migration of prey populace in a two patchy environment. Both patch-1 and patch-2 are free fishing sectors for prey but not predator. We assumed that each sector is to be homogeneous. The growth of the prey populace in the truancy of predator populace in view of growth is logistic. Partial harvesting is permitted in the patches of prey population, and the predator population are not participated in harvesting. A pictorial representation of this biological phenomenon is in the Fig. 100.



By considering these in view, the dynamics of the system may be governed by the following equations.

$$p_1'(t) = r_1p_1\left(1 - \frac{p_1}{\theta K}\right) - \frac{m_1p_1}{\theta K} + \frac{m_2p_2}{(1-\theta)K} - \frac{a_1p_1p_3}{\beta + p_1} - q_1E_1p_1 \tag{1}$$

$$p_2'(t) = r_2p_2\left(1 - \frac{p_2}{(1-\theta)K}\right) + \frac{m_1p_1}{\theta K} - \frac{m_2p_2}{(1-\theta)K} - q_2E_2p_2 \tag{2}$$

$$p_3'(t) = \frac{a_2p_1p_3}{\beta + p_1} - \eta p_3^2 - dp_3 \tag{3}$$

with initial densities

$$p_1(0) \geq 0,\ p_2(0) \geq 0,\ p_3(0) \geq 0 \tag{4}$$

Here all parameters are assumed to be positive. Let p_1, p_3 denote densities of prey and predator population respectively, and p_2 denote density of the prey population inner part of the reserved zone, where there is an absence of predation considered at time t. In model (1–3), r_1, r_2 denotes intrinsic growth rates of prey population inside zone-1 and zone-2 respectively. K denotes the environmental carrying capacity of the total prey population. η is the self-limitation on predator species. θ is the part of an area under protection. m_1 and m_2 are migration rates of the prey in Zone-1 and Zone-2 respectively. a_1 is the capturing rate of predators and a_2 is the conversion rate of predators. β is the half-saturating constant. d is the death rate of the predator population $q_i(i = 1,2)$ are the constant catch ability coefficients of prey species in zone-1 and zone-2 respectively and $E_i(i = 1,2)$ is the combined effort applied to harvest prey species in zone-1 and zone-2. Throughout our analysis, we assume that $r_1 - q_1E_1 > 0$; $r_2 - q_2E_2 > 0$.

3. Boundedness of the solution

Theorem. *All the non-negative solutions of the model system* (1–3) *that initiate in* $\Re^3_+$ *are uniformly bounded.*

Proof. Let $p_1(t)$, $p_2(t)$, $p_3(t)$ be any solution of the system (1)–(3) *with non-negative initial condition such that*

$$w(t) = p_1(t) + p_2(t) + p_3(t) \tag{5}$$

Differentiate (5) with respect to t, we obtain $w'(t) = p_1'(t) + p_2'(t) + p_3'(t)$

Then

$$w'(t) = (r_1 - q_1E_1)p_1 - \frac{r_1p_1^2}{\theta K} + (r_2 - q_2E_2)p_2 - \frac{r_2p_2^2}{(1-\theta)K} - \frac{(a_1 - a_2)p_1p_3}{\beta + p_1} - \eta p_3^2 - dp_3$$

Since from a biological point of view, the conversion rate from prey to predator cannot exceed the predators maximum attack rate, hence we have $a_1 \geq a_2$, and obtain

$w'(t) + \xi w \leq r_1\theta K + r_2(1-\theta)K - \eta = \mu$ where $\xi = \min\{r_1 - q_1E_1, r_2 - q_2E_2, d\}$.

Applying a Lemma on differential inequalities, we obtain $0 \leq w(p_1,p_2,p_3) \leq (\mu/\xi)(1 - e^{-\xi t}) + (w(p_1(0), p_2(0), p_3(0))/e^{\xi t})$ and for $t \to \infty$

We have $0 \leq w(p_1, p_2, p_3) \leq (\mu/\xi)$.

Therefore the solution of system (1–3) is represented as

$$\Gamma = \{(p_1, p_2, p_3) \in R^3_+ : 0 \leq w \leq (\mu/\xi) + \varepsilon,\ \forall\ \varepsilon > 0\} \tag{6}$$

This completes the Proof.

4. Existence of equilibrium points with feasible condition

For the system (1)–(3) we have the equilibrium points as

(i) $E_0(0,0,0)$ which always exists.
(ii) $E_1(\overline{p}_1, 0,0)$. where $\overline{p}_1 = \frac{\theta K}{r_1}\left[r_1 - q_1E_1 - \frac{m_1}{\theta K}\right]$.

For $\overline{p}_1$ to be positive, we have $r_1 - q_1E_1 > \dfrac{m_1}{\theta K}$ (7)

(iii) $E_2(p_1^\phi, 0, p_3^\phi)$where $p_3^\phi = \frac{1}{\eta}\left[\frac{a_2p_1^\phi}{\beta + p_1^\phi} - d\right]$.

For p_3^ϕ to be positive,

$$\text{we must have } \frac{a_2p_1^\phi}{\beta + p_1^\phi} > d, \tag{8}$$

From equation (1) we have $A(p_1^\phi)^2 + Bp_1^\phi + C = 0$

Where $A = r_1 > 0$; $B = r_1\beta - \theta\left(r_1 - q_1E_1 - \frac{m_1}{\theta K}\right)$. ;

$C = \theta K\beta\left(r_1 - q_1E_1 - \dfrac{m_1}{\theta K}\right) - a_1p_3^\phi\theta K.$

For p_1^ϕ to be positive and unique, we must have

$$r_1\beta < \theta\left(r_1 - q_1E_1 - \frac{m_1}{\theta K}\right) \tag{9}$$

$$\text{and } \theta K\beta\left(r_1 - q_1E_1 - \frac{m_1}{\theta K}\right) < a_1p_3^\phi\theta K \tag{10}$$

(iv) $E_3(p_1^*, p_2^*, p_3^*)$where
$p_3^* = \frac{1}{\eta}\left[\frac{a_2p_1^*}{\beta + p_1^*} - d\right]$, $p_2^* = \frac{(1-\theta)K}{(r_2 + m_2)}\left[(r_2 - q_2E_2) + \frac{m_1p_1^*}{\theta K}\right]$

For p_3^*, p_2^* to be positive we must have $\frac{a_2p_1^*}{\beta + p_1^*} > d_1^*$ (11) and from equation (1), we have $A_1(p_1^\phi)^2 + B_1p_1^\phi + C_1 = 0$ and for p_1^* to be positive and unique, we must have $r_1 = \frac{\theta KL}{\beta}$ and $p_3^* = \frac{\beta L}{a_1}$ where

$$L = r_1 - q_1E_1 - \frac{m_1}{\theta K} - \frac{m_2p_2^*}{(1 - \theta K)} \tag{12}$$

5. Stability analysis for positive interior steady state point

Theorem. *The positive interior steady state* $E_3(p_1^*, p_2^*, p_3^*)$ *is asymptotically locally stable if it satisfies the condition* $A_1 > 0, A_2 > 0, A_1A_2 - A_3 > 0$

Proof. *Let the Jacobian matrix of the system* (1–3) *evaluated at the equilibrium point* E_3 *be* $J(E_3(p_1^*, p_2^*, p_3^*)) = (a_{ij})_{3X3}$, *where*

$$a_{11} = \frac{-r_1p_1^*}{\theta K} - \frac{m_2p_2^*}{p_1^*(1-\theta)K} + \frac{a_1p_1^*p_*^3}{(\beta + p_1^*)^2},\ a_{12} = \frac{m_2}{(1-\theta)K},$$

$$a_{13} = \frac{-a_1p_1^*}{(\beta + p_1^*)^2},$$

$$a_{21} = \frac{m_1}{\theta K},\ a_{22} = \frac{-r_2p_2^*}{(1-\theta)K} - \frac{m_1p_1^*}{p_2^*\theta K},\ a_{23} = 0,\ a_{31} = \frac{a_2\beta p_3^*}{(\beta + p_1^*)^2},$$

$$a_{32} = 0,\ a_{33} = -\eta p_3^*$$

Thus the characteristic equation of the Jacobian matrix at E_3 is obtained as

$$\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = 0 \tag{13}$$

where
$A_1 = -(a_{11} + a_{22} + a_{33})$, $A_2 = a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{12}a_{21} - a_{13}a_{31}$,
$A_3 = a_{12}a_{21}a_{33} + a_{13}a_{22}a_{31} - a_{11}a_{22}a_{33}$

Using Routh-Hurwitz criteria, it follows that all eigen values of the characteristic equation (13) have negative real parts if and only if;

$$A_1 > 0,\ A_2 > 0,\ A_1A_2 - A_3 > 0 \tag{14}$$

6. Delay analysis

In this segment 6 we analyse the model system (1–3) with an elementary constraint of time τ (represents the delay in predator population response function). Then the proposed structure (1–3) is shaped as

$$\frac{dp_1}{dt} = r_1p_1\left(1 - \frac{p_1}{\theta K}\right) - \frac{m_1p_1}{\theta K} + \frac{m_2p_2}{(1-\theta)K} - \frac{a_1p_1p_3}{\beta + p_1} - q_1E_1p_1 \tag{15}$$

$$\frac{dp_2}{dt} = r_2p_2\left(1 - \frac{p_2}{(1-\theta)K}\right) + \frac{m_1p_1}{\theta K} - \frac{m_2p_2}{(1-\theta)K} - q_2E_2p_2 \tag{16}$$

$$\frac{dp_3}{dt} = \frac{a_2 p_1(t-\tau) p_3}{\beta + p_1(t-\tau)} - \eta p_3^2 - dp_3 \tag{17}$$

with the initial densities $p_1(\theta) \geq 0,\ p_2(\theta) \geq 0,\ p_3(\theta) \geq 0,\ \theta \in (-\tau, 0),\ \tau \neq 0$.

The remarkable note of this segment is to discuss the behaviour analysis of $E_3(p_1^*, p_2^*, p_3^*)$ in the presence of discrete delay ($\tau \neq 0$). Now to prove the stability behaviour of $E_3(p_1^*, p_2^*, p_3^*)$ for the system (15–17), first we linearize the system (15–17) by using the following transformation

$p_1(t) = p_1^* + x_1(t),\ p_2(t) = p_2^* + y_1(t),\ p_3(t) = p_3^* + z_1(t)$. The linear system is given by

$$\dot{x_1}(t) = a_{11}x_1(t) + a_{12}y_1(t) + a_{13}z_1(t),\ \ \dot{y_1}(t) = a_{21}x_1(t) + a_{22}y_1(t) + a_{23}z_1(t)$$

$$\dot{z_1}(t) = c_{31}x_1(t-\tau) + a_{22}y_1(t) + a_{33}z_1(t)$$

where

$$a_{11} = \frac{-r_1 p_1^*}{\theta K} - \frac{m_2 p_2^*}{p_1^*(1-\theta)K} + \frac{a_1 p_1^* p_3^*}{(\beta + p_1^*)^2},\ a_{12} = \frac{m_2}{(1-\theta)K},\ a_{13} = \frac{-a_1 p_1^*}{(\beta + p_1^*)^2},\ a_{21} = \frac{m_1}{\theta K}$$

$$a_{22} = \frac{-r_2 p_2^*}{(1-\theta)K} - \frac{m_1 p_1^*}{p_2^* \theta K},\ a_{23} = 0,\ c_{31} = \frac{a_2 \beta p_3^*}{(\beta + p_1^*)^2},\ a_{22} = 0,\ a_{33} = -\eta p_3^*$$

We look for solution of the model (15–17) of the form $A(\tau) = \rho e^{-\lambda\tau},\ \rho \neq 0$, which leads to the characteristic equation

$$\Delta(\lambda, \tau) = (\lambda^3 + u_1\lambda^2 + u_2\lambda + u_3) + (u_4\lambda + u_5)e^{-\lambda\tau} = 0 \tag{18}$$

where $u_1 = -a_{11} - a_{22} - a_{33}$, $u_2 = a_{11}a_{22} - a_{21}a_{12} + a_{11}a_{33} + a_{33}a_{22}$, $u_3 = a_{12}a_{21}a_{33} - a_{11}a_{22}a_{33}$, $u_4 = -a_{13}c_{31}$, $u_5 = a_{13}a_{22}c_{31}$

The eigenvalues are the roots of the characteristic equation (18) of the system (15–17) that has infinitely many solutions. We wish to find a periodic solution of the system (15–17); for the periodic solution eigenvalues will be purely imaginary.

Substituting $\lambda = i\omega$ in equation (18) we get $[-i\omega^3 - u_1\omega^2 + iu_2\omega + u_3] + [iu_4\omega + u_5]e^{-i\omega\tau} = 0$. Comparing real and imaginary parts,

We obtain $u_1\omega^2 - u_3 = (u_5 \cos\omega\tau + \omega\ u_4 \sin\omega\tau)$ and $u_2\ \omega - \omega^3 = -\omega\ u_4 \cos\omega\tau + (u_5 \sin\ \omega\tau)$.

Squaring and adding we obtain,

$$\omega^6 + S_1\omega^4 + S_2\omega^2 + S_3 = 0 \tag{19}$$

where $S_1 = u_1^2 - 2u_2$, $S_2 = u_2^2 - 2u_3u_5 - u_4^2$, $S_3 = u_3^2 - u_5^2$. Substituting $\omega^2 = \delta$ in (19) we obtain,

$$f(\delta) = \delta^3 + S_1\delta^2 + S_2\delta + S_3 = 0 \tag{20}$$

Now equation (20) will be positive if

$$S_1 > 0,\ \ S_3 < 0 \tag{21}$$

By the Descartes rule of sign, the cubic equation (20) has at least one positive root. Consequently the stability criteria of the system for $\tau = 0$, will not necessarily ensure the stability of system for $\tau \neq 0$.

The critical value of delay is given as

$$\cos\omega\tau = \frac{(\omega^4(u_4) - u_5u_3) + \omega^2(u_1u_5 - u_2u_4)}{(u_5^2 + u_4^2\omega^2)}.$$

So corresponding to $\lambda = i\omega_0$, there exists τ_0^* such that

$$\tau^* = \frac{1}{\omega_0}\left[\cos^{-1}\left[\frac{(\omega^4(u_4) - u_5u_3) + \omega^2(u_1u_5 - u_2u_4)}{(u_5^2 + u_4^2\omega^2)}\right]\right] + \frac{2n\pi}{\omega_0},$$

$n = 0,1,2,3.......$

6.1. Hopf bifurcation role play

We observe that the conditions for Hopf bifurcation are satisfied yielding the required periodic solution, that is $\left[\frac{d}{d\tau}(\mathrm{Re}\,\lambda)\right]_{\tau=\tau_0} \neq 0$. This signifies that there exists at least one eigenvalue with a positive real part for $\tau > \tau^*$. Now we show the existence of Hopf bifurcation near $E_2(p_1^*, p_2^*, p_3^*)$ by taking τas bifurcating parameter.

Differentiating equation (18) with respect to τ

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{3\lambda^2 + 2u_1\lambda + u_2}{\lambda(u_4\lambda + u_5)e^{-\lambda\tau}} + \frac{u_4}{\lambda(u_4\lambda + u_5)} - \frac{\tau}{\lambda}$$
$$= \frac{2\lambda^3 + u_1\lambda^2 - u_3 - (u_4\lambda + u_5)e^{-\lambda\tau}}{\lambda^2(u_4\lambda + u_5)e^{-\lambda\tau}} + \frac{u_4\lambda}{\lambda^2(u_4\lambda + u_5)} - \frac{\tau}{\lambda}$$
$$= \frac{(2\lambda^3 + u_1\lambda^2 - u_3)}{-\lambda^2(\lambda^3 + u_1\lambda^2 + u_2\lambda + u_3)} + \frac{-u_5}{\lambda^2(u_4\lambda + u_5)} - \frac{\tau}{\lambda}$$

Taking $\lambda = i\omega_0$ in the above equation, we obtain

$$\left(\frac{d\lambda}{d\tau}\right)^{-1}_{\lambda=i\omega_0} = \frac{2(i\omega_0)^3 + u_1(i\omega_0)^2 - u_3}{-(i\omega_0)^2((i\omega_0)^3 + u_1(i\omega_0)^2 + u_2(i\omega_0) + u_3)} + \frac{-u_5}{(i\omega_0)^2(u_4(i\omega_0) + u_5)} + \frac{i\tau}{\omega_0}$$
$$= \left[\frac{-[(u_1\omega_0^2) + 2i\omega_0^3 + u_3]}{\omega_0^2[(u_3 - u_1\omega_0^2) + i(u_2w_0 - w_0^3)}\cdot\frac{(u_3 - u_1\omega_0^2) - i(u_2w_0 - w_0^3)}{(u_3 - u_1\omega_0^2) - i(u_2w_0 - w_0^3)}\right]$$
$$+ \left[\frac{u_5}{\omega_0^2((u_5 + iu_4\omega_0)}\cdot\frac{(u_5 - iu_4\omega_0}{(u_5 - iu_4\omega_0}\right] + \frac{\tau i}{w_0}$$

$$\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}_{\lambda=iw_0} = \left[\frac{2\omega_0^3(\omega_0^3 - u_2\omega_0) - ((u_1\omega_0^2)^2 - u_3^2)}{[\omega_0^2(u_3 - u_1\omega_0^2)^2 + (u_2\omega_0 - \omega_0^3)^2]}\right] + \frac{(u_5)^2}{\omega_0^2[(u_5)^2 + u_4^2\omega_0^2]}$$

Thus we obtain $Re\left(\frac{d\lambda}{d\tau}\right)^{-1}_{\lambda=i\omega_0} > 0$. Therefore the transversity condition holds and hence Hopf bifurcation occurs at $\tau = \tau^*$. This signifies that there exists at least or equal value with positive real part for $\tau > \tau^*$.

Theorem. *If E_3 exists with the condition* (11-12) *and $\delta = \omega_0^2$ is a positive root of* (19), *then there exists a $\tau = \tau^*$ such that* (*i*) *E_3 is locally asymptotically stable for $0 \leq \tau < \tau^*$;* (*ii*) *E_3 is unstable for $\tau > \tau^*$;* (*iii*) *The system* (15–17) *undergoes a Hopf –bifurcation around E_2 at $\tau = \tau^*$, $\tau^* = \min h(\omega_0)$. Where $h(\omega_0) = \tau_0^* = \frac{1}{\omega_0}\left[\cos^{-1}\left[\frac{(\omega^4(u_4) - u_5u_3) + \omega^2(u_1u_5 - u_2u_4)}{(u_5^2 + u_4^2\omega^2)}\right]\right] + \frac{2n\pi}{\omega_0}$, n =0,1,2,3......., and the minimum is taken over all positive ω_0, such that $\delta = \omega_0^2$ is a solution of* (19).

7. Stochasticity role play

This segment is mainly focused on the stochastic approach for the proposed structure, and determines the impact of random noise in view of steadiness and other dynamics. The stochastic approach for the proposed structure [1–3] is defined as

$$p_1'(t) = r_1 p_1\left(1 - \frac{p_1}{\theta k_1}\right) - \frac{m_1 p_1}{\theta k_1} + \frac{m_2 p_2}{(1-\theta)k_1} - \frac{\alpha_1 p_1 p_3}{\beta_1 + p_1} - q_1 E_1 p_1 + \xi_1\psi_1(t)$$

$$p_2'(t) = r_2 p_2\left(1 - \frac{p_2}{(1-\theta)k_1}\right) + \frac{m_1 p_1}{\theta k_1} - \frac{m_2 p_2}{(1-\theta)k_1} - q_2 E_2 p_2 + \xi_2\psi_2(t) \tag{22}$$

$$p_3'(t) = -\eta p_3^2 + \frac{\alpha_2 p_1 p_3}{\beta_1 + p_1} - dp_3 + \xi_3\psi_3(t)$$

Where ξ_1, ξ_2, ξ_3 are the real constants and $\psi_i(t) = [\psi_1(t), \psi_2(t), \psi_3(t)]$ is a three dimensional Gaussian white noise process satisfying $E(\psi_i(t)) = 0;\ i = 1, 2, 3;\ E[\psi_i(t)\psi_j(t)] = \delta_{ij}\delta(t - t');\ i = j = 1, 2, 3$ where δ_{ij} is the Kronecker symbol and δ is the δ-Dirac function. All other parameters have their usual meanings (section 1)

$$\text{Let } p_1(t) = u_1(t) + S^*;\ p_2(t) = u_2(t) + P^*;\ p_3(t) = u_3(t) + T^*;\ \text{Then } p_1'(t) = u_1'(t);\ p_2'(t) = u_2'(t);\ p_3'(t) = u_3'(t) \tag{23}$$

Using (23), the linear parts of (22)

$$u_1'(t) = -\frac{r_1}{\theta K}u_1(t)S^* - \alpha_1 u_3(t)S^* + \xi_1\psi_1(t)$$
$$u_2'(t) = -\frac{r_2}{(1-\theta)k_1}u_2(t)P^* + \xi_2\psi_2(t) \quad (24)$$
$$u_3'(t) = -\eta u_3(t)T^* + \alpha_2 u_1(t)T^* + \xi_3\psi_3(t)$$

Taking the Fourier transform on both sides of (24) we get,

$$\left(i\omega + \frac{r_1 S^*}{\theta K}\right)\tilde{u}_1(\omega) + \alpha_1 S^*\tilde{u}_3(\omega) = \xi_1\tilde{\psi}_1(\omega)$$
$$\left(i\omega + \frac{r_2}{(1-\theta)K}P^*\right)\tilde{u}_2(\omega) = \xi_2\tilde{\psi}_2(\omega) \quad (25)$$
$$(i\omega + \eta T^*)\tilde{u}_3(\omega) - \alpha_2 T^*\tilde{u}_1(\omega) = \xi_3\tilde{\psi}_3(\omega)$$

The matrix form of (25) is

$$M(\omega)\tilde{u}(\omega) = \tilde{\psi}(\omega) \quad (26)$$

where

$$M(\omega) = \begin{pmatrix} A_1(\omega) & B_1(\omega) & C_1(\omega) \\ A_2(\omega) & B_2(\omega) & C_2(\omega) \\ A_3(\omega) & B_3(\omega) & C_3(\omega) \end{pmatrix}; \ \tilde{u}(\omega) = \begin{bmatrix} \tilde{u}_1(\omega) \\ \tilde{u}_2(\omega) \\ \tilde{u}_3(\omega) \end{bmatrix}; \ \tilde{\psi}(\omega) = \begin{bmatrix} \xi_1\tilde{\psi}_1(\omega) \\ \xi_2\tilde{\psi}_2(\omega) \\ \xi_3\tilde{\psi}_3(\omega) \end{bmatrix};$$

where $A_1(\omega) = i\omega + \frac{r_1 S^*}{\theta K}$, $B_1(\omega) = 0$, $C_1(\omega) = \alpha_1 S^*$,

$$A_2(\omega) = 0, \ B_2(\omega) = i\omega + \frac{r_2 P^*}{(1-\theta)K}, \ C_2(\omega) = 0,$$

$$A_3(\omega) = -\alpha_2 T^*, \ B_3(\omega) = 0; \ C_3(\omega) = i\omega + \eta T^*,$$

Equation (26) can also be written as

$$\tilde{u}(\omega) = [M(\omega)]^{-1}\tilde{\psi}(\omega) \quad (27)$$

Where

$$[M(\omega)]^{-1} = \frac{1}{R(\omega) + iI(\omega)}\begin{pmatrix} D_1 & D_2 & D_3 \\ E_1 & E_2 & E_3 \\ F_1 & F_2 & F_3 \end{pmatrix} \quad (28)$$

and where $D_1 = \left[\left(-\omega^2 + \frac{r_2\gamma P^* T^*}{(1-\theta)K}\right) + i\left(\frac{r_2 P^*\omega}{(1-\theta)K} + \omega\gamma T^*\right)\right]$; $E_1 = 0$,

$$F_1 = \left[\left(\frac{\alpha_2 r_2 T^* P^*}{(1-\theta)K}\right) + i\left(\frac{\omega\alpha_2 T^*}{(1-\theta)K}\right)\right]; \ D_2 = 0,$$

$$E_2 = \left[\left(-\omega^2 + \frac{r_1 P^* T^*\eta}{\theta K} + \alpha_1\alpha_2 S^* T^*\right) + i\left(\frac{r_1\omega P^*}{\theta K} + \eta\omega T^*\right)\right], \ F_2 = 0,$$

$$D_3 = \left[\left(-\frac{\alpha_1 r_2 S^* P^*}{(1-\theta)K}\right) + i(-\alpha_1\omega S^*)\right], \ E_3 = 0,$$

$$F_3 = \left[\frac{r_1 r_2 S^* P^*}{\theta(1-\theta)K^2} - \omega^2\right] + i\left[\frac{r_1\omega S^*}{\theta K} + \frac{r_2\omega P^*}{(1-\theta)K}\right]$$

Here $|D_1|^2 = X_1^2 + Y_1^2$; $|D_2|^2 = X_2^2 + Y_2^2$; $|D_3|^2 = X_3^2 + Y_3^2$; $|E_1|^2 = X_4^2 + Y_4^2$; $|E_2|^2 = X_5^2 + Y_5^2$; $|E_3|^2 = X_6^2 + Y_6^2$; $|F_1|^2 = X_7^2 + Y_7^2$; $|F_2|^2 = X_8^2 + Y_8^2$; $|F_3|^2 = X_9^2 + Y_9^2$;

where $X_1 = \left(-\omega^2 + \frac{r_2\gamma P^* T^*}{(1-\theta)K}\right)$; $Y_1 = \left(\frac{r_2 P^*\omega}{(1-\theta)K} + \omega\gamma T^*\right)$; $X_2 = 0$; $Y_2 = 0$;

$$X_3 = \left(-\frac{\alpha_1 r_2 S^* P^*}{(1-\theta)K}\right); \ Y_3 = (-\alpha_1\omega S^*); \ X_4 = 0; \ Y_4 = 0;$$

$$X_5 = \left(-\omega^2 + \frac{r_1 P^* T^*\eta}{\theta K} + \alpha_1\alpha_2 S^* T^*\right); \ Y_5 = \left(\frac{r_1\omega P^*}{\theta K} + \eta\omega T^*\right);$$

$$X_6 = 0; \ Y_6 = 0; \ X_7 = \left(\frac{\alpha_2 r_2 T^* P^*}{(1-\theta)K}\right); \ Y_7 = \left(\frac{\omega\alpha_2 T^*}{(1-\theta)K}\right);$$

$$X_8 = 0; \ Y_8 = 0; \ X_9 = \left(\frac{r_1 r_2 S^* P^*}{\theta(1-\theta)K^2} - \omega^2\right); \ Y_9 = \left(\frac{r_1\omega S^*}{\theta K} + \frac{r_2\omega P^*}{(1-\theta)K}\right) \quad (29)$$

$|M(\omega)|^2 = [R(\omega)]^2 + [I(\omega)]^2$ where

$$R(\omega) = \frac{\eta r_1 r_2 S^* P^* T^*}{\theta(1-\theta)K^2} + \frac{\alpha_1\alpha_2 S^* P^* T^*}{(1-\theta)K} - \frac{\omega^2 r_1 S^*}{\theta K} - \frac{\omega^2 r_2 P^*}{(1-\theta)K} - \omega^2\eta T^*$$

and

$$I(\omega) = \left[\frac{r_2\eta\omega P^* T^*}{(1-\theta)K} + \frac{r_1\omega\eta S^* T^*}{\theta K} + \frac{r_1 r_2\omega S^* P^*}{\theta(1-\theta)K^2} + \alpha_1\alpha_2\omega S^* T^* - \omega^3\right]$$

If the function $Y(t)$ has a zero mean value, then the fluctuation intensity (variance) of its components in the frequency interval $[\omega, \omega + d\omega]$ is $S_Y(\omega)d\omega$. Where $S_Y(\omega)$ is the spectral density of Y and is defined as

$$S_Y(\omega) = \lim_{\tilde{T}\to\infty}\frac{|\tilde{Y}(\omega)|^2}{\tilde{T}} \quad (30)$$

If Y has a zero mean value, the inverse transform of $S_Y(\omega)$ is the auto covariance function

$$C_Y(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_Y(\omega)e^{i\omega\tau}d\omega \quad (31)$$

The corresponding variance of fluctuations in $Y(t)$ is given by

$$\sigma_Y^2 = C_Y(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_Y(\omega)d\omega \quad (32)$$

and the auto correlation function is the normalized auto covariance

$$P_Y(\tau) = \frac{C_Y(\tau)}{C_Y(0)} \quad (33)$$

For a Gaussian white noise process, it is

$$S_{\psi_i\psi_j}(\omega) = \lim_{\tilde{T}\to+\infty}\frac{E[\tilde{\psi}_i(\omega)\tilde{\psi}_j(\omega)]}{\tilde{T}} = \lim_{\hat{T}\to+\infty}\frac{1}{\hat{T}}\int_{-\frac{\tilde{T}}{2}}^{\frac{\tilde{T}}{2}}\int_{-\frac{\tilde{T}}{2}}^{\frac{\tilde{T}}{2}} E[\tilde{\psi}_i(t)\tilde{\psi}_j(t')]\,e^{-i\omega(t-t')}dt\,dt' = \delta_{ij} \quad (34)$$

From (28), we have $\tilde{u}_i(\omega) = \sum_{j=1}^{3} K_{ij}(\omega)\,\tilde{\psi}_j(\omega)$; $i = 1,2,3$ (35)

From (28), we have $\tilde{u}_i(\omega) = \sum_{j=1}^{3} K_{ij}(\omega)\,\tilde{\psi}_j(\omega)$; $i = 1,2,3 S_{u_i}(\omega)$

$$= \sum_{j=1}^{3}\xi_j\,|K_{ij}(\omega)|^2 \ ; \ i = 1,2,3 \quad (36)$$

where $K_{ij}(\omega) = [M(\omega)]^{-1}$

Hence by (35) and (36), the intensities of fluctuations in the variable u_i ; $i = 1,2,3$ are given by

$$\sigma_{u_i}^2 = \frac{1}{2\pi}\sum_{j=1}^{3}\int_{-\infty}^{\infty}\xi_j\,|K_{ij}(\omega)|^2 d\omega; \ i = 1,2,3 \quad (37)$$

and from (28), (29), (37) we obtain

$$\sigma_{u_1}^2 = \frac{1}{2\pi}\left\{\int_{-\infty}^{\infty}\frac{1}{R^2(\omega) + I^2(\omega)}[\xi_1(X_1^2 Y_1^2) + \xi_2(X_2^2 + Y_2^2) + \xi_3(X_3^2 + Y_3^2)]d\omega\right\} \quad (38)$$

$$\sigma_{u_2}^2 = \frac{1}{2\pi}\left\{\int_{-\infty}^{\infty} \frac{1}{R^2(\omega) + I^2(\omega)}[\xi_1(X_4^2 + Y_4^2) + \xi_2(X_5^2 + Y_5^2) + \xi_3(X_6^2 + Y_6^2)]d\omega\right\} \tag{39}$$

$$\sigma_{u_3}^2 = \frac{1}{2\pi}\left\{\int_{-\infty}^{\infty} \frac{1}{R^2(\omega) + I^2(\omega)}[\xi_1(X_7^2 + Y_7^2) + \xi_2(X_8^2 + Y_8^2) + \xi_3(X_9^2 + Y_9^2)]d\omega\right\} \tag{40}$$

where $|M(\omega)| = R(\omega) + iI(\omega)$. If we are interested in the dynamics of system (22) with either $\alpha_1 = 0$ or $\alpha_2 = 0\ \alpha_3 = 0$, then the population variances are as follows.

If $\xi_1 = 0,\ \xi_2 = 0$, then

$$\sigma_{u_1}^2 = \frac{\xi_3}{2\pi}\int_{-\infty}^{\infty} \frac{(X_3^2 + Y_3^2)}{R^2(\omega) + I^2(\omega)}d\omega;\ \sigma_{u_2}^2 = \frac{\xi_3}{2\pi}\int_{-\infty}^{\infty} \frac{(X_6^2 + Y_6^2)}{R^2(\omega) + I^2(\omega)}d\omega;$$

$$\sigma_{u_3}^2 = \frac{\xi_3}{2\pi}\int_{-\infty}^{\infty} \frac{(X_9^2 + Y_9^2)}{R^2(\omega) + I^2(\omega)}d\omega;$$

If $\xi_2 = 0,\ \xi_3 = 0$, then

$$\sigma_{u_1}^2 = \frac{\xi_1}{2\pi}\int_{-\infty}^{\infty} \frac{(X_1^2 + Y_1^2)}{R^2(\omega) + I^{21}(\omega)}d\omega;\ \sigma_{u_2}^2 = \frac{\xi_1}{2\pi}\int_{-\infty}^{\infty} \frac{(X_4^2 + Y_4^2)}{R^2(\omega) + I^2(\omega)}d\omega;$$

$$\sigma_{u_3}^2 = \frac{\xi_1}{2\pi}\int_{-\infty}^{\infty} \frac{(X_7^2 + Y_7^2)}{R^2(\omega) + I^2(\omega)}d\omega$$

If $\xi_3 = 0,\ \xi_1 = 0$, then

$$\sigma_{u_1}^2 = \frac{\xi_2}{2\pi}\int_{-\infty}^{\infty} \frac{(X_2^2 + Y_2^2)}{R^2(\omega) + I^2(\omega)}d\omega;\ \sigma_{u_2}^2 = \frac{\xi_2}{2\pi}\int_{-\infty}^{\infty} \frac{(X_5^2 + Y_5^2)}{R^2(\omega) + I^2(\omega)}d\omega;$$

$$\sigma_{u_3}^2 = \frac{\xi_2}{2\pi}\int_{-\infty}^{\infty} \frac{(X_8^2 + Y_8^2)}{R^2(\omega) + I^2(\omega)}d\omega$$

The equations (38)-(40) give three variations of the inhabitants. The integrations over the real line can be estimated, which gives the variations of the inhabitants.

8. Numerical simulations

In this section, the proposed model (1–3) has been numerically explored for the dynamical behaviour using MATLAB software. The main feature of the simulation is considered from a qualitative point of

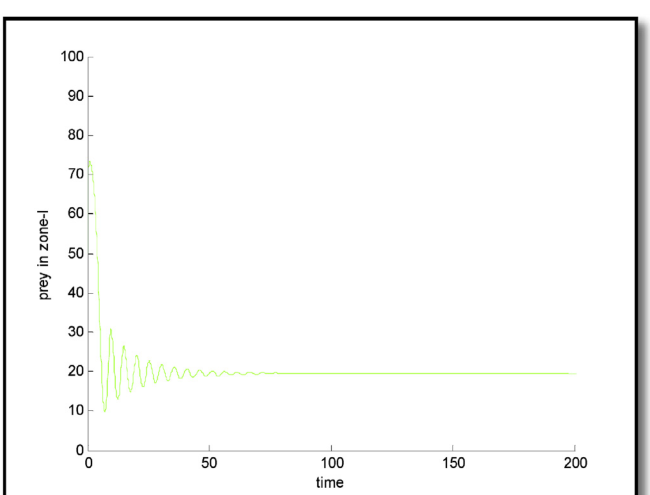


Fig. 1. Time series evolution of prey in zone-1.

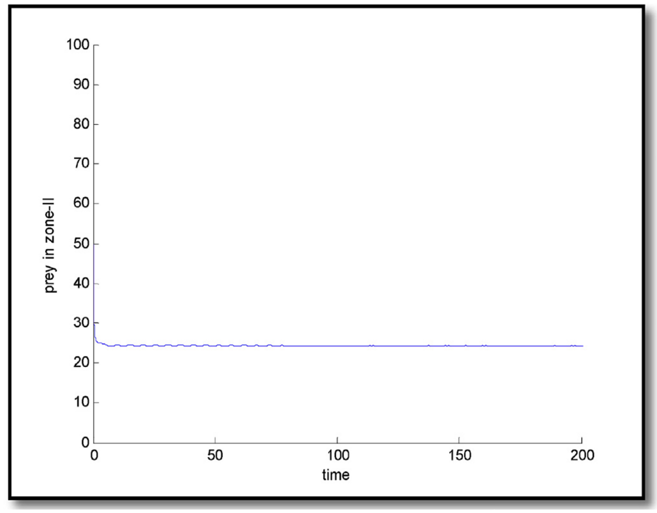


Fig. 2. Time series evolution of prey in zone-II.

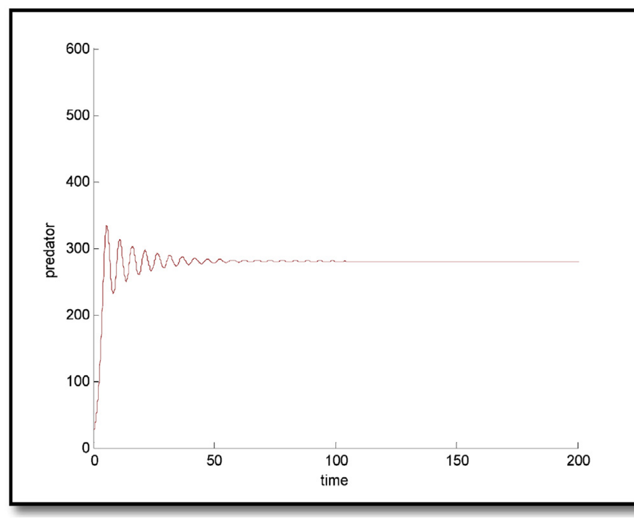


Fig. 3. Time series evolution of predator.

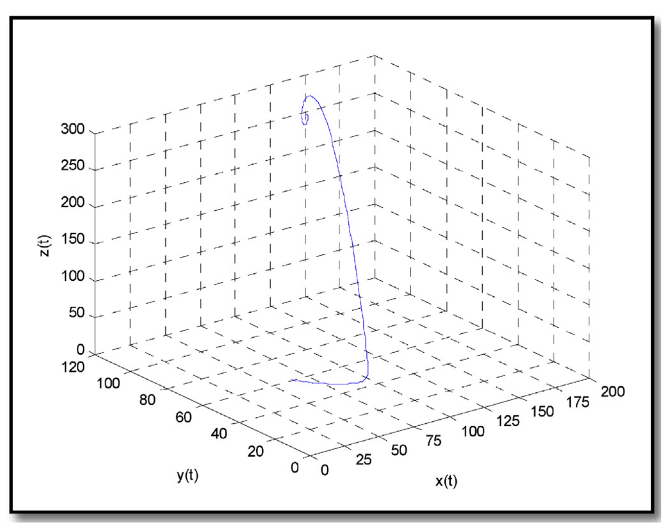


Fig. 4. Phase portrait of the system (1–3).

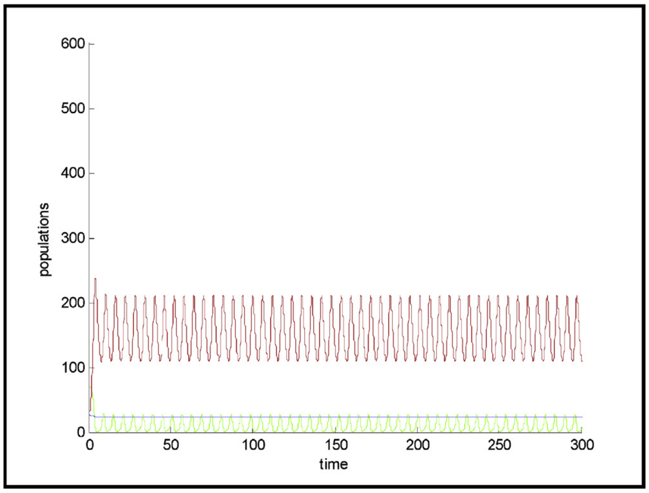


Fig. 5. Super critical bifurcation of the system (1–3).

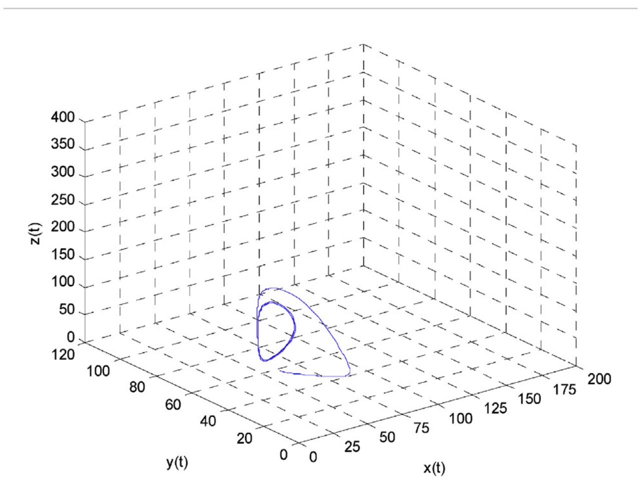


Fig. 6. Phase portrait of the system.

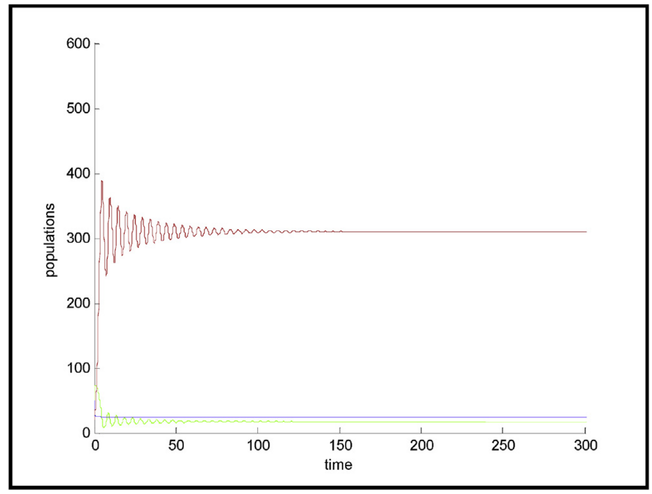


Fig. 7. Stable behavior of the delayed system.

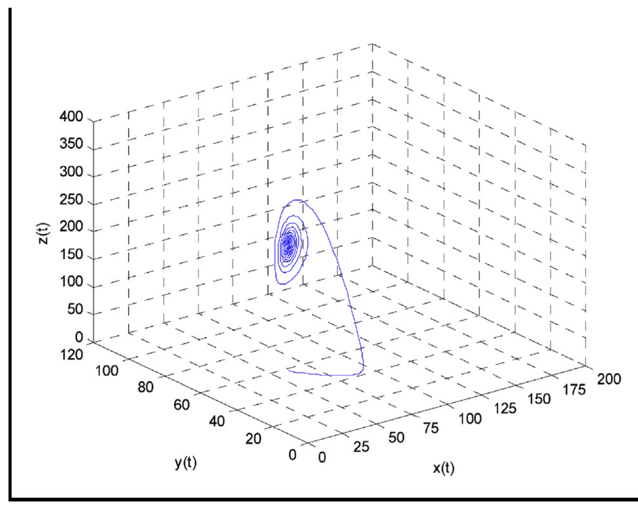


Fig. 8. Phase portrait of the delayed system.

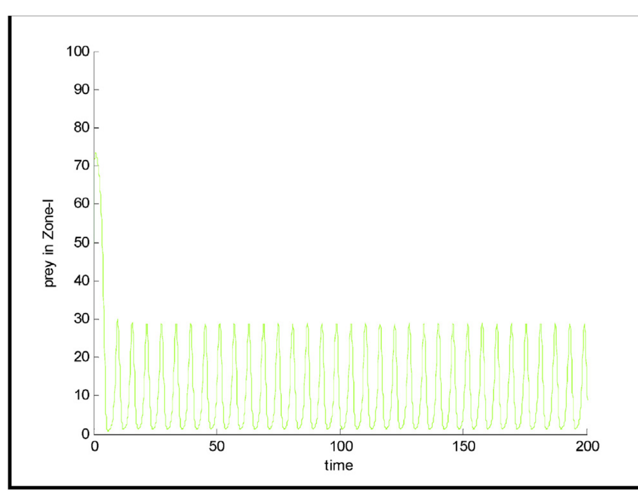


Fig. 9. Bifurcation diagram of prey in zone-1.

view. However, various data sets for the biological feasible parameters were tested and the results were collected.

Consider the following data

$r_1 = 5; r_2 = 3; K = 100; \beta = 30; a_1 = 0.75; a_2 = 1.5; \theta = 0.69;$

$m_1 = 2.5; m_2 = 2.5; d = 0.31; q_1 = 0.3;$

$q_2 = 0.3; E_1 = 2; E_2 = 2; \eta = 0.001$

From Figs. 1-3, the positive equilibrium point is locally asymptotically stable. This shows the coexistence occurs between the prey in Zone-1, prey in Zone-2 and predator respectively. Fig. 4 represents phase portraits of the system.

Considering the data $r_1 = 5; r_2 = 3; K = 100; \beta = 30; a_2 = 1.5; \theta = 0.69; m_1 = 2.5; m_2 = 2.5; d = 0.31;$

$q_1 = 0.3; q_2 = 0.3; E_1 = 2; E_2 = 2; \eta = 0.001$

Now the capturing rate of the prey to predator in zone-I is considered as a bifurcation parameter and the dynamical behaviour of the model (1–3) is analysed. Considering a_1 as a bifurcating parameter, periodic oscillations occur due to Hopf-bifurcation which is shown in Fig. 5. If the value of a_1 is gradually increased, keeping other parameters fixed, the stability of the system vanishes at the point where a_1 crosses its critical value $a_1^* = 0.85$. The corresponding phase portrait is shown in Fig. 6.

Now a delay is included and the dynamical behaviour of the system (15–17) is investigated. Considering the data $r_1 = 5; r_2 = 3; K = 100; \beta = 30; a_1 = 0.85; a_2 = 1.75; \theta = 0.69; m_1 = 2.5; m_2 = 2.5; d = 0.31; q_1 = 0.3; q_2 = 0.3; E_1 = 2; E_2 = 2; \eta = 0.001$ and the critical value of bifurcation parameter $\tau = \tau^* = 9.1$

If τ is set to be 8.5, the equilibrium point is locally asymptotically stable, for populations (p_1^*, p_2^*, p_3^*) converging to their steady state in

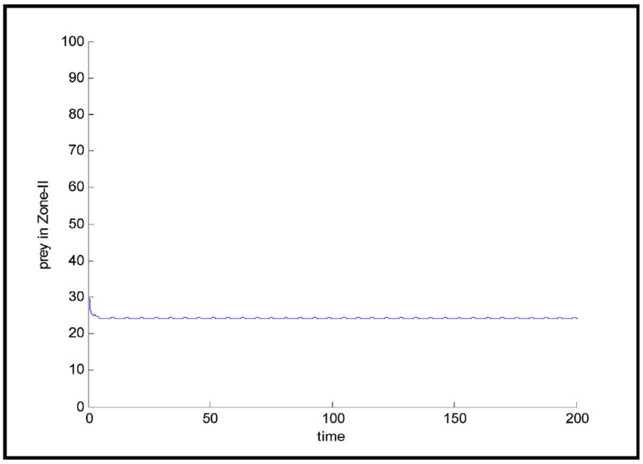


Fig. 10. Bifurcation diagram of prey in zone-2.

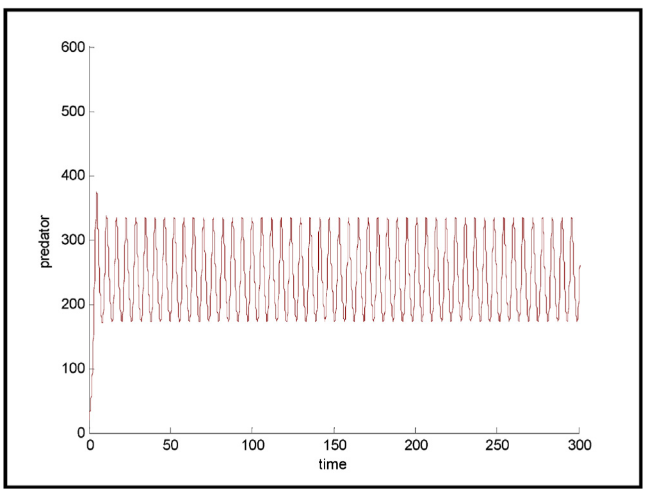


Fig. 11. Bifurcation diagram of predator.

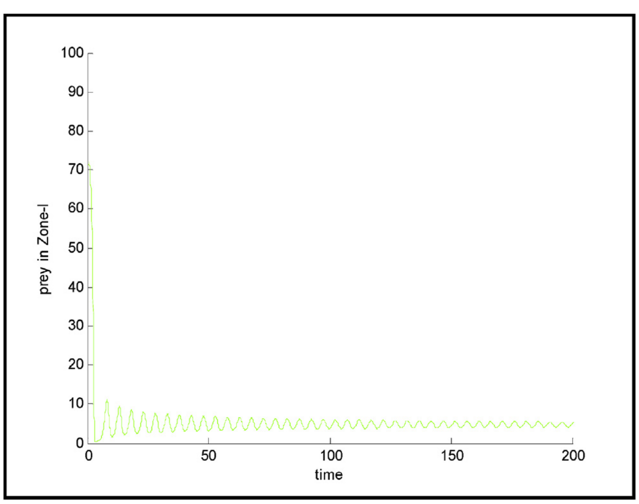


Fig. 12. Unstable solution of system (15–17) prey in zone-1.

finite time, see Fig. 7, with the corresponding phase portrait being shown in Fig. 8.

Now a gradual increase in the value of τ with other parameters fixed can reach the critical value $\tau = \tau^* = 9.1$, where the stability disappears and Hopf-bifurcation occurs - see Figs. 9-11. Finally, an inflation in the value of the delay $\tau = 10.64 > \tau^*$ proves that the positive interior

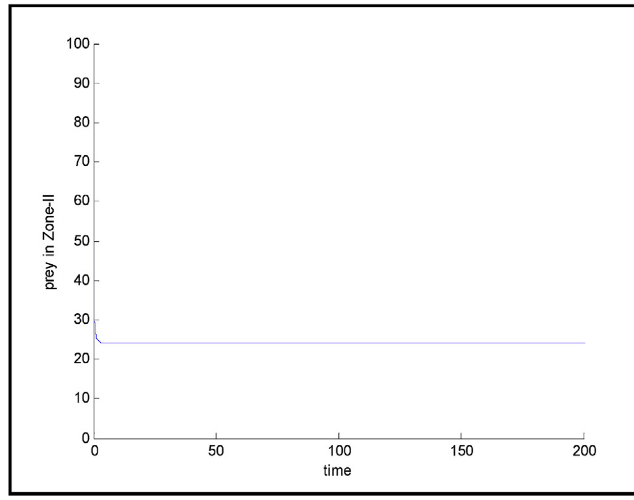


Fig. 13. Unstable solution of system (15–17) in zone-2.

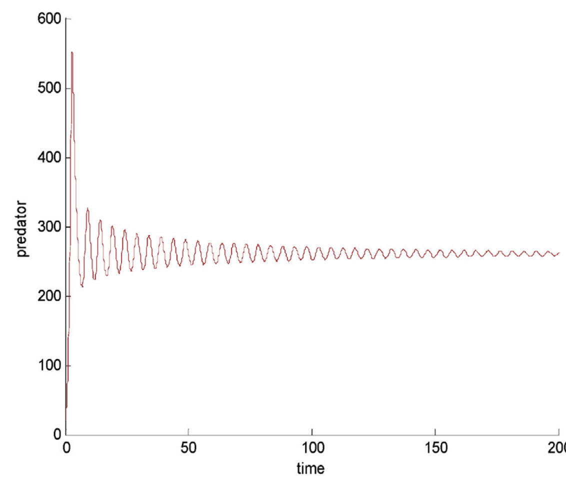


Fig. 14. Unstable solution of system (15–17) in predator.

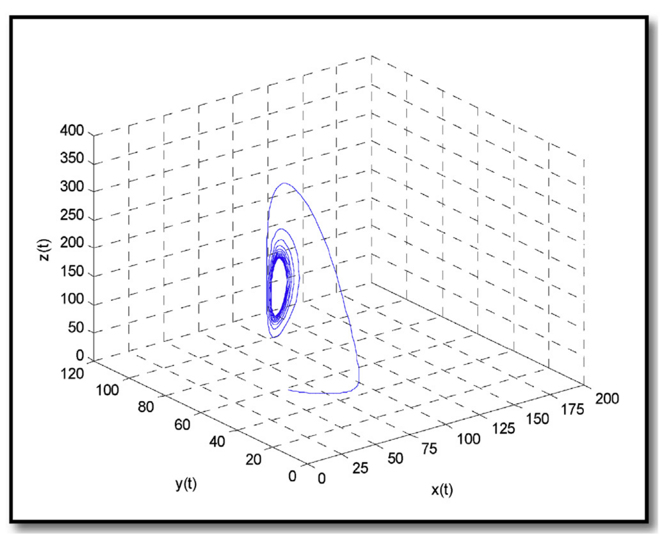


Fig. 15. Phase portrait of the system (15–17).

equilibrium point is unstable and a periodic orbit exists near the equilibrium point, which is evident from Figs. 12–14 and the corresponding phase portrait shown in Fig. 15.

Figs. 1-4 represents time series evolution of prey in zone-1, prey in zone-2 and predator, phase portrait of species of the system. Figs. 5-6 represent that supercritical bifurcation exists for the parameter and corresponding phase portrait. Figs. 7-8 represent convergence of the co-existence equilibrium point by considering a delay in the system, with corresponding phase portrait of the system. Figs. 9-11 represent bifurcation diagrams of prey in zone-1, prey in zone-2, predator species with $\tau = 8.5$. Figs. 12–14 represent bifurcation diagrams of prey in zone-1, prey in zone-2, predator species with $\tau = 10.64$. Fig. 15 shows the phase portrait diagram.

9. Concluding remarks

In this paper, the authors consider a differential algebraic bio-ecological model with a time delay, and the dynamical behaviour of the model system is developed in the presence of two patchy environments for population under the influence of environmental factor noise also.

In the above segments, the proposed structure is studied under the delay constraint, Hopf bifurcation analysis and behavioural analysis with stochastic approach. If the value $a_1 < 0.75$, then the system is observed as stable, if $a_1 > 0.75$ the system exhibits oscillatory behaviour and hence the system is observed as unstable. Similarly when $1.5 < a_2 < 2.5$ and $a_2 < 2.5$, the system shows periodic oscillatory behaviour and hence the system is unstable.

The overall study of the current work is initially focused on a deterministic approach, and in later segments, special focus on delay and stochastic analysis played major role in our theoretical study as well as in the numerical illustrations. On the other hand, the model may show rich dynamics by considering a spatiotemporal system of equations and its dynamics. The future scope of the current research problem may include simulation of a spatiotemporal model to analyse its dynamics.

Ethical statement

This paper is followed the ethical of journal procedure

Conflicts of interest

We don't have any conflict to publish our article in Informatics in Medicine Unlocked

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