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Investigating relations between discrete Painlevé equations: The multistep approach

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We show how, starting from a mapping where the independent variable advances one step at a time, one can obtain versions of the mapping corresponding to a multi-step evolution. The same procedure is applied to discrete Painlevé equations, and we proceed to establish Miura relations between the single-step and the multi-step versions (in the present study "multi" referring to double, triple, and quintuple). These Miura relations are discrete Painlevé equations on their own right. We show that, while in some cases it is impossible to obtain a multi-step equation for a single variable, deriving a Miura system is still possible. We perform our analysis for equations associated with the affine Weyl groups $E_8^{(1)}$, $E_7^{(1)}$, $E_6^{(1)}$, and $A_4^{(1)}$. *Published by AIP Publishing.*
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I. INTRODUCTION

One of the main characteristics of integrable systems is the existence of a wealth of interrelations which allow us to obtain a better grasp of these all-important systems. Foremost among such relations are the Miura transformations. The name Miura transformations comes from the analogy of such relations with the one discovered in 1968 by Miura[1] between the Korteweg-de Vries equation and the modified Korteweg-de Vries one. Miura transformations do also exist in the domain of Painlevé equations. The best known Miura transformation is the one related to Painlevé II. It is better expressed as a system

$$\frac{dx}{dz} = p - x^2 - z/2, \tag{1a}$$

$$\frac{dp}{dz} = 2px + \alpha. \tag{1b}$$

Eliminating p between the two equations, one obtains the P_{II} equation $x'' = 2x^3 + zx + \alpha - 1/2$, while the prime denotes the derivative with respect to z. Eliminating x leads to the equation $p'' = p'^2/(2p) + 2p^2 - zp - \alpha^2/(2p)$, which is equation XXXIV in the canonical Painlevé-Gambier list.[2]

The parallel between continuous and discrete Painlevé equations being quasi-perfect, it is natural for Miura transformations to exist for discrete Painlevé equations as well. For instance, the analog of the Miura introduced above does exist and was obtained by two of the present authors in Ref. 3. We start by introducing the system

$$x_{n+1} = 1 - \frac{y_n + (z_n + z_{n+1})/2}{1 + x_n}, \tag{2a}$$

$$y_{n-1} = \frac{m + y_n(1 - x_n)}{1 + x_n}, \tag{2b}$$

which is more conveniently written as

$$y_n = (1 + x_n)(1 - x_{n+1}) - \frac{z_n + z_{n+1}}{2}, \tag{3a}$$

$$x_n = \frac{m + y_n - y_{n-1}}{y_n + y_{n-1}}, \tag{3b}$$

with $z_n = \alpha n + \beta$. Eliminating y from this system, we obtain for x the d-P$_{\rm II}$ equation $x_{n+1} + x_{n-1} = (2z_n x_n + m - (z_{n+1} - z_n)/2)/(1 - x_n^2)$, while eliminating x, we find $(y_{n+1} + y_n)(y_n + y_{n-1}) = (4y_n^2 - m^2)/(y_n+(z_n+z_{n+1})/2)$, which, as shown in Ref. 4, is the discrete form of equation XXXIV encountered above.

In a recent paper,[5] two of the present authors in collaboration with Willox have considered equations constructed for discrete Painlevé equations associated with the affine Weyl group $E_8^{(1)}$. The method used there was to start from a discrete Painlevé equation in $E_8^{(1)}$ previously derived in trihomographic form, consider its autonomous limit, and using the results of Ref. 6, write the simplest possible mapping. Next, starting from the invariant of the latter and introducing homographic transformations, we obtained all possible canonical forms of the said invariant and the mappings corresponding to these forms. One interesting result obtained there was that we were able to obtain equations where we skipped one step out of two.

This is better illustrated through an example. We start from the autonomous equation

$$X_{n+1}X_{n-1} = A\frac{X_n - 1}{X_n}, \tag{4}$$

with invariant

$$K = \frac{(X_nX_{n-1} + A)(X_n + X_{n-1} - 1)}{X_nX_{n-1}}. \tag{5}$$

In order to obtain the mapping for a double step evolution, the simplest way is to solve (4) for X_n and obtain an invariant involving X_{n+1} and X_{n-1}. We obtain thus

$$K = \frac{(X_nX_{n-2} - X_n - A)(X_nX_{n-2} - X_{n-2} - A)}{X_nX_{n-2} - A}, \tag{6}$$

and, putting $A = a^2$ and rescaling X, the mapping

$$(X_nX_{n+2} - 1)(X_nX_{n-2} - 1) = \frac{X_n}{a^2X_n - a}. \tag{7}$$

Several such double-step evolutions were obtained, and the corresponding mappings were subsequently deautonomised leading to discrete Painlevé equations.

In this paper, we shall focus on one of the equations examined in Ref. 5, and show that there exist Miura-type relations between the equations obtained for the single step and the equations for double or triple steps. Moreover we shall show that in the case at hand one can obtain an equation also when one considers one step out of five (but in this case no equation for a single variable can be obtained). In order to derive the Miura transformations, we introduce a method, based on the singularity structure of the equations, which is easy to implement and particularly powerful.

II. FROM SINGLE-STEP TO MULTI-STEP EQUATIONS

In a recent paper, we have investigated the discrete Painlevé equations coming from specific equations associated with the $E_8^{(1)}$ affine Weyl group.[6] The equation we are going to work with in what follows, written in trihomographic expression, is

$$\frac{x_{n+1} - (5t_n - \alpha)^2}{x_{n+1} - (t_n + \alpha)^2}\,\frac{x_{n-1} - (5t_n + \alpha)^2}{x_{n-1} - (t_n - \alpha)^2}\,\frac{x_n - t_n^2}{x_n - 49t_n^2} = 1, \tag{8}$$

where we consider only secular behavior in the parameters. Furthermore, we take the autonomous limit of (8) taking $\alpha = 0$ and $t_n = 1$. We find thus

$$\frac{x_{n+1} - 25}{x_{n+1} - 1}\,\frac{x_{n-1} - 25}{x_{n-1} - 1}\,\frac{x_n - 1}{x_n - 49} = 1. \tag{9}$$

Introducing a new dependent variable X by

$$\frac{1}{2}X_n = \frac{x_n - 25}{x_n - 1} \tag{10}$$

leads to

$$X_{n+1}X_{n-1} = A(X_n - 1), \tag{11}$$

where $A = 4$. By assuming A to be a genuine parameter and a function of n, we can deautonomise (11). The application of the singularity confinement yields the relation

$$A_{n+3}A_{n-2} = A_{n+1}A_n, \tag{12}$$

which is integrated to $\log A_n = \alpha n + \beta + \phi_2(n) + \phi_3(n)$. Here we use the notation $\phi_m(n)$ in order to indicate a periodic function with period m, i.e., $\phi_m(n + m) = \phi_m(n)$. Its precise expression is $\phi_m(n) = \sum_{l=1}^{m-1} \epsilon_l^{(m)} \exp\left(\frac{2i\pi ln}{m}\right)$. Note that, since the summation starts at 1 and not at 0, $\phi_m(n)$ introduces only $m - 1$ parameters. The corresponding discrete Painlevé equation is associated with the affine Weyl group $A_4^{(1)}$. Pursuing with the autonomous case, we remark that (11) being of QRT (Quispel-Roberts-Thompson)-type has an invariant of the form

$$K = \frac{(X_nX_{n-1} + A(X_n + X_{n-1}) - A)(X_n + X_{n-1} + A)}{X_nX_{n-1}}. \tag{13}$$

Using invariant (13), we can obtain a double step evolution. We solve (11) for X_n and, downshifting the indices, we obtain the invariant

$$K = \frac{(X_nX_{n-2} - aX_n - 1 + a^2)(X_nX_{n-2} - aX_{n-2} - 1 + a^2)}{X_nX_{n-2} - 1}, \tag{14}$$

where we have taken $A = -a^2$ and have rescaled X so as to absorb a factor of a. From (14), we obtain the mapping

$$(X_nX_{n+2} - 1)(X_nX_{n-2} - 1) = a^2(1 - aX_n), \tag{15}$$

the deautonomisation of which was presented in Ref. 5. Triple-step evolutions can also be obtained. We find in this case the invariant

$$K = \frac{(AX_nX_{n-3} - A - 1)(AX_nX_{n-3} - X_n - X_{n-3} - A - 2)}{X_nX_{n-3} - 1}, \tag{16}$$

where we have absorbed a factor of A in X bringing the denominator to a canonical form. The corresponding mapping is

$$(X_nX_{n-3} - 1)(X_nX_{n+3} - 1) = \frac{1}{A^2}\frac{(X_n + 1)^2}{X_n - 1/A}, \tag{17}$$

and its deautonomisation was given in Ref. 7.

All the discrete Painlevé equations presented up to this point are associated with the group $A_4^{(1)}$. However as we have shown in Ref. 5, it is not difficult to construct the equations associated with higher affine Weyl groups, namely, $E_6^{(1)}$, $E_7^{(1)}$, and $E_8^{(1)}$. In Sec. III, we shall use these results in order to construct Miura transformations for all these discrete Painlevé equations.

III. MIURA TRANSFORMATIONS

One of the main motivations of this paper is the discovery of the Miura transformations studied in Ref. 8 that relate equations associated with $E_7^{(1)}$ which had different canonical forms and where one of the two was a double-step equation. The starting point is the equation

$$\left(\frac{x_n + x_{n+1} - z_{n+1}}{x_n + x_{n+1}}\right)\left(\frac{x_n + x_{n-1} - z_n}{x_n + x_{n-1}}\right) = \frac{x_n - z_{n+1} - z_n}{x_n}, \tag{18}$$

already identified in Ref. 9, with $z_n = \alpha n + \beta + \phi_3(n) + \phi_5(n)$, while the double step equation is

$$\frac{(y_n - y_{n+2} + (z_{n-1}+z_{n+3})^2)(y_n - y_{n-2} + (z_{n+1}+z_{n-3})^2) + 4y_n(z_{n-1}+z_{n+3})(z_{n+1}+z_{n-3})}{(z_{n+1}+z_{n-3})(y_n - y_{n+2} + (z_{n-1}+z_{n+3})^2) + (z_{n-1}+z_{n+3})(y_n - y_{n-2} + (z_{n+1}+z_{n-3})^2)} = \frac{y_n + (z_{n+1}+2z_n+z_{n-1})(z_{n+1}+z_{n-1})}{z_{n+1}+z_n+z_{n-1}} \quad (19)$$

identified in Ref. 10. In Ref. 8, we obtained the Miura between these two equations by using an auxiliary one

$$w_{n+1}(z_n + 2z_{n-1} - w_n) + w_{n-1}(z_n + 2z_{n+1} + w_n) - w_n(z_{n+1}+z_{n-1}) + z_n(z_{n+1}-z_{n-1}) = 0, \quad (20)$$

which originates from the deautonomisation of a mapping of the HKY (Hirota-Kimura-Yahagi) type.[11] The Miura between (18) and (19) can be obtained by eliminating the variable w_n from the Miuras relating w_n and x_n on the one hand

$$x_n = \frac{(w_{n+1}+z_{n+1})(z_n - w_n)}{2(w_{n+1} - w_n - z_{n+1} - z_n)} \quad \text{and} \quad w_n = z_n \frac{x_{n-1} - x_n}{x_{n-1} + x_n} \quad (21)$$

and w_n and y_n on the other hand

$$(w_{n-1} - w_{n+1} + z_{n+1} + 2z_n + z_{n+1})y_n = 2w_{n-1}w_{n+1}(z_{n+1}+z_n+z_{n+1}) + w_{n-1}(z_{n+1}^2 - z_{n-1}^2 - 2z_nz_{n+1}) + w_{n+1}(z_{n-1}^2 - z_{n+1}^2 - 2z_nz_{n-1}) + (z_{n+1}+z_{n+1})(z_{n+1}^2+z_{n+1}^2) + 2z_n(z_{n+1}+z_{n+1})^2 - 2z_nz_{n+1}z_{n+1}$$

and

$$w_n = -\frac{y_{n+1} - y_{n-1}}{2(z_{n+2}+z_{n-2})} + \frac{1}{2}(z_{n+2} - z_{n-2}). \quad (22)$$

This is a somewhat roundabout way to present the Miura in question. Fortunately, it turns out that one can obtain simple expressions for this Miura. We find indeed

$$\left(y_n - (2x_n - z_{n+1} + z_{n-1})^2\right)\left(y_{n+1} - (2x_n + z_{n+2} - z_n)^2\right) = 16x_n(x_n + z_{n+2})(x_n + z_{n-1})(x_n - z_n - z_{n+1}) \quad (23a)$$

complemented by

$$\frac{y_n + (2x_n - z_{n+1} + z_{n-1})(2x_{n-1} + z_{n+1} - z_{n-1})}{x_n + x_{n-1}} = \frac{y_n - (z_{n+1}+z_{n-1})(z_{n+1}+2z_n+z_{n-1})}{z_n}. \quad (23b)$$

Note that the Miura does not allow the construction of a single-step equation for y_n: only a double-step equation is possible.

No multiplicative equation was considered in Ref. 8 and, in fact, it is not clear whether it is possible to find an auxiliary variable like w. However the construction of a Miura transformation between a q equation associated with $E_7^{(1)}$, and a double-step equation is perfectly feasible. Our starting point is the multiplicative analog of (18),

$$\left(\frac{x_nx_{n+1} - z_{n+1}^2}{x_nx_{n+1} - 1}\right)\left(\frac{x_nx_{n-1} - z_n^2}{x_nx_{n-1} - 1}\right) = \frac{x_n - z_{n+1}^2z_n^2}{x_n - 1}, \quad (24)$$

where $\log z_n = \alpha n + \beta + \phi_3(n) + \phi_5(n)$. The Miura equation has now the form

$$\left(y_{n+1} - \frac{x_nz_{n+2}}{z_n} - \frac{z_n}{x_nz_{n+2}}\right)\left(y_n - \frac{x_nz_{n-1}}{z_{n+1}} - \frac{z_{n+1}}{x_nz_{n-1}}\right) = \frac{z_{n+2}z_{n-1}}{z_{n+1}z_n}\frac{(x_n-1)(x_n - z_n^2z_{n+1}^2)(x_n - 1/z_{n-1}^2)(x_n - 1/z_{n+2}^2)}{x_n^2}, \quad (25a)$$

$$\frac{y_n - x_nz_{n-1}/z_{n+1} - x_{n-1}z_{n+1}/z_{n-1}}{x_nx_{n-1} - 1} = \frac{y_n - z_{n+1}z_n^2z_{n-1} - 1/(z_{n+1}z_{n-1})}{z_n^2 - 1}. \quad (25b)$$

Eliminating x between the two equations of (25), we obtain the double-step equation,

$$\frac{(y_{n+2}z_{n-1}z_{n+3}-y_n)(y_{n-2}z_{n+1}z_{n-3}-y_n)-(z_{n+1}^2z_{n-3}^2-1)(z_{n-1}^2z_{n+3}^2-1)}{(y_{n+2}-z_{n-1}z_{n+3}y_n)(y_{n-2}-z_{n+1}z_{n-3}y_n)-(z_{n+1}^2z_{n-3}^2-1)(z_{n-1}^2z_{n+3}^2-1)/(z_{n+1}z_{n-3}z_{n-1}z_{n+3})}$$
$$=z_{n+3}z_{n-3}\frac{y_n-z_{n+1}z_{n-1}(z_n^2+1)}{y_nz_{n+1}z_n^2z_{n-1}-(z_n^2+1)}, \quad (26)$$

which again has a form usually encountered in equations associated with the group $E_8^{(1)}$.

In the examples that will follow we shall not seek an auxiliary variable like w but proceed in a more direct approach. We start with the simple case of Eq. (11) which we write in nonautonomous form as

$$x_{n+1}x_{n-1}=z_n(1-x_n), \quad (27)$$

where $\log z_n=\alpha n+\beta+\phi_2(n)+\phi_3(n)$. We have seen in Sec. II that one can obtain a double-step equation at the autonomous limit. Here we are interested in a non-autonomous form, i.e., a genuine discrete Painlevé equation. As pointed out in Ref. 5, the latter was derived in Ref. 7. It has the form (written admittedly with some hindsight)

$$(y_ny_{n+2}-1)(y_ny_{n-2}-1)=\frac{(1-z_{n+1}y_n/c_n)(1-z_{n-2}y_n/c_n)}{1-y_n/c_n}, \quad (28)$$

where $\log z_n=2\alpha n+\beta+\phi_2(n)+\phi_3(n)$ and $c(n+2)c(n)=z(n+1)z(n)$ leading to $\log c_n=\alpha(2n-1)+\beta+\phi_3(n+1)$.

We claim that the two discrete Painlevé equations are related by a Miura transformation. In order to construct it, we start by introducing the quantity

$$R_n=\left(\frac{1-x_n}{x_{n-1}}\right)\left(\frac{1-x_{n-1}}{x_n}\right). \quad (29)$$

This form is dictated by the singularity structure of (27).[12] Using the equation obeyed by z_n, we find that the singularity pattern of (27) is $\{1,0,z_{n+2},\infty,\infty,1/z_{n+7},0,1\}$. This leads to the following pattern for R_n: $\{0,f,\infty,1-1/z_{n+2},1,1-z_{n+7},\infty,f',0\}$, where f,f' are two finite quantities depending on the initial conditions. These two finite values are precisely due to the existence of the factors $(1-x_{n-1})/x_n$ and $(1-x_n)/x_{n-1}$. Using the quantity R_n, we proceed to construct the variable y_n expressed as a homography of the former. Again our guide is the singularity structure. Taking one out of two values in R_n (since y_n advances with double steps), we find the succession $\{0,\infty,1,\infty,0\}$ which we must match to the singularity pattern of (28), namely, $\{c_n,\infty,0,\infty,c_{n+8}\}$ provided c_n and z_n that are defined as above. It suffices thus to introduce the relation $y_n=c_n(1-R_n)$ in order to obtain the desired singularity pattern. The first half of the Miura is thus

$$(y_nx_n-c_n)(y_nx_{n-1}-c_n)=c_n(c_n-y_n). \quad (30a)$$

The second one can easily be obtained using (28). We find thus

$$z_n(y_nx_n-c_n)(y_{n+1}x_n-c_{n+1})=(1-x_n)c_nc_{n+1}. \quad (30b)$$

Note that (30) can be brought to a canonical form provided we define y_n/c_n as a new variable. Again the Miura does not allow the construction of a single-step equation for y_n but only a double-step one.

Next we turn to the $E_6^{(1)}$-associated equation obtained in[5]

$$(x_nx_{n+1}-1)(x_nx_{n-1}-1)=(1-x_n)(1-z_nx_n), \quad (31)$$

where $\log z_n=2\alpha n+\beta+\phi_2(n)+\phi_5(n)$. Based on the singularity structure of (31), we introduce the quantity

$$R_n=\left(\frac{x_n-1/z_n}{x_{n-1}-z_n}\right)\left(\frac{x_{n-1}-1/z_{n-1}}{x_n-z_{n-1}}\right), \quad (32)$$

and the auxiliary variable y_n by $R_n=(y_n/\zeta_n-1)/(y_nz_nz_{n-1}/\zeta_n-1)$. We obtain thus the Miura

$$(z_nx_ny_n/\zeta_n-1)(z_nx_ny_{n+1}/\zeta_{n+1}-1)=\frac{(x_n-z_{n-1})(x_n-z_{n+1})(z_nx_n-1)}{z_{n+1}z_{n-1}(x_n-1)} \tag{33a}$$

and

$$(z_nx_ny_n/\zeta_n-1)(z_{n-1}x_{n-1}y_n/\zeta_n-1)=(1-y_n/\zeta_n)(1-y_nz_nz_{n-1}/\zeta_n), \tag{33b}$$

where $\log\zeta_n=\alpha(2n-1)+\beta+\phi_5(n)+\phi_5(n-1)-\phi_5(n+2)$ and we have $\zeta_{n+1}\zeta_{n-1}=z_nz_{n-1}$. Eliminating x from (33), we find

$$(y_{n+2}y_n-1)(y_ny_{n-2}-1)=\frac{(cy_n-1)(y_n/c-1)(y_n/\zeta_n-1)}{z_nz_{n-1}y_n/\zeta_n-1}, \tag{34}$$

where $\log c=5\alpha+\phi_2(n)$, which is a constant since n has always the same parity.

IV. TRIPLE- AND QUINTUPLE-STEP EQUATIONS

Before proceeding further we must make here an important remark. The Miura transformations we present are always written as a system of two first-order mappings. This system, as it stands, is an integrable non-autonomous discrete system and, in fact, a discrete Painlevé equation on its own right. Whether one can eliminate both ways, establishing the relation between two different discrete Painlevé equations is ultimately immaterial. The Miura construction starts from an equation for the variable, say x, and obtains a system involving x and a new variable y which is a new discrete Painlevé equation. In what follows, we shall encounter several such examples.

We start with the triple-step mapping obtained from (11). In Sec. II, we gave it its form as (17) but as pointed out in Ref. 5 this form is too simple to allow deautonomisation. Thus we rewrite it as

$$(x_nx_{n-3}-1)(x_nx_{n+3}-1)=a\frac{(x_n-b)(x_b-1/b)}{x_n-c}, \tag{35}$$

which is deautonomised to

$$(x_nx_{n-3}-1)(x_nx_{n+3}-1)=dz_nz_{n+1}\frac{(x_n-b)(x_b-1/b)}{x_n-z_n}, \tag{36}$$

where b, d are constant and $\log z_n=\alpha n+\beta+\phi_2(n)$, an equation related to the group $A_4^{(1)}$. The construction of the Miura is the same as for the case of the double-step equation (28), but it turns out that the result is trivial. It amounts to define as new variable y, one of the two factors which appears in the left-hand side of (36). Thus we proceed to consider the next case, i.e., the triple-step equation associated with the group $E_6^{(1)}$. Its multiplicative form was obtained in[5]

$$\left(\frac{x_{n+3}-z_{n+2}x_n}{x_{n+3}z_{n+1}-x_n}\right)\left(\frac{x_{n-3}-z_{n-2}x_n}{x_{n-3}z_{n-1}-x_n}\right)=\frac{(x_nz_n-1)(x_n-1)}{(x_n-z_{n-1})(x_n-z_{n+1})}, \tag{37}$$

where $\log z_n=\alpha n+\beta+\phi_2(n)+\phi_5(n)$. We introduce the quantity R_n just as in Eq. (32) and construct an auxiliary variable y_n by

$$R_n=\frac{y_n-1}{z_nz_{n-1}y_n-1}. \tag{38}$$

Note that given (38) and the definition of R_n, y_n is to be understood as a variable at the position (n – 1/2). We find thus the Miura system

$$\left(\frac{y_{n+2}z_{n+1}-x_n}{y_{n+2}z_{n+1}z_{n+2}-x_n}\right)\left(\frac{y_{n+2}z_{n+2}-x_{n+3}}{y_{n+2}z_{n+1}z_{n+2}-x_{n+3}}\right)=\frac{y_{n+2}-1}{y_{n+2}z_{n+1}z_{n+2}-1}, \tag{39a}$$

$$\left(\frac{y_{n+2}z_{n+1}-x_n}{y_{n+2}z_{n+1}z_{n+2}-x_n}\right)\left(\frac{y_{n-1}z_{n-1}-x_n}{y_{n-1}z_{n-1}z_{n-2}-x_n}\right)=\frac{x_nz_n-1}{z_{n+2}z_{n-2}(x_n-1)}, \tag{39b}$$

but given the form of (39), it is not possible to obtain an equation for y alone.

A triple-step equation associated with $E_7^{(1)}$ does also exist as shown in Ref. 5. Working with the multiplicative case we find, starting from (24)

$$\left(\frac{x_{n+3}x_n - z_{n+1}^2 z_{n+3}^2}{x_{n+3}x_n - 1/z_{n+2}^2}\right)\left(\frac{x_{n-3}x_n - z_n^2 z_{n-2}^2}{x_{n-3}x_n - 1/z_{n-1}^2}\right) = \frac{(x_n - z_{n-2}^2)(x_n - z_{n+3}^2)(x_n - z_n^2 z_{n+1}^2)}{(x_n - 1)(x_n - 1/z_{n-1}^2)(x_n - 1/z_{n+2}^2)}. \tag{40}$$

We shall not enter into the details of the construction of the Miura but give directly the result. We obtain the system

$$\begin{aligned}(y_{n-1} - x_n/(z_n z_{n-2}) - z_n z_{n-2}/x_n)\,(y_{n+2} - x_n/(z_{n+1}z_{n+3}) - z_{n+1}z_{n+3}/x_n)\\ = \frac{(x_n - 1)(x_n/(z_n z_{n+1}) - z_n z_{n+1})(x_n/z_{n-2} - z_{n-2})(x_n/z_{n+3} - z_{n+3})}{x_n^2},\end{aligned} \tag{41a}$$

$$\frac{x_{n+3} + x_n - z_{n+1}z_{n+3}y_{n+2}}{x_{n+3}x_n - z_{n+1}^2 z_{n+3}^2} = \frac{z_{n+1}z_{n+2}^2 z_{n+3}y_{n+2} - z_{n+2}^2 - 1}{z_{n+1}^2 z_{n+2}^2 z_{n+3^2} - 1}. \tag{41b}$$

Again, we remark that while it is possible to eliminate y between the two equations of (41) and get back to (40), it is not possible to obtain a single equation for y_n.

We turn now to the case of quintuple-step equations and start with the equation obtained from (11). In contrast to the double- and triple-step equations, here it is not possible to obtain an equation for a single variable. Still we can introduce the Miura transformation (29) and define an auxiliary variable y by $y_n = 1 - R_n$ which is to be understood as a variable in position $(n - 1/2)$. We shall not enter into the details of the calculation (which are straightforward and can be performed with the help of computer algebra) and just give the final result. We find thus the following system:

$$\left(\frac{y_{n+3}z_{n+1} - x_n}{y_{n+3}z_{n+1}z_{n+2} - x_n}\right)\left(\frac{y_{n+3}z_{n+4} - x_{n+5}}{y_{n+3}z_{n+3}z_{n+4} - x_{n+5}}\right) = 1 - y_{n+3}, \tag{42a}$$

$$\left(\frac{y_{n+3}z_{n+1} - x_n}{y_{n+3}z_{n+1}z_{n+2} - x_n}\right)\left(\frac{y_{n-2}z_{n-1} - x_n}{y_{n-2}z_{n-1}z_{n-2} - x_n}\right) = \frac{z_n}{z_{n+2}z_{n-2}}(1 - x_n). \tag{42b}$$

No elimination of either of the variables is possible but system (42) does define a discrete Painlevé equation associated with the group $A_4^{(1)}$.

The case of the $E_6^{(1)}$-associated equation can be treated along the same lines. We start from R_n defined by (32) and introduce a variable y by $R_n = (y_n/(z_n z_{n-1}) - 1)/(y_n - 1)$. Using the definition of y_n and Eq. (31) for x, we find the system

$$(x_{n+5}y_{n+3} - 1)(x_n y_{n+3} - 1) = \frac{(1 - y_{n+3})(1 - ay_{n+3})(1 - by_{n+3})}{1 - y_{n+3}/(z_{n+2}z_{n+3})}, \tag{43a}$$

$$(x_n y_{n+3} - 1)(x_n y_{n-2} - 1) = \frac{(1 - x_n)(1 - x_n/a)(1 - x_n/b)(1 - x_n z_n)}{(1 - x_n/z_{n-1})(1 - x_n/z_{n+1})}. \tag{43b}$$

We remind here that the variable z_n obeys the relation $z_{n+4}z_{n-3} = z_{n+2}z_{n-1}$. As a consequence, the quantities z_{n+4}/z_{n+2} and z_{n+1}/z_{n+3} appearing in the numerators of Eq. (43) are constant when one advances by a step of 5. We therefore simplify our notations by introducing two constants $a \equiv z_{n+4}/z_{n+2}$ and $b \equiv z_{n+1}/z_{n+3}$. Equation (43) is a discrete Painlevé equation first identified in Ref. 7 [Equation (43) in that paper] albeit presented with a slightly different gauge choice.

Finally we turn to the case related to $E_7^{(1)}$. The auxiliary variable y is defined again by (25b), while x is given by Eq. (24). No quintuple-step equation for a single variable is possible but, just as for the $A_4^{(1)}$ and $E_6^{(1)}$ cases, one can obtain a system involving x and y. We find

$$\begin{aligned}\left(y_{n+3} - x_n z_{n+2}z_{n+4} - \frac{1}{x_n z_{n+2}z_{n+4}}\right)\left(y_{n-2} - x_n z_{n-1}z_{n-3} - \frac{1}{x_n z_{n-1}z_{n-3}}\right)\\ = \frac{(x_n z_{n+4}^2 - z_{n+1}^2)(x_n z_{n-3}^2 - z_n^2)(x_n z_{n+2}^2 - 1)(x_n z_{n-1}^2 - 1)(x_n - 1)}{z_{n+4}z_{n+2}z_{n-1}z_{n-3}x_n^2(x_n - z_{n+1}^2 z_n^2)},\end{aligned} \tag{44a}$$

$$\frac{y_{n+3} - z_{n+4}z_{n+2}(x_{n+5} + x_n)}{x_{n+5}x_n - 1/(z_{n+4}^2 z_{n+2}^2)} = z_{n+4}^2 z_{n+2}^2 \frac{z_{n+4}z_{n+2}y_{n+3}^2 - Ay_{n+3} - z_{n+4}z_{n+2}B}{z_{n+4}z_{n+2}(z_{n+5}^2 z_{n+4}^2 z_{n+3}^2 z_{n+2}^2 z_{n+1}^2 - 1)y_{n+3} - C}, \tag{44b}$$

where $A = z_{n+4}^2 z_{n+3}^2 z_{n+2}^2 + z_{n+5}^2 z_{n+4}^2 + z_{n+4}^2 z_{n+2}^2 + z_{n+2}^2 z_{n+1}^2$, $B = z_{n+5}^2 z_{n+4}^2 z_{n+3}^2 z_{n+2}^2 z_{n+1}^2 - z_{n+5}^2 z_{n+4}^2 z_{n+3}^2 - z_{n+4}^2 z_{n+3}^2 z_{n+2}^2 - z_{n+3}^2 z_{n+2}^2 z_{n+1}^2 - z_{n+5}^2 z_{n+4}^2 - z_{n+5}^2 z_{n+1}^2 - z_{n+2}^2 z_{n+1}^2 + 1$, and $C = z_{n+5}^2 z_{n+4}^2 z_{n+3}^2 z_{n+2}^2 z_{n+1}^2 + z_{n+5}^2 z_{n+4}^4 z_{n+3}^2 + z_{n+4}^2 z_{n+3}^2 z_{n+2}^4 z_{n+1} + z_{n+5}^2 z_{n+4}^2 z_{n+2}^2 z_{n+1}^2 - z_{n+4}^2 z_{n+3}^2 z_{n+2}^2 - z_{n+5}^2 z_{n+4}^2 - z_{n+4}^2 z_{n+2}^2 - z_{n+2}^2 z_{n+1}^2$.

Equation (44) is a q-discrete Painlevé equation which has not been previously derived.

V. $\mathbf{E_8^{(1)}}$-ASSOCIATED EQUATIONS

The case of $E_8^{(1)}$-associated equations is, quite expectedly, very cumbersome. Deriving the explicit forms of the Miuras in the fully deautonomised case turns out not to be always possible in practice (and even in the cases where this can be done the expressions often turn out to be too lengthy to be reproduced in the paper). With this caveat in mind, we proceed to the examination of the simplest case, namely, that of the Miura between two single-step equations. Our starting point is the trihomographic equation

$$\frac{x_{n+1} - (z_{n+1} + 2z_n + 2z_{n-1})^2}{x_{n+1} - z_{n+1}^2} \frac{x_{n-1} - (2z_{n-1} + 2z_n + z_{n-1})^2}{x_{n-1} - z_{n-1}^2} = \frac{x_n - (2z_{n-1} + 3z_n + 2z_{n-1})^2}{x_n - z_n^2}, \tag{45}$$

where $z_n = \alpha n + \beta + \phi_2(n) + \phi_3(n) + \phi_5(n)$. We start by introducing an auxiliary variable ζ related to z through $z_n = \zeta_{n+6} + \zeta_{n-6} - \zeta_n$. Given the form of z_n, we have for ζ the expression $\zeta_n = \alpha n + \beta + \phi_2(n) + \phi_3(n) - \phi_5(n+1) - \phi_5(n-1) - 2\phi_5(n)$. The Miura transformation is constructed by introducing first the quantity

$$R_n = \left(\frac{x_n - (2z_{n-1} + 3z_n + 2z_{n-1})^2}{x_{n-1} - (2z_{n-1} + 2z_n + z_{n-1})^2}\right)\left(\frac{x_{n-1} - (2z_n + 3z_{n-1} + 2z_{n-2})^2}{x_n - (z_n + 2z_{n-1} + 2z_{n-2})^2}\right) \tag{46}$$

and defining the new dependent variable y by

$$y_n = \frac{a_n^2 R_n - b_n^2}{R_n - 1}, \tag{47}$$

where $a_n = -\zeta_{n-4} + \zeta_{n-3} + \zeta_{n-2} + \zeta_{n-1} + \zeta_n + \zeta_{n+1} + \zeta_{n+2} - \zeta_{n+3}$ and $b_n = a_n + 2(\zeta_{n-4} + \zeta_{n+3})$. We obtain thus the system

$$\frac{(y_n - x_n + \zeta_{n-4}^2)(y_n - x_{n-1} + \zeta_{n+3}^2) + 4y_n\zeta_{n-4}\zeta_{n+3}}{\zeta_{n+3}(y_n - x_n + \zeta_{n-4}^2) + \zeta_{n-4}(y_n - x_{n-1} + \zeta_{n+3}^2)} = \frac{y_n - (\zeta_{n-3} + \zeta_{n-2} + \zeta_{n-1} + \zeta_n + \zeta_{n+1} + \zeta_{n+2})^2}{\zeta_{n-4} + \zeta_{n+3}} + \zeta_{n-4} + \zeta_{n+3}, \tag{48a}$$

$$\frac{(x_n - y_n + \zeta_{n-4}^2)(x_n - y_{n+1} + \zeta_{n+4}^2) + 4x_n\zeta_{n-4}\zeta_{n+4}}{\zeta_{n+4}(x_n - y_n + \zeta_{n-4}^2) + \zeta_{n-4}(x_n - y_{n+1} + \zeta_{n+4}^2)} = \frac{x_n^2 - Ax_n - B}{(\zeta_{n-4} + \zeta_{n+4})x_n - C}, \tag{48b}$$

where A, B, C are very lengthy expressions in terms of ζ_n, given below.

$$\begin{aligned} A = 2\Big(&\zeta_{n-4}\zeta_{n-3} - \zeta_{n-4}\zeta_{n-2} - \zeta_{n-4}\zeta_{n-1} + \zeta_{n-4}\zeta_{n+3} - \zeta_{n-4}\zeta_{n+2} - \zeta_{n-4}\zeta_{n+1} - \zeta_{n-4}\zeta_n + \zeta_{n-3}^2 + \zeta_{n-3}\zeta_{n+4} + \zeta_{n-2}^2 \\ &+2\zeta_{n-2}\zeta_{n-1} - \zeta_{n-2}\zeta_{n+4} + 2\zeta_{n-2}\zeta_{n+2} + 2\zeta_{n-2}\zeta_{n+1} + 2\zeta_{n-2}\zeta_n + \zeta_{n-1}^2 - \zeta_{n-1}\zeta_{n+4} + 2\zeta_{n-1}\zeta_{n+2} + 2\zeta_{n-1}\zeta_{n+1} + 2\zeta_{n-1}\zeta_n \\ &+\zeta_{n+4}\zeta_{n+3} - \zeta_{n+4}\zeta_{n+2} - \zeta_{n+4}\zeta_{n+1} - \zeta_{n+4}\zeta_n + \zeta_{n+3}^2 + \zeta_{n+2}^2 + 2\zeta_{n+2}\zeta_{n+1} + 2\zeta_{n+2}\zeta_n + \zeta_{n+1}^2 + 2\zeta_{n+1}\zeta_n + \zeta_n^2\Big), \end{aligned}$$

$$\begin{aligned} B = &(2\zeta_{n-4} + \zeta_{n-3} - \zeta_{n-2} - \zeta_{n-1} + 2\zeta_{n+4} + \zeta_{n+3} - \zeta_{n+2} - \zeta_{n+1} - \zeta_n)(\zeta_{n-3} + \zeta_{n-2} + \zeta_{n-1} + \zeta_{n+3} + \zeta_{n+2} + \zeta_{n+1} + \zeta_n) \\ &\times(\zeta_{n-3} + \zeta_{n-2} + \zeta_{n-1} - \zeta_{n+3} + \zeta_{n+2} + \zeta_{n+1} + \zeta_n)(-\zeta_{n-3} + \zeta_{n-2} + \zeta_{n-1} + \zeta_{n+3} + \zeta_{n+2} + \zeta_{n+1} + \zeta_n), \end{aligned}$$

$$
\begin{aligned}
C = {} & \zeta_{n-4}\zeta_{n-3}^2 + 2\zeta_{n-4}\zeta_{n-3}\zeta_{n-2} + 2\zeta_{n-4}\zeta_{n-3}\zeta_{n-1} - 2\zeta_{n-4}\zeta_{n-3}\zeta_{n+3} + 2\zeta_{n-4}\zeta_{n-3}\zeta_{n+2} + 2\zeta_{n-4}\zeta_{n-3}\zeta_{n+1} + 2\zeta_{n-4}\zeta_{n-3}\zeta_n \\
& + \zeta_{n-4}\zeta_{n-2}^2 + 2\zeta_{n-4}\zeta_{n-2}\zeta_{n-1} + 2\zeta_{n-4}\zeta_{n-2}\zeta_{n+3} + 2\zeta_{n-4}\zeta_{n-2}\zeta_{n+2} + 2\zeta_{n-4}\zeta_{n-2}\zeta_{n+1} + 2\zeta_{n-4}\zeta_{n-2}\zeta_n + \zeta_{n-4}\zeta_{n-1}^2 \\
& + 2\zeta_{n-4}\zeta_{n-1}\zeta_{n+3} + 2\zeta_{n-4}\zeta_{n-1}\zeta_{n+2} + 2\zeta_{n-4}\zeta_{n-1}\zeta_{n+1} + 2\zeta_{n-4}\zeta_{n-1}\zeta_n + \zeta_{n-4}\zeta_{n+3}^2 + 2\zeta_{n-4}\zeta_{n+3}\zeta_{n+2} + 2\zeta_{n-4}\zeta_{n+3}\zeta_{n+1} \\
& + 2\zeta_{n-4}\zeta_{n+3}\zeta_n + \zeta_{n-4}\zeta_{n+2}^2 + 2\zeta_{n-4}\zeta_{n+2}\zeta_{n+1} + 2\zeta_{n-4}\zeta_{n+2}\zeta_n + \zeta_{n-4}\zeta_{n+1}^2 + 2\zeta_{n-4}\zeta_{n+1}\zeta_n + \zeta_{n-4}\zeta_n^2 + \zeta_{n-3}^2\zeta_{n+4} \\
& + 2\zeta_{n-3}\zeta_{n-2}\zeta_{n+4} + 4\zeta_{n-3}\zeta_{n-2}\zeta_{n+3} + 2\zeta_{n-3}\zeta_{n-1}\zeta_{n+4} + 4\zeta_{n-3}\zeta_{n-1}\zeta_{n+3} - 2\zeta_{n-3}\zeta_{n+4}\zeta_{n+3} + 2\zeta_{n-3}\zeta_{n+4}\zeta_{n+2} \\
& + 2\zeta_{n-3}\zeta_{n+4}\zeta_{n+1} + 2\zeta_{n-3}\zeta_{n+4}\zeta_n + 4\zeta_{n-3}\zeta_{n+3}\zeta_{n+2} + 4\zeta_{n-3}\zeta_{n+3}\zeta_{n+1} + 4\zeta_{n-3}\zeta_{n+3}\zeta_n + \zeta_{n-2}^2\zeta_{n+4} + 2\zeta_{n-2}\zeta_{n-1}\zeta_{n+4} \\
& + 2\zeta_{n-2}\zeta_{n+4}\zeta_{n+3} + 2\zeta_{n-2}\zeta_{n+4}\zeta_{n+2} + 2\zeta_{n-2}\zeta_{n+4}\zeta_{n+1} + 2\zeta_{n-2}\zeta_{n+4}\zeta_n + \zeta_{n-1}^2\zeta_{n+4} + 2\zeta_{n-1}\zeta_{n+4}\zeta_{n+3} + 2\zeta_{n-1}\zeta_{n+4}\zeta_{n+2} \\
& + 2\zeta_{n-1}\zeta_{n+4}\zeta_{n+1} + 2\zeta_{n-1}\zeta_{n+4}\zeta_n + \zeta_{n+4}\zeta_{n+3}^2 + 2\zeta_{n+4}\zeta_{n+3}\zeta_{n+2} + 2\zeta_{n+4}\zeta_{n+3}\zeta_{n+1} + 2\zeta_{n+4}\zeta_{n+3}\zeta_n + \zeta_{n+4}\zeta_{n+2}^2 \\
& + 2\zeta_{n+4}\zeta_{n+2}\zeta_{n+1} + 2\zeta_{n+4}\zeta_{n+2}\zeta_n + \zeta_{n+4}\zeta_{n+1}^2 + 2\zeta_{n+4}\zeta_{n+1}\zeta_n + \zeta_{n+4}\zeta_n^2.
\end{aligned}
$$

Eliminating y between the two equations of (48) leads back to (45) for x. However given the form of (48b), it is impossible to eliminate x and obtain a single equation for y.

Next we consider the double-step evolution. Starting from (45), we can obtain an equation relating x_n to $x_{n\pm2}$. We find thus the equation

$$\frac{(x_n - x_{n+2} + 16z^2)(x_n - x_{n-2} + 16z^2) + 64x_nz^2}{4z(x_n - x_{n+2} + 16z^2) + 4z(x_n - x_{n-2} + 16z^2)} = \frac{(x_n + 95z^2)(x_n - z^2)}{8z(x_n + 5z^2)} - 2z. \tag{49}$$

Deautonomising this equation becomes a manageable task provided we introduce the parametrisation we introduced in Ref. 13. We can show then that the non-autonomous form of (49) is equivalent to the case 4.3.4 of that Ref. 13. Concentrating on the Miura transformation, we limit ourselves, at first, to the autonomous case (in which case the variables z and ζ introduced above do coincide). The Miura transformation is always given by (48) and the quantities A, B, and C are now equal to $42z^2$, $175z^4$, and $110z^3$, respectively. It turns out that when we consider a double-step evolution we can eliminate x from the two equations of (48) and obtain a single equation for y. Its autonomous form is

$$\frac{(y_n - y_{n+2} + 16z^2)(y_n - y_{n-2} + 16z^2) + 64y_nz^2}{4z(y_n - y_{n+2} + 16z^2) + 4z(y_n - y_{n-2} + 16z^2)} = \frac{(y_n + 36z^2)(y_n - 16z^2)}{8z(y_n + 11z^2)} + 8z. \tag{50}$$

Deautonomising this equation turns out to be feasible, and it results in Eq. (4.4.1) we first derived in our Ref. 13.

Triple- and quintuple-step evolutions can also be considered. The triple-step equation for x alone is

$$\frac{(x_n - x_{n+3} + 36z^2)(x_n - x_{n-3} + 36z^2) + 144x_nz^2}{6z(x_n - x_{n+3} + 36z^2) + 6z(x_n - x_{n-3} + 36z^2)} = \frac{x_n^3 + 229x_n^2z^2 + 2803x_nz^4 + 1575z^6}{12z(x_n^2 + 46x_nz^2 + 145z^4)}. \tag{51}$$

Again, for its deautonomisation we proceed as for the double-step equation, and find that the non-autonomous form is equivalent to Eq. (5.2.5) of Ref. 13. The Miura transformation can be obtained from the elementary one (48). We thus find the system

$$\frac{(y_{n+2} - x_n + 9z^2)(y_{n+2} - x_{n+3} + 9z^2) + 36y_nz^2}{3z(y_{n+2} - x_n + 9z^2) + 3z(y_{n+2} - x_{n+3} + 9z^2)} = \frac{y_{n+2} + 32z^2}{6z}, \tag{52a}$$

$$\frac{(x_n - y_{n-1} + 9z^2)(x_n - y_{n+2} + 9z^2) + 36x_nz^2}{3z(x_n - y_{n-1} + 9z^2) + 3z(x_n - y_{n+2} + 9z^2)} = \frac{(x_n - 9z^2)(x_n + 11z^2)}{6z(x_n + z^2)} + 6z. \tag{52b}$$

Clearly no elimination of x is possible in this case, and thus we cannot obtain a single equation for y.

Finally we turn to the quintuple-step evolution. Here no single equation does exist either for x or for y. Still it is possible to use the Miura (48) in order to obtain the system

$$\frac{(y_{n+3} - x_n + 25z^2)(y_{n+3} - x_{n+5} + 25z^2) + 100y_nz^2}{5z(y_{n+3} - x_n + 25z^2) + 5z(y_{n+3} - x_{n+5} + 25z^2)} = \frac{(y_{n+3} + 176z^2)(y_{n+3} + 8z^2)}{2z(5y_{n+3} + 112z^2)} + 4z, \tag{53a}$$

$$\frac{(x_n - y_{n-2} + 25z^2)(x_n - y_{n+3} + 25z^2) + 100x_n z^2}{5z(x_n - y_{n-2} + 25z^2) + 5z(x_n - y_{n+3} + 25z^2)} = \frac{x_n^3 + 149x_n^2 z^2 + 827x_n z^4 + 175z^6}{2z(5x_n^2 + 128x_n z^2 + 155z^4)}. \tag{53b}$$

In all the cases above it is, *a priori*, possible to use the Miura (48), involving the full expressions for A, B, C, and obtain the non-autonomous forms for (52) and (53). However the resulting expressions would have been prohibitively lengthy and, in that sense, not of great use. Thus we prefer to limit ourselves to the autonomous forms.

VI. CONCLUSIONS

In this paper, we have addressed the problem of the existence of relations between discrete Painlevé equations corresponding to evolutions with different steps. The possibility of the existence of double- or triple-step evolution equations besides the single-step ones was already presented in Ref. 5 in collaboration with Willox. In that paper we obtained several instances of such multi-step equations. The aim of the present paper is different from that of Ref. 5. Instead of just deriving such multi-step equations, we asked ourselves whether there existed Miura-like transformations relating to these systems.

The work of the present paper was centred around one particular system, a discrete Painlevé equation associated with the affine Weyl group $E_8^{(1)}$, the coefficients of which had periodicities involving periods 2, 3, and 5. Starting from this equation and following the method introduced in Ref. 5, we could derive systems associated with the groups $E_7^{(1)}$, $E_6^{(1)}$, and $A_4^{(1)}$. The important ingredient of our approach is the Miura seed, denoted by R in the text, the form of which is dictated by the singularity structure of the system. Once R is known, the first half of the Miura introduces a new variable, say y, which is expressed simply as a homography of R. Having the basic Miura, one can then proceed to the construction of the double, triple, and quintuple-step equations. In the first case, one can always obtain an equation for the variable y, but this is not possible for the case of triple- or quintuple-step evolutions. Still the result is most interesting since the equations obtained using the basic Miura are discrete Painlevé equations on their own right.

One important remark concerns Eqs. (25), (41), and (44). They have forms which have never before been encountered in our studies of discrete Painlevé equations. In fact, when we first obtained them we wondered whether these forms were canonical or not. It turned out, as we showed in Ref. 14, that they are indeed canonical forms corresponding to cases that were missing in the previous classification[15] of the canonical forms of the QRT[16] mapping. This discovery opened another interesting track for the exploration of discrete Painlevé equations.

The application of the method presented here is not in any way tailored to the system mentioned in the second paragraph. It can be, in fact, in combination with the approach developed in Ref. 5, applied to any of the $E_8^{(1)}$-associated discrete Painlevé equations derived in Ref. 5. Extending the method to multiplicative and/or asymmetric (in the QRT sense) systems is, in principle, feasible. We intend to address such questions in future publications of ours.

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