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Locating-total domination in graphs

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ABSTRACT

In this paper, we continue the study of locating-total domination in graphs. A set S of vertices in a graph G is a total dominating set in G if every vertex of G is adjacent to a vertex in S. We consider total dominating sets S which have the additional property that distinct vertices in $V(G) \setminus S$ are totally dominated by distinct subsets of the total dominating set. Such a set S is called a locating-total dominating set in G, and the locating-total domination number of G is the minimum cardinality of a locating-total dominating set in G. We obtain new lower and upper bounds on the locating-total domination number of a graph. Interpolation results are established, and the locating-total domination number in special families of graphs, including cubic graphs and grid graphs, is investigated.

1. Introduction

The problem of placing monitoring devices, such as surveillance cameras or fire alarms, in a system such that every site in the system (including the monitoring devices themselves) is adjacent to a monitor can be modeled by total domination in graphs. Applications where it is also important that if there is a problem in the system its location can be uniquely identified by the set of monitors, can be modeled by a combination of total domination and locating sets.

Let $G = (V, E)$ be a graph with vertex set V, edge set E and no isolated vertex. A *total dominating set*, abbreviated TD-set, of G is a set S of vertices of G such that every vertex is adjacent to a vertex in S. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a TD-set. The literature on this subject has been surveyed and detailed in the domination book by Haynes et al. [7]. A recent survey of total domination in graphs can be found in [9].

The study of locating-dominating sets in graphs was pioneered by Slater [12,13] and this concept was later extended to total domination in graphs. A *locating-total dominating set*, abbreviated LTD-set, in G is a TD-set S with the property that distinct vertices in $V \setminus S$ are totally dominated by distinct subsets of S. Every graph G with no isolated vertex has a LTD-set, since V is such a set. The *locating-total domination number*, denoted $\gamma_t^L(G)$, of G is the minimum cardinality of a LTD-set of G. A LTD-set of cardinality $\gamma_t^L(G)$ is called a $\gamma_t^L(G)$-set. This concept of locating-total domination in graphs was first studied by Haynes et al. [8] and has been studied, for example, in [1–5] and elsewhere.

1.1. Notation

For notation and graph theory terminology, we in general follow [7]. Specifically, let G be a graph with vertex set $V(G) = V$ of order $|V| = n$ and size $|E(G)| = m$, and let v be a vertex in V. The *open neighborhood* of v is $N_G(v) = \{u \in V \mid uv \in E(G)\}$ and the *closed neighborhood of* v is $N_G[v] = \{v\} \cup N(v)$. The degree of v is $d_G(v) = |N_G(v)|$. If the graph G is clear from the

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context, we simply write $N(v)$ and $d(v)$ rather than $N_G(v)$ and $d_G(v)$, respectively. For a set $S \subseteq V$, its *open neighborhood* is the set $N(S) = \cup_{v \in S} N(v)$, and its *closed neighborhood* is the set $N[S] = N(S) \cup S$. Thus a set $S \subseteq V$ is a TD-set in G if $N(S) = V$, while S is a LTD-set if it is a TD-set and for every pair of distinct vertices u and v in $V \setminus S$, we have $N(u) \cap S \neq N(v) \cap S$. For sets $A, B \subseteq V$, we say that *A dominates B* if $B \subseteq N[A]$, while *A totally dominates B* if $B \subseteq N(A)$. The maximum distance among all pairs of vertices of G is the *diameter* of G, which is denoted by $\mathrm{diam}(G)$.

A *cycle* on n vertices is denoted by C_n, while a *path* on n vertices is denoted by P_n. We denote by K_n the *complete graph* on n vertices and by $K_{m,n}$ the *complete bipartite graph* with one partite set of cardinality m and the other of cardinality n. A *star* is a complete bipartite graph of the form $K_{1,n}$. A vertex of degree one is called a *leaf*, and its neighbor is called a *support vertex*. We denote the set of leaves of G by $L(G)$. An edge incident with a leaf is called a *pendant edge*. The *corona*, $\mathrm{cor}(G)$, of a graph G is that graph obtained from G by adding a pendant edge to each vertex of G. For a subset $S \subseteq V$, the subgraph induced by S is denoted by $G[S]$. The *girth* of G is the length of a shortest cycle in G, which we denote by $g(G)$.

If X and Y are two vertex disjoint subsets of V, then we denote the set of all edges of G that join a vertex of X and a vertex of Y by $[X, Y]$. Further, if all edges are present between the vertices in X and the vertices in Y, we say that $[X, Y]$ is *full*, while if there are no edges between the vertices in X and the vertices in Y, we say that $[X, Y]$ is *empty*.

For graphs G and H, the *Cartesian product* $G \Box H$ is the graph with vertex set $V(G) \times V(H)$ where two vertices (u_1, v_1) and (u_2, v_2) are adjacent if and only if either $u_1 = u_2$ and $v_1 v_2 \in E(H)$ or $v_1 = v_2$ and $u_1 u_2 \in E(G)$.

1.2. Known results and observations

Every LTD-set of a graph is also a TD-set of the graph, implying the following observation.

Observation 1 *([8]). $\gamma_t^L(G) \geq \gamma_t(G)$ for every graph G.*

In the special case when G is a path, every TD-set of G is also a LTD-set of G. Thus the locating-total domination number of a path is precisely its total domination number.

Observation 2 *([8]). For $n \geq 2$, $\gamma_t^L(P_n) = \gamma_t(P_n) = \lfloor n/2 \rfloor + \lceil n/4 \rceil - \lfloor n/4 \rfloor$.*

It is also a simple exercise to determine the locating-total domination number of certain well-studied families of graphs.

Observation 3. *The following hold.*

(a) *For $n \geq 3$, $\gamma_t^L(C_n) = \gamma_t(C_n) = \lfloor n/2 \rfloor + \lceil n/4 \rceil - \lfloor n/4 \rfloor$.*
(b) *For $n \geq 2$, $\gamma_t^L(K_{1,n}) = n$.*
(c) *For $m \geq n \geq 2$, $\gamma_t^L(K_{m,n}) = m + n - 2$.*
(d) *For $n \geq 3$, $\gamma_t^L(K_n) = n - 1$.*

A lower bound on the locating-total domination number of a tree in terms of its order is given in [8] and the extremal trees achieving equality in the bound are also characterized.

Theorem 4 *([8]). If T is a tree of order $n \geq 2$, then $\gamma_t^L(T) \geq 2(n + 1)/5$.*

Chen and Sohn [6] established the following lower and upper bounds on the locating-total domination number of a tree in terms of its order and number of leaves and support vertices. Furthermore they constructively characterize the extremal trees achieving the bounds.

Theorem 5 *([6]). If T is a tree of order $n \geq 3$ with ℓ leaves and s support vertices, then $(n+\ell+1)/2 - s \leq \gamma_t^L(T) \leq (n+\ell)/2$.*

We remark that the concept of a locating-paired dominating set, where we require that the paired-dominating set (a dominating set that contains a perfect matching) is also a locating set, has been studied in [11]. Although every graph with no isolated vertex has a LTD-set, not every graph with no isolated vertex has a locating-paired dominating set. However using an identical proof as in Proposition 6 in [11], we have the following result.

Theorem 6 *([11]). If G is a graph of order $n \geq 3$ and maximum degree $\Delta \geq 2$ with no isolated vertex, then $\gamma_t^L(G) \geq 2n/(\Delta+2)$, and this bound is sharp.*

The following observation follows readily from the definition of a LTD-set in a graph.

Observation 7. *Let S be a LTD-set in a graph G and let X be a subset of vertices of G.*

(a) *If $N[u] = N[v]$ for every pair $u, v \in X$, then $|S \cap X| \geq |X| - 1$.*
(b) *If $N(u) = N(v)$ for every pair $u, v \in X$, then $|S \cap X| \geq |X| - 1$.*

2. Results

2.1. Lower bounds and interpolation results

We first establish a lower bound on the locating-total domination number of a graph in terms of its order.

Lemma 8. *If G is a connected graph of order $n \geq 2$ with $\gamma_t^L(G) = a$, then $n \leq 2^a + a - 1$.*

Proof. Let S be a $\gamma_t^L(G)$-set. Then, $|S| = a \geq 2$. For each $v \in V \setminus S$, let $N_v = N(v) \cap S$. Then, N_v is a non-empty subset of the set S. Since there are $2^a - 1$ distinct non-empty subsets of an a-element set, and since $N_u \neq N_v$ for every pair of distinct vertices u and v in $V \setminus S$, we have that $n - a = |V \setminus S| \leq 2^a - 1$, or, equivalently, $n \leq 2^a + a - 1$. □

Corollary 9. *If G is a connected graph of order $n \geq 2$, then $\gamma_t^L(G) \geq \lfloor \log_2 n \rfloor$.*

Proof. Let $\gamma_t^L(G) = a$, where $a \geq 2$. By Lemma 8, $n \leq 2^a + a - 1$. For $a \geq 2$, we have that $a - 1 < 2^a$, and so $n < 2 \cdot 2^a = 2^{a+1}$. Thus, $a > (\log_2 n) - 1$, implying that $\gamma_t^L(G) = a \geq \lfloor \log_2 n \rfloor$. □

By Lemma 8, if G is a graph with $\gamma_t^L(G) = a$ for some integer $a \geq 1$, then the order of G is at most $2^a + a - 1$. We prove next the following interpolation result for the locating-total domination number of a graph.

Theorem 10. *For every two integers a, b with $2 < a + 1 \leq b \leq 2^a + a - 1$, there exists a connected graph G of order b with $\gamma_t^L(G) = a$.*

Proof. Let a and b be integers with $a \geq 2$ and $a + 1 \leq b \leq 2^a + a - 1$. If $b = a + 1$, then we simply take $G = K_{1,a}$. In this case, G has order b and, by Observation 3, $\gamma_t^L(G) = a$. Suppose that $a + 2 \leq b \leq 2a - 1$. Then, $1 \leq b - (a + 1) \leq a - 2$ and we let G be the graph obtained from a star $K_{1,a}$ by subdividing $b - (a + 1)$ edges exactly once. Note that G has $2a - b + 1$ leaves that have a common neighbor. Every $\gamma_t^L(G)$-set contains the $b - a$ support vertices of G as well as $2a - b$ leaves that have a common neighbor. Thus, G has order b and $\gamma_t^L(G) = (b - a) + (2a - b) = a$.

Finally suppose that $2a \leq b \leq 2^a + a - 1$. Let G_a be the corona $\text{cor}(K_a)$ of a complete graph K_a and let S be the set of a vertices of the complete graph. We note that the set S has $2^a - a - 1$ distinct subsets of cardinality 2 or more. Select $b - 2a$ such distinct non-empty subsets of S, and let G be the graph obtained from G_a by adding $b - 2a$ new vertices corresponding to these $b - 2a$ distinct subsets of S and joining each element of S to those new vertices corresponding to subsets it is a member of. Then, G has order b. By construction, distinct vertices not in the set S have distinct intersections with the set S, implying that the set S is a LDT-set of G, and so $\gamma_t^L(G) \leq |S|$. However, every LTD-set in G contains the set S, and so $\gamma_t^L(G) \geq |S|$. Consequently, $\gamma_t^L(G) = |S| = a$. □

As a special case of Theorem 10, we note that, for every integer $a \geq 2$, there exists a connected graph G of order $n = 2^a + a - 1$ with $\gamma_t^L(G) = a = \lfloor \log_2 n \rfloor$. Hence the lower bound in Corollary 9 is sharp. Next, we obtain lower bound for the locating-total domination number in terms of the diameter $\text{diam}(G)$ of a graph G.

Theorem 11. *If G is a connected graph of order at least 2, then $\gamma_t^L(G) \geq (\text{diam}(G) + 1)/2$.*

Proof. Let $d = \text{diam}(G)$, and let x and y be two vertices of G with $d(x, y) = d$. For $i = 0, 1, 2, \ldots, d$, let V_i be the set of all vertices of G at distance i from x. Let S be an LTD-set. Let $X_0 = V_0 \cup V_1 \cup V_2$, and for $i = 1, \ldots, \lfloor (d - 2)/4 \rfloor$, let $X_i = V_{4i-1} \cup V_{4i} \cup V_{4i+1} \cup V_{4i+2}$. If $d \not\equiv 2 \pmod 4$, let

$$X_{\lceil \frac{d-2}{4} \rceil} = \bigcup_{i=4\lfloor \frac{d-2}{4} \rfloor + 3}^{d} V_i.$$

In order to totally dominate the vertices in $V_0 \cup V_1$ we have that $|S \cap X_0| \geq 2$. For $i = 1, \ldots, \lfloor (d - 2)/4 \rfloor$, in order to totally dominate the vertices in $V_{4i} \cup V_{4i+1}$ we have that $|S \cap X_i| \geq 2$. If $d \equiv 0 \pmod 4$, then in order to totally dominate the vertices in V_d we have that $|S \cap X_{\lceil (d-2)/4 \rceil}| \geq 1$. If $d \equiv 1 \pmod 4$, then in order to totally dominate the vertices in $V_{d-1} \cup V_d$ we have that $|S \cap X_{\lceil (d-2)/4 \rceil}| \geq 2$. Therefore the following holds. If $d \equiv 0 \pmod 4$, then $|S| \geq 2 + 2\lfloor (d - 2)/4 \rfloor + 1 = (d + 2)/2$. If $d \equiv 1 \pmod 4$, then $|S| \geq 2 + 2\lfloor (d - 2)/4 \rfloor + 2 = (d + 3)/2$. If $d \equiv 2 \pmod 4$, then $|S| \geq 2 + 2\lfloor (d - 2)/4 \rfloor = (d + 2)/2$. If $d \equiv 3 \pmod 4$, then $|S| \geq 2 + 2\lfloor (d - 2)/4 \rfloor = (d + 1)/2$. In all four cases, we have that $|S| \geq (d + 1)/2$. Since S is an arbitrary LTD-set in G, the desired lower bound follows. □

That the bound of Theorem 11 is sharp may be seen as follows. Let $G = P_n$, where $n \geq 4$ and $n \equiv 0 \pmod 4$. Then, $\text{diam}(G) = n - 1$ and by Observation 2, $\gamma_t^L(G) = n/2$. Consequently, $\gamma_t^L(G) = (\text{diam}(G) + 1)/2$.

2.2. Upper bounds

In this section, we present upper bounds in the locating-total domination number of a graph. Our first result characterizes graphs with large locating-total domination numbers.

Theorem 12. *Let G be a connected graph of order $n \geq 3$. Then, $\gamma_t^L(G) \leq n-1$, with equality if and only if G is a star or a complete graph.*

Proof. Let $G = (V, E)$ be a connected graph of order $n \geq 3$ and let v be a vertex of minimum degree in G. Then, $V \setminus \{v\}$ is a LTD-set in G, and so $\gamma_t^L(G) \leq n-1$. By Observation 3, if G is a star or a complete graph of order $n \geq 3$, then $\gamma_t^L(G) = n-1$. This establishes the sufficiency.

To prove the necessity, let $G = (V, E)$ be a connected graph of order $n \geq 3$ satisfying $\gamma_t^L(G) = n-1$. For the sake of contradiction, assume that G is neither a star nor a complete graph. Let u and v be two vertices at maximum distance apart in G, and so $d(u, v) = \mathrm{diam}(G)$. Since G is not a complete graph, $\mathrm{diam}(G) \geq 2$. If $\mathrm{diam}(G) \geq 3$, then $V \setminus \{u, v\}$ is a LTD-set in G, and so $\gamma_t^L(G) \leq n-2$, a contradiction. Hence, $\mathrm{diam}(G) = 2$. Let w be a common neighbor of u and v. Suppose $d(u) = 1$. Then, w is adjacent to every vertex in G. Since G is not a star, there are two neighbors of w, say x and y, that are adjacent. But then $V \setminus \{u, x\}$ is a LTD-set in G, and so $\gamma_t^L(G) \leq n-2$, a contradiction. Hence, $\delta(G) \geq 2$. If there is a vertex $x \in V$ such that $N(x) = \{u, w\}$, then the set $S = V \setminus \{v, x\}$ is a TD-set in G. In this case, we note that $u \in N(x) \cap S$ but $u \notin N(v) \cap S$, and so $N(x) \cap S \neq N(v) \cap S \neq \emptyset$. Thus, S is a LTD-set in G, a contradiction. Hence there is no vertex $x \in V$ such that $N(x) = \{u, w\}$. Since $\delta(G) \geq 2$, the set $S = V \setminus \{u, w\}$ is therefore a TD-set in G. However, $v \in N(w) \cap S$ but $v \notin N(u) \cap S$, and so $N(u) \cap S \neq N(w) \cap S \neq \emptyset$. Thus, S is a LTD-set in G, once again a contradiction. Therefore, G is either a star or a complete graph. $\square$

We show next that even if we impose a minimum degree condition and structural requirements in the statement of Theorem 12, then the upper bound of Theorem 12 can only be improved slightly.

Theorem 13. *Let G be a connected bipartite graph of order n with minimum degree $\delta(G) = \delta \geq 2$. Then, $\gamma_t^L(G) \leq n-2$, with equality if and only if $G = C_6$ or $G = K_{\delta,n-\delta}$.*

Proof. Let G be a connected bipartite graph of order n with minimum degree $\delta(G) = \delta \geq 2$. By Theorem 12, $\gamma_t^L(G) \leq n-2$. If $G = C_6$, then $\gamma_t^L(G) = 4 = |V(G)|-2$, while if $G = K_{\delta,n-\delta}$, then by Observation 3(c), $\gamma_t^L(G) = (n-\delta)+\delta-2 = |V(G)|-2$. This establishes the sufficiency.

To prove the necessity, suppose that $G = (V, E)$ is a connected bipartite graph of order n with minimum degree $\delta(G) = \delta \geq 2$ satisfying $\gamma_t^L(G) = n-2$. Let u and v be two vertices at maximum distance apart in G, and so $d(u, v) = \mathrm{diam}(G)$. Let $P\colon u = v_0, v_1, \ldots, v_k = v$ be a u-v path of length $\mathrm{diam}(G)$, and so $k = \mathrm{diam}(G)$. For $i = 0, 1, 2, \ldots, k$, let $V_i = \{x \mid d(u, x) = i\}$. Then, $V_0 = \{u\}$, $V_1 = N(u)$ and for $i = 2, \ldots, k$, we note that $v_i \in V_i$. Further for $0 \leq i < j \leq k$, if $j - i \geq 2$, then $[V_i, V_j]$ is empty. Since G is a bipartite graph, each set V_i, $0 \leq i \leq k$, is an independent set in G.

If $k \geq 4$, then since each set V_i is an independent set in G and since $\delta \geq 2$, the set $S = V \setminus \{v_0, v_1, v_k\}$ is a LTD-set in G, and so $\gamma_t^L(G) \leq |S| = n-3$, a contradiction. Hence, $k \leq 3$. Further since G is a bipartite graph and $\delta \geq 2$, the graph G is not a complete graph, and so $k \in \{2, 3\}$.

Suppose that $k = 3$. We consider the sets $N(u)$ and $N(v)$. As observed earlier, $N(u) = V_1$. Since V_1 is an independent set, we note that $N(x) \setminus \{u\} \subseteq V_2$ for each vertex $x \in V_1$ and since V_3 is an independent set, we note that $N(x) \subseteq V_2$ for each vertex $x \in V_3$. In particular, $N(v) \subseteq V_2$. Further since $\delta \geq 2$, each vertex in V_1 has at least one neighbor in V_2, while each vertex in V_3 has at least two neighbors in V_2.

Suppose that $[N(u), N(v)]$ is full. Then the set $S = V \setminus \{u, v_1, v_2\}$ is a TD-set in G. Further, $N(u) \cap S = V_1 \setminus \{v_1\}$, $N(v_1) \cap S = V_2 \setminus \{v_2\}$, while $N(v_2) \cap S \subseteq (V_1 \setminus \{v_1\}) \cup \{v\}$. Thus, S is a LTD-set of G, and so $\gamma_t^L(G) \leq |S| = n-3$, a contradiction. Hence, $[N(u), N(v)]$ is not full. Let x and y be two nonadjacent vertices, where $x \in N(u)$ and $y \in N(v)$.

If $S_u = V \setminus \{u, x, y\}$ is a TD-set in G, then S_u is a LTD-set of G, and so $\gamma_t^L(G) \leq |S| = n-3$, a contradiction. Hence, S_u is not a TD-set in G, implying that there is a vertex $y' \in V_1$ of degree 2 such that $N(y') = \{u, y\}$ (and so the vertex y' is not totally dominated by S_u). Analogously, considering the set $S_v = V \setminus \{v, x, y\}$, there is a vertex $x' \in V_2$ of degree 2 such that $N(x') = \{v, x\}$. Hence, $F = G[\{u, v, x, x', y, y'\}]$ is an induced 6-cycle in G.

If $d(x) \geq 3$, then let $D = V \setminus \{u, x, x'\}$. If $d(y) \geq 3$, then let $D = V \setminus \{v, y, y'\}$. If $d(u) \geq 3$, then let $D = V \setminus \{u, x, y'\}$. If $d(v) \geq 3$, then let $D = V \setminus \{v, x', y\}$. In all four cases, the set D is a LTD-set of G, and so $\gamma_t^L(G) \leq n-3$, a contradiction. Hence, $d(u) = d(v) = d(x) = d(y) = 2$. Thus every vertex of the induced 6-cycle F has degree 2 in G, implying by the connectivity of G that $G = F = C_6$.

Suppose that $k = 2$. Let x be an arbitrary vertex in V_1 and let y be an arbitrary vertex in V_2. Since both V_1 and V_2 are independent sets, the vertices x and y have no common neighbor. However $\mathrm{diam}(G) = 2$, implying that x and y are adjacent. Hence, $[V_1, V_2]$ is full. Therefore, G is a complete bipartite graph with partite sets $V_0 \cup V_2$ and V_1. Thus, $G = K_{a,b}$ for some integers a, b, where $a \geq b \geq 2$. Equivalently since $n = a + b$ and $\delta = b$, we have that $G = K_{\delta,n-\delta}$. $\square$

Let G be a connected graph of large order $n \geq 3$. By Theorem 12, if $\mathrm{diam}(G) = 1$, then $\gamma_t^L(G) = n-1$. By Theorem 13, if $\mathrm{diam}(G) = 2$, then it is possible that $\gamma_t^L(G) = n-2$. For large minimum degree and large diameter, we have the following upper bound on the locating-total domination number.

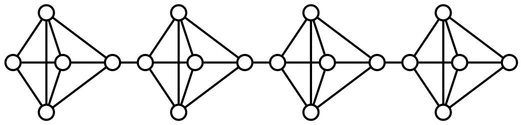

Fig. 1. A graph in the family $\mathcal{F}_{11}$.

Theorem 14. *Let G be a connected graph of order n with minimum degree at least 3 and diameter $\operatorname{diam}(G) = d \geq 3$. Then, $\gamma_t^L(G) \leq n - \lfloor d/2 \rfloor - 1$.*

Proof. Let $G = (V, E)$ and let u and v be two vertices at maximum distance apart in G, and so $d(u, v) = \operatorname{diam}(G)$. Let $P: u = v_0, v_1, \ldots, v_d = v$ be a u–v path of length $\operatorname{diam}(G)$, and so $d = \operatorname{diam}(G)$. We now consider the induced path $P = P_{d+1}$ on $d + 1$ vertices. Let

$$S = \bigcup_{i=0}^{\lfloor d/2 \rfloor} \{v_{2i}\}.$$

Then, $|S| = \lfloor d/2 \rfloor + 1$. We now consider the set $D = V \setminus S$. Let $X = V \setminus V(P)$. Then, $D = X \cup (V(P) \setminus S)$, and so $X \subset D$. Since $\delta(G) \geq 3$, every vertex on the path P has at least one neighbor in X, and so the set D dominates V. In particular every vertex of D on the path P has at least one neighbor in X and is therefore totally dominated by D. Every vertex in X that has a neighbor in X is totally dominated by D. Further, if v is an isolated vertex in $G[X]$, then by our choice of the path P and the minimum degree requirement we must have that $d_G(v) = 3$ and that the three neighbors of v are consecutive vertices on P. However, since D contains one vertex from every two consecutive vertices on P, the vertex v is totally dominated by D. Therefore the set D is a TD-set in G. Let x and y be two arbitrary vertices in $V \setminus D$. If x and y are consecutive vertices on P, then either x or y belongs to the set D, a contradiction. Hence, renaming x and y, if necessary, we may assume that $x = v_i$ and $y = v_j$, where $0 \leq i \leq j - 2 \leq d$. If $i < j - 2$, then $v_{i+1} \in N(x) \cap D$ but $v_{i+1} \notin N(y) \cap D$, and so x and y are totally dominated by distinct subsets of D. If $i = j - 2$, then either $i \geq 1$, in which case $v_{i-1} \in N(x) \cap D$ but $v_{i-1} \notin N(y) \cap D$, or $i = 0$, in which case $v_3 \in N(y) \cap D$ but $v_3 \notin N(x) \cap D$. Once again, x and y are totally dominated by distinct subsets of D. Hence, D is a LTD-set of G, implying that $\gamma_t^L(G) \leq |D| = n - |S| = n - \lfloor d/2 \rfloor - 1$. □

The bound in Theorem 14 is asymptotically best possible, as may be seen as follows. Let $k \geq 3$ and $\delta \geq 3$ be a fixed integers and let $d = 3k - 1$. Let $\mathcal{F}_d$ denote the family of graphs that can be obtained from a path $v_0 v_1 v_2 \ldots v_d$ of length d by replacing each vertex v_i, $0 \leq i \leq d$, with a clique A_i, where $|A_i| = 1$ if $i \not\equiv 1 \pmod 3$ and $|A_i| = \delta$ if $i \equiv 1 \pmod 3$, and adding all edges between A_i and A_{i+1}. In particular, we note that $A_i = \{v_i\}$ for $i \not\equiv 1 \pmod 3$. (A graph in the family $\mathcal{F}_{11}$ with $\delta = 3$, for example, is illustrated in Fig. 1.)

Let $F \in \mathcal{F}_d$ have order n and let S be a LTD-set in F. Let $Q: v_0 = u_0, u_1, u_2, \ldots, u_d = v_d$ be a v_0-v_d path in F. Necessarily, $u_i \in A_i$ for $i = 0, 1, \ldots, d$. By Observation 7(a), $|S \cap A_i| \geq |A_i| - 1$ for every i with $|A_i| = \delta$. Renaming vertices if necessary, we may assume that $A_i \setminus \{u_i\} \subseteq S \cap A_i$ for every i with $|A_i| = \delta$. Hence the only possible vertices of F not in the LTD-set S are the $3k$ vertices on the path Q. For $i = 0, 1, \ldots, k - 1$, let $X_i = \{u_{3i}, u_{3i+1}, u_{3i+2}\}$. Thus, $(X_0, X_1, \ldots, X_{k-1})$ is a partition of $V(Q)$. In order for u_0 and u_1 (respectively, u_{3k-2} and u_{3k-1}) to be totally dominated by distinct subsets of S we must have $|S \cap X_0| \geq 1$ and $|S \cap X_{k-1}| \geq 1$. Let $i \in \{1, 2, \ldots, k-2\}$. If $S \cap X_i = \emptyset$, then in order for u_{3i} and u_{3i+1} to be totally dominated by distinct subsets of S we must have $u_{3i-1} \in S$ and in order for u_{3i+1} and u_{3i+2} to be totally dominated by distinct subsets of S we must have $u_{3i+3} \in S$. Hence, if $|S \cap X_i| = 0$, then $\{u_{3i-1}, u_{3i+3}\} \subset S$. Let $R \subset V(Q)$ consist of four consecutive vertices on the path Q. Suppose that $R \cap S = \emptyset$. If $X_i \subset R$ for some i, $0 \leq i \leq k - 1$, we get a contradiction. Hence, $R = \{v_{3i+1}, v_{3i+2}, v_{3i+3}, v_{3i+4}\}$ for some i, $0 \leq i \leq k - 2$. In order for u_{3i+1} and u_{3i+2} (respectively, u_{3i+3} and u_{3i+4}) to be totally dominated by distinct subsets of S we must have $u_{3i} \in S$ (respectively, $u_{3i+5} \in S$). Hence at most four consecutive vertices on the path Q are not in S. Further, $|S \cap X_0| \geq 1$ and $|S \cap X_{k-1}| \geq 1$. Therefore, $|S \cap V(Q)| \geq d/5$, implying that $|S| = |V(F)| - |V(Q) \setminus S| \geq |V(F)| - 4d/5 = n - 4d/5$. This is true for every LTD-set S in F, implying that $\gamma_t^L(F) \geq n - 4d/5$.

2.3. Cubic graphs

We show next that the locating-total domination number and the total domination number of a connected cubic graph can differ significantly. The complete graph on four vertices minus one edge is called a diamond, sometimes written as $K_4 - e$.

Lemma 15. *For every integer $k \geq 1$, there exists a connected cubic graph G satisfying $\gamma_t^L(G) - \gamma_t(G) \geq 2k$.*

Proof. Let $k \geq 1$ be a given fixed integer. Let G_k be the connected cubic graph constructed as follows. Take $4k$ disjoint copies $F_1, F_2, \ldots, F_{4k}$ of a diamond, where $V(F_i) = \{a_i, b_i, c_i, d_i\}$ and where $a_i b_i$ is the missing edge in F_i. Let G_k be obtained from the disjoint union of these $4k$ diamonds by adding the edges $\{a_i b_{i+1} \mid i = 1, 2, \ldots, 4k - 1\}$ and adding the edge $a_{4k} b_1$. The graph G_1, for example, is illustrated in Fig. 2.

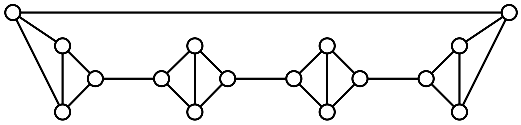

Fig. 2. The graph G_1.

For $i = 0, 1, \ldots, k-1$, let $Y_i = V(F_{4i+1}) \cup V(F_{4i+2}) \cup V(F_{4i+3}) \cup V(F_{4i+4})$ and let $X_i = \{a_{4i+1}, a_{4i+2}, b_{4i+3}, b_{4i+4}, c_{4i+1}, c_{4i+4}\}$. Then, $(Y_0, Y_1, \ldots, Y_{k-1})$ is a partition of $V(G_k)$. Since X_i totally dominates the set Y_i for each i, $0 \le i \le k-1$, we have that $X = \cup_{i=0}^{k-1} X_i$ is a TD-set in G_k, implying that $\gamma_t(G_k) \le |X| = 6k$.

Let S be a LTD-set in G_k. For each j, $1 \le j \le 4k$, we note that in the graph G_k we have $N[c_j] = N[d_j]$. Hence by Observation 7(a), we have that $|S \cap \{c_j, d_j\}| \ge 1$ for all $j = 1, 2, \ldots, 4k$. Renaming vertices if necessary, we may assume that $C \subseteq S$, where $C = \cup_{j=1}^{4k}\{c_j\}$. For each vertex c_j, $1 \le j \le 4k$, let c_j' be a vertex in S that totally dominates c_j, and so c_jc_j' is an edge in G_k. Since the vertices in the set C are pairwise at distance at least 3 apart in G_k, we note that $c_i' \neq c_j'$ for $1 \le i < j \le 4k$. Hence, $|S| \ge 2|C| = 8k$. This is true for every LTD-set S in G_k, implying that $\gamma_t^L(G_k) \ge 8k$. Hence, $\gamma_t^L(G_k) - \gamma_t(G_k) \ge 8k - 6k = 2k$. $\square$

Let $\mathcal{G}_n$ denote the family of all connected cubic graphs of order n. We define

$$\xi(n) = \max\left\{\frac{\gamma_t^L(G)}{\gamma_t(G)}\right\},$$

where the maximum is taken over all graphs $G \in \mathcal{G}_n$. If $G \in \mathcal{G}_4$, then $G = K_4$ and $\gamma_t^L(G) = 3$ and $\gamma_t(G) = 2$, and so $\xi(4) = 3/2$. If $G \in \mathcal{G}_6$, then either $G = K_{3,3}$, in which case $\gamma_t^L(G) = 4$ and $\gamma_t(G) = 2$, or G is the prism $C_3 \,\square\, K_2$, in which case $\gamma_t^L(G) = 3$ and $\gamma_t(G) = 2$. Thus, $\xi(6) = 2$. For $n \ge 16$, the family G_k of connected cubic graphs constructed in the proof of Lemma 15 yields the following result.

Lemma 16. *For $n \equiv 0 \pmod{16}$, we have $\xi(n) \ge \frac{4}{3}$.*

We pose the following two open questions that we have yet to settle.

Question 1. *Is it true that for n sufficiently large, we have $\xi(n) \le \frac{4}{3}$?*

Question 2. *Is it true that if G is a connected cubic graph of order $n \ge 8$, then $\gamma_t^L(G) \le n/2$?*

2.4. Grid graphs

In this section we investigate the locating-total domination number in a grid graph $P_m \,\square\, P_n$ for small m.

Theorem 17. *If $n \equiv r \pmod 5$, where $0 \le r < 5$, then*

$$\gamma_t^L(P_2 \square P_n) = \begin{cases} 4\left\lfloor \frac{n}{5} \right\rfloor + r & \text{if } r \neq 1 \\ 4\left\lfloor \frac{n}{5} \right\rfloor + 2 & \text{if } r = 1. \end{cases}$$

Proof. We proceed by induction on $n \ge 1$. It is a routine exercise to verify that $\gamma_t^L(P_2 \,\square\, P_1) = \gamma_t^L(P_2 \,\square\, P_2) = 2$, $\gamma_t^L(P_2 \,\square\, P_3) = 3$, and $\gamma_t^L(P_2 \,\square\, P_4) = \gamma_t^L(P_2 \,\square\, P_5) = 4$. This establishes the base cases. Suppose then that $n \ge 6$ and that the result holds for all grids $P_2 \,\square\, P_{n'}$, where $1 \le n' < n$. Let $G = P_2 \,\square\, P_n$ and let $V(G) = \cup_{i=1}^{n}\{a_i, b_i\}$, where $a_1a_2 \ldots a_n$ and $b_1b_2 \ldots b_n$ are paths P_n and a_ib_i is an edge for $i = 1, 2, \ldots, n$. For $i = 1, 2, \ldots, n$, let $X_i = \{a_i, b_i\}$. Further let $X_{\ge i} = \cup_{j=i}^{n} X_j$ and let $X_{\le i} = \cup_{j=1}^{i} X_j$. Let $F = G[X_{\ge 6}]$, and so $F = P_2 \,\square\, P_{n-5}$.

Among all $\gamma_t^L(G)$-set, let S be chosen so that

(1) $|S \cap X_{\le 5}|$ is a minimum.
(2) Subject to (1), $|S \cap X_1|$ is a minimum.
(3) Subject to (2), $|S \cap X_2|$ is a minimum.
(4) Subject to (3), $|S \cap X_3|$ is a minimum.
(5) Subject to (4), $|S \cap X_4|$ is a minimum.

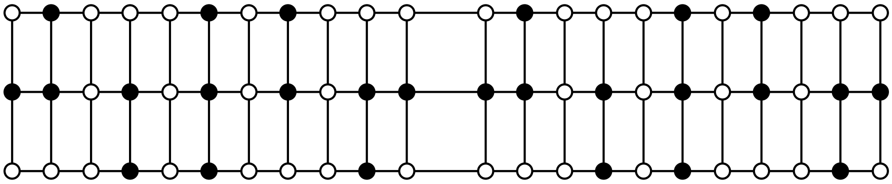

Fig. 3. A LTD-set for the grid $P_3 \,\square\, P_{22}$.

Suppose $X_1 \subset S$. If $X_2 \subset S$, then $(S \setminus X_1) \cup X_3$ is a LTD-set of G, contradicting our choice of the set S. Hence, $|X_2 \cap S| \le 1$. Suppose that $|X_2 \cap S| = 1$. By symmetry, we may assume that $a_2 \in S$, and so $b_2 \notin S$. But then $(S \setminus \{b_1\}) \cup \{b_3\}$ is a LTD-set of G, contradicting our choice of the set S. Hence, $X_2 \cap S = \emptyset$. But then $(S \setminus X_1) \cup X_2$ is a LTD-set of G, contradicting our choice of the set S. Therefore, $|X_1 \cap S| \le 1$.

Suppose $|X_1 \cap S| = 1$. By symmetry, we may assume that $a_1 \in S$, and so $b_1 \notin S$. Therefore, $a_2 \in S$ in order to totally dominate a_1. If $b_2 \in S$, then $(S \setminus \{a_1\}) \cup \{a_3\}$ is a LTD-set of G, contradicting our choice of the set S. Hence, $b_2 \notin S$. By our choice of the set S, the set $S' = (S \setminus \{a_1\}) \cup \{b_2\}$ is not a LTD-set of G. This implies that $a_3 \notin S$ and that a_1 and a_3 are not totally dominated by distinct subsets of S', and so $N(a_1) \cap S' = N(a_3) \cap S' = \{a_2\}$. Thus, $b_3 \notin S'$ and $a_4 \notin S'$. Therefore, $\{b_2, b_3, a_3, a_4\} \cap S = \emptyset$. But then $N(b_2) \cap S = N(a_3) \cap S = \{a_2\}$, contradicting the fact that b_2 and a_3 are totally dominated by distinct subsets of S. Hence, $X_1 \cap S = \emptyset$. In order to totally dominate X_1, we have that $X_2 \subset S$.

If $X_3 \subset S$, then $(S \setminus X_3) \cup X_4$ is a LTD-set of G, contradicting the minimality of S. Hence, $|X_3 \cap S| \le 1$. Suppose that $|X_3 \cap S| = 1$. By symmetry, we may assume that $a_3 \in S$, and so $b_3 \notin S$. If $b_4 \in S$, then $(S \setminus \{a_3\}) \cup \{a_4\}$ is a LTD-set of G, contradicting our choice of the set S. Hence, $b_4 \notin S$. By our choice of the set S, the set $D = (S \setminus \{a_3\}) \cup \{b_4\}$ is not a LTD-set of G. This implies that a_1 and a_3 are not totally dominated by distinct subsets of D, and so $N(a_1) \cap D = N(a_3) \cap D = \{a_2\}$. Thus, $b_3 \notin D$ and $a_4 \notin D$, implying that $\{b_3, b_4, a_4\} \cap S = \emptyset$. Therefore, $b_5 \in S$ in order to totally dominate b_4. Suppose that $a_5 \notin S$. Then, $b_6 \in S$ in order to totally dominate b_5. Further, $a_6 \in S$ in order for b_4 and a_5 to be totally dominated by distinct subsets of S. But then $(S \setminus \{a_3, b_5\}) \cup X_4$ is a LTD-set of G, contradicting our choice of the set S. Hence, $a_5 \in S$. If $X_6 \cap S \ne \emptyset$, then removing the vertices in $X_5 \cup (X_6 \cap S) \cup \{a_3\}$ from the set S, and replacing them with the four vertices in the set $X_4 \cup X_6$, produces a new LTD-set of G that contradicts our choice of the set S. Hence, $X_6 \cap S = \emptyset$. Thus, $b_7 \in S$ in order for b_4 and b_6 to be totally dominated by distinct subsets of S. If $a_7 \in S$, then $(S \setminus \{a_3, a_5, b_5\}) \cup (X_4 \cup \{a_6\})$ is a LTD-set of G, contradicting our choice of the set S. Hence, $a_7 \notin S$, and so $b_8 \in S$ in order to totally dominate the vertex b_7. But then $(S \setminus \{a_3, a_5, b_5\}) \cup (X_4 \cup \{a_7\})$ is a LTD-set of G, contradicting our choice of the set S. Hence, $X_3 \cap S = \emptyset$.

In order for a_1 and a_3 to be totally dominated by distinct subsets of S, we have that $a_4 \in S$. Analogously, $b_4 \in S$ in order for b_1 and b_3 to be totally dominated by distinct subsets of S. Therefore, $X_4 \subset S$. If $X_5 \subset S$, then $(S \setminus X_5) \cup X_6$ is a LTD-set of G, contradicting the minimality of S. Hence, $|X_5 \cap S| \le 1$. Suppose that $|X_5 \cap S| = 1$. By symmetry, we may assume that $a_5 \in S$, and so $b_5 \notin S$. But then the set $(S \setminus \{a_5\}) \cup \{b_6\}$ is a LTD-set of G, contradicting our choice of the set S. Hence, $X_5 \cap S = \emptyset$.

Since $S \cap X_{\le 5} = X_2 \cup X_4$, the restriction of the set S to F is a LTD-set of F, implying that $\gamma_t^L(F) \le |S \cap V(F)| = |S| - 4$, or, equivalently, $\gamma_t^L(G) = |S| \ge \gamma_t^L(F) + 4$. Conversely every $\gamma_t^L(F)$-set can be extended to a LTD-set of G by adding to it the set $X_2 \cup X_4$, implying that $\gamma_t^L(G) \le \gamma_t^L(F) + 4$. Consequently, $\gamma_t^L(G) = \gamma_t^L(F) + 4$. The desired result now follows by applying the inductive hypothesis to the grid $F = P_2 \,\square\, P_{n-5}$. $\square$

For $m \ge 3$, we have yet to determine the locating-total domination number in the grid graph $P_m \,\square\, P_n$. We consider here the special case when $m = 3$. For $k \ge 1$, let $G_k = P_3 \,\square\, P_n$, where $n = 11k$, and let $V(G_k) = \cup_{i=1}^{n} \{a_i, b_i, c_i\}$, where $a_1 a_2 \ldots a_n$, $b_1 b_2 \ldots b_n$ and $c_1 c_2 \ldots c_n$ are paths P_n and where $a_i b_i c_i$ is a path P_3 for $i = 1, 2, \ldots, n$. Let

$$A_k = \bigcup_{i=0}^{k-1} \{a_{11i+2}, a_{11i+6}, a_{11i+8}\} \quad \text{and} \quad C_k = \bigcup_{i=0}^{k-1} \{c_{11i+4}, c_{11i+6}, c_{11i+10}\}$$

and let

$$B_k = \bigcup_{i=0}^{k-1} \{b_{11i+1}, b_{11i+2}, b_{11i+4}, b_{11i+6}, b_{11i+8}, b_{11i+10}, b_{11i+11}\}.$$

Then, $S_k = A_k \cup B_k \cup C_k$ is a LTD-set in G_k, and so $\gamma_t^L(G_k) \le 13k = 13n/11$. In the special case when $k = 2$, the LTD-set is indicated in Fig. 3, albeit without the vertex labels. Hence we have the following observation.

Observation 18. *For $n \equiv 0 \pmod{11}$, we have $\gamma_t^L(P_3 \,\square\, P_n) \le \frac{13}{11} n$.*

For small values of n, namely $1 \le n \le 12$, we can show that $\gamma_t^L(P_3 \,\square\, P_n) = \left\lceil \frac{13}{11} n \right\rceil$. However we have yet to determine[1] the locating-total domination number of $P_3 \,\square\, P_n$ for $n \ge 13$.

[1] We remark that subsequent to our paper being accepted Ville Junnila [10] informed us that they have determined the optimal density of the infinite grid of height 3 to be 7/18.

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