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Multiobjective Fractional Symmetric Duality in Mathematical Programming with (C, G_f)-Invexity Assumptions

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Abstract: In this paper, a new class of (C, G_f)-invex functions introduce and give nontrivial numerical examples which justify exist such type of functions. Also, we construct generalized convexity definitions (such as, (F, G_f)-invexity, C-convex etc.). We consider Mond–Weir type fractional symmetric dual programs and derive duality results under (C, G_f)-invexity assumptions. Our results generalize several known results in the literature.

Keywords: symmetric duality; multiobjective; fractional programming; (C, G_f)-invexity

1. Introduction

The goal of optimization is to find the best value for each variable in order to achieve satisfactory performance. Optimization is an active and fast growing research area and has a great impact on the real world. In most real life problems, decisions are made taking into account several conflicting criteria, rather than by optimizing a single objective. Such a problem is called multiobjective programming. Problems of multiobjective programming are widespread in mathematical modelling of real world systems problems for a very broad range of applications.

In 1981, Hanson [1] introduced the concept of invexity which is an extension of differentiable convex function and proved the sufficiency of Kuhn-Tucker conditions. Antczak [2] introduced the concept of G-invex functions and derived some optimality conditions for constrained optimization problems under G-invexity. In [3], Antczak extended the above notion by defining a vector valued G_f-invex function and proved necessary and sufficient optimality conditions for a multiobjective nonlinear programming problem. Recently, Kang et al. [4] defined G-invexity for a locally Lipchitz function and obtained optimality conditions for multiobjective programming using these functions. Many researchers have worked related to the same area [5–7].

In the last several years, various optimality and duality results have been obtained for multiobjective fractional programming problems. Bector and Chandra Motivated by various concepts of generalized convexity. Ferrara and Stefaneseu [8] used the (ϕ, ρ)-invexity to discuss the optimality conditions and duality results for multiobjective programming problem. Further, Stefaneseu and Ferrara [9] introduced a new class of $(\phi, \rho)^{\omega}$-invexity for a multiobjective program and established optimality conditions and duality theorems under these assumptions.

In this article, we have introduced various definitions (C, G_f)-invexity/(F, G_f)-invexity and constructed nontrivial numerical examples illustrates the existence of such functions. We considered a pair of multiobjective Mond–Weir type symmetric fractional primal-dual problems. Further, under the (C, G_f)-invexity assumptions, we derive duality results.

2. Preliminaries and Definitions

Consider the following vector minimization problem:

$$\text{(MP)} \qquad \text{Minimize } f(x) = \Big\{ f_1(x), f_2(x), ..., f_k(x) \Big\}^T$$

$$\text{Subject to } X^0 = \{x \in X \subset R^n : g_j(x) \leq 0,\ j = 1, 2, ..., m\}$$

where $f = \{f_1, f_2, ..., f_k\} : X \to R^k$ and $g = \{g_1, g_2, ..., g_m\} : X \to R^m$ are differentiable functions defined on X.

Definition 1 ([10])**.** *A point $\bar{x} \in X^0$ is said to be an efficient solution of (MP) if there exists no other $x \in X^0$ such that $f_r(x) < f_r(\bar{x})$, for some $r = 1, 2, ..., k$ and $f_i(x) \leq f_i(\bar{x})$, for all $i = 1, 2, ..., k$.*

Let $f = (f_1, ..., f_k) : X \to R^k$ be a differentiable function defined on open set $\phi \neq X \subseteq R^n$ and $I_{f_i}(X), i = 1, 2, ..., k$ be the range of f_i.

Definition 2 ([11])**.** *Let $C : X \times X \times R^n \to R$ $(X \subseteq R^n)$ be a function which satisfies $C_{x,u}(0) = 0, \forall (x, u) \in X \times X$. Then, the function C is said to be convex on R^n with respect to third argument iff for any fixed $(x, u) \in X \times X$,*

$$C_{x,u}(\lambda x_1 + (1-\lambda)x_2) \leq \lambda C_{x,u}(x_1) + (1-\lambda) C_{x,u}(x_2),\ \forall \lambda \in (0, 1),\ \forall x_1, x_2 \in R^n.$$

Now, we introduce the definition of C-convex function:

Definition 3 ([12])**.** *The function f is said to be C-convex at $u \in X$ such that $\forall x \in X$,*

$$f_i(x) - f_i(u) \geq C_{x,u}[\nabla_x f_i(u)],\ \forall\, i = 1, 2, ..., k.$$

If the above inequality sign changes to $\leq$, then f is called C-concave at $u \in X$.

Definition 4. *The function f is said to be G_f-convex at $u \in X$ if there exist a differentiable function $G_f = (G_{f_1}, G_{f_2}, ..., G_{f_k}) : R \to R^k$ such that every component $G_{f_i} : I_{f_i}(X) \to R$ is strictly increasing on the range of I_{f_i} such that $\forall\, x \in X$,*

$$G_{f_i}(f_i(x)) - G_{f_i}(f_i(u)) \geq (x-u) G'_{f_i}(f_i(u)) \nabla_x f_i(u),\ \forall\, i = 1, 2, ..., k.$$

If the above inequality sign changes to $\leq$, then f is called G_f-concave at $u \in X$.

Definition 5. *A functional $F : X \times X \times R^n \to R$ is said to be sublinear with respect to the third variable if for all $(x, u) \in X \times X$,*

(i) $\quad F_{x,u}(a_1 + a_2) \leq F_{x,u}(a_1) + F_{x,u}(a_2)$, *for all* $a_1,\ a_2 \in R^n$,

(ii) $F_{x,u}(\alpha a) = \alpha F_{x,u}(a)$, *for all* $\alpha \in R_+$ *and* $a \in R^n$.

Now, we introduce the definition of a differentiable vector valued (F, G_f)-invex function.

Definition 6. *The function* f*is said to be* (F, G_f)*-invex at* $u \in X$ *if there exist sublinear functional* F *and a differentiable function* $G_f = (G_{f_1}, G_{f_2}, ..., G_{f_k}) : R \to R^k$ *such that every component* $G_{f_i} : I_{f_i}(X) \to R$ *is strictly increasing on the range of* I_{f_i} *such that* $\forall\ x \in X$,

$$G_{f_i}(f_i(x)) - G_{f_i}(f_i(u)) \geq F_{x,u}[G'_{f_i}(f_i(u))\nabla_x f_i(u)],\ \forall\ i = 1, 2, ..., k.$$

If the above inequality sign changes to $\leq$*I* f *is called* (F, G_f)*-incave at* $u \in X$.

Next, we introduce the definition of (C, G_f)-invex function:

Definition 7. *The function* f *is said to be* (C, G_f)*-invex at* $u \in X$ *if there exist convex function* C *and a differentiable function* $G_f = (G_{f_1}, G_{f_2}, ..., G_{f_k}) : R \to R^k$ *such that every component* $G_{f_i} : I_{f_i}(X) \to R$ *is strictly increasing on the range of* I_{f_i} *such that* $\forall\ x \in X$,

$$G_{f_i}(f_i(x)) - G_{f_i}(f_i(u)) \geq C_{x,u}[G'_{f_i}(f_i(u))\nabla_x f_i(u)],\ \forall\ i = 1, 2, ..., k.$$

Definition 8. *Let* $f : X \to R^k$ *be a vector-valued differentiable function. If there exist sublinear functional* F *and a differentiable function* $G_f = (G_{f_1}, G_{f_2}, ..., G_{f_k}) : R \to R^k$ *such that every component* $G_{f_i} : I_{f_i}(X) \to R$ *is strictly increasing on the range of* I_{f_i} *and a vector valued function* $\eta : X \times X \to R^n$ *such that* $\forall\ x \in X$ *and* $p_i \in R^n$,

$$F_{x,u}[G'_{f_i}(f_i(u))\nabla_x f_i(u)] \geq 0 \Rightarrow G_{f_i}(f_i(x)) - G_{f_i}(f_i(u)) \geq 0, \text{ for all } i = 1, 2, ..., k,$$

then f *is called* (F, G_f)*-pseudoinvex at* $u \in X$ *with respect to* η.

If the above inequalities sign changes to $\leq$, *then* f *is called* (F, G_f)*-incave/* (F, G_f)*-pseudoincave at* $u \in X$.

Definition 9. *Let* $f : X \to R^k$ *be a vector-valued differentiable function. If there exist convex function* C *and a differentiable function* $G_f = (G_{f_1}, G_{f_2}, ..., G_{f_k}) : R \to R^k$ *such that every component* $G_{f_i} : I_{f_i}(X) \to R$ *is strictly increasing on the range of* I_{f_i} *and a vector valued function* $\eta : X \times X \to R^n$ *such that* $\forall\ x \in X$ *and* $p_i \in R^n$,

$$C_{x,u}[G'_{f_i}(f_i(u))\nabla_x f_i(u)] \geq 0 \Rightarrow G_{f_i}(f_i(x)) - G_{f_i}(f_i(u)) \geq 0, \text{ for all } i = 1, 2, ..., k,$$

then f *is called* (C, G_f)*-pseudoinvex at* $u \in X$.

If the above inequalities sign changes to $\leq$, *then* f *is called* (C, G_f)*-incave/* (C, G_f)*-pseudoincave at* $u \in X$.

Now, we give a nontrivial example which is (C, G_f)-invex function, but on the either side the function f cannot hold the definitions like as (F, G_f)-invex, F-convex and C-convex.

Example 1. *Let* $f : [-1, 1] \to R^2$ *be defined as*

$$f(x) = \{f_1(x), f_2(x)\}$$

where $f_1(x) = x^4$, $f_2(x) = $ *arc (tan x) and* $G_f = \{G_{f_1}, G_{f_2}\} : R \to R^2$ *be defined as:*

$$G_{f_1}(t) = t^9 + t^7 + t^3 + 1 \text{ and } G_{f_2}(t) = \tan t.$$

Let $C : X \times X \times R^2 \to R$ *be given as:*

$$C_{x,u}(a) = a^2(x-u).$$

Now, we will show that f *is* (C, G_f)*-invex at* $u = 0$*. For this, we have to claim that*

$$\tau_i = G_{f_i}(f_i(x)) - G_{f_i}(f_i(u)) - C_{x,u}[G'_{f_i}(f_i(u))\nabla_x f_i(u)] \geq 0, \text{ for } i = 1, 2.$$

Substituting the values of f_1, f_2, G_{f_1} *and* G_{f_2} *in the above expressions, we obtain*

$$\tau_1 = x^{36} + x^{28} + x^{12} + 1 - (u^{36} + u^{28} + u^{12} + 1) - C_{x,u}\left[(36u^{35} + 28u^{27} + 12u^{11}) \times 4u^3\right],$$

and

$$\tau_2 = x - u - C_{x,u}\left[1 \times \frac{1}{(1+u^2)}\right]$$

which at $u = 0$ *yield*

$$\tau_1 = x^{36} + x^{28} + x^{12} \text{ and } \tau_2 = 0.$$

Obviously, $\tau_1 \geq 0$*, and* $\tau_2 \geq 0$, $\forall\, x \in [-1, 1]$.
Hence, f *is* (C, G_f)*-invex at* $u = 0 \in [-1, 1]$.
Now, suppose

$$\delta = f_2(x) - f_2(u) - C_{x,u}\left(\nabla_x f_2(u)\right)$$

or

$$\delta = arc\ (tan\ x) - arc\ (tan\ u) - C_{x,u}\left[\frac{1}{(1+u^2)}\right]$$

which at $u = 0$ *yields*

$$\delta = arc\ (tan\ x) - 1.$$

This expression may not be non-negative for all $x \in [-1, 1]$*. For instance at* $x = 1 \in [-1, 1]$,

$$\delta = \frac{\pi}{4} - 1 < 0.$$

Therefore, f_2 *is not C-convex at* $u = 0$*. Hence,* $f = (f_1,\ f_2)$ *is not C-convex at* $u = 0 \in [-1, 1]$.
Finally, $C_{x,u}$ *is not sublinear in its third position. Hence, function* f *is neither F nor* (F, G_f)*-invex functions.*

3. G-Mond-Weir Type Primal-Dual Model

In this section, we consider the following pair of multiobjective fractional symmetric primal-dual programs:

(MFP) Minimize $L(x,y) = \left\{\frac{G_{f_1}(f_1(x,y))}{G_{g_1}(g_1(x,y))}, \frac{G_{f_2}(f_2(x,y))}{G_{g_2}(g_2(x,y))}, ..., \frac{G_{f_k}(f_k(x,y))}{G_{g_k}(g_k(x,y))}\right\}$

subject to

$$\sum_{i=1}^{k} \lambda_i \left[G'_{f_i}(f_i(x,y))\nabla_y f_i(x,y) - \frac{G_{f_i}(f_i(x,y))}{G_{g_i}(g_i(x,y))}(G'_{g_i} g_i(x,y))\nabla_y g_i(x,y)\right] \leq 0,$$

$$y^T\sum_{i=1}^{k}\lambda_i\left[G'_{f_i}(f_i(x,y))\nabla_y f_i(x,y)-\frac{G_{f_i}(f_i(x,y))}{G_{g_i}(g_i(x,y))}(G'_{g_i}(g_i(x,y))\nabla_y g_i(x,y))\right]\geq 0,$$

$$\lambda>0,\ \lambda^T e=1.$$

(MFD) Maximize $M(u,v)=\left\{\frac{G_{f_1}(f_1(u,v))}{G_{g_1}(g_1(u,v))},\ \frac{G_{f_2}(f_2(u,v))}{G_{g_2}(g_2(u,v))},...,\frac{G_{f_k}(f_k(u,v))}{G_{g_k}(g_k(u,v))}\right\}$
subject to

$$\sum_{i=1}^{k}\lambda_i\left[G'_{f_i}(f_i(u,v))\nabla_x f_i(u,v)-\frac{G_{f_i}(f_i(u,v))}{G_{g_i}(g_i(u,v))}(G'_{g_i}(g_i(u,v))\nabla_x g_i(u,v))\right]\geq 0,$$

$$u^T\sum_{i=1}^{k}\lambda_i\left[G'_{f_i}(f_i(u,v))\nabla_x f_i(u,v)-\frac{G_{f_i}(f_i(u,v))}{G_{g_i}(g_i(u,v))}(G'_{g_i}(g_i(u,v))\nabla_x g_i(u,v))\right]\leq 0,$$

$$\lambda>0,\ \lambda^T e=1.$$

$G_{f_i}:I_{f_i}\to R$ and $G_{g_i}:I_{g_i}\to R$ are differentiable strictly increasing functions on their domains. It is assumed that in the feasible regions, the numerators are nonnegative and denominators are positive.

Now, Let $U=(U_1,U_2,...,U_k)$ and $V=(V_1,V_2,...,V_k)$. Then, we can express the programs (MFP) and (MFD) equivalently as:

$(MFP)_U$ Minimize U
subject to

$$G_{f_i}(f_i(x,y))-U_iG_{g_i}(g_i(x,y))=0,\ i=1,2,...,k, \tag{1}$$

$$\sum_{i=1}^{k}\lambda_i\left[(G'_{f_i}(f_i(x,y))\nabla_y f_i(x,y))-U_i\left(G'_{g_i}(g_i(x,y))\nabla_y g_i(x,y)\right)\right]\leq 0, \tag{2}$$

$$y^T\sum_{i=1}^{k}\lambda_i\left[(G'_{f_i}(f_i(x,y))\nabla_y f_i(x,y))-U_i\left(G'_{g_i}(g_i(x,y))\nabla_y g_i(x,y)\right)\right]\geq 0, \tag{3}$$

$$\lambda>0,\ \lambda^T e=1. \tag{4}$$

$(MFD)_V$ Minimize V
subject to

$$G_{f_i}(f_i(u,v))-V_i\left(G_{g_i}(g_i(u,v))\right)=0,\ i=1,2,...,k, \tag{5}$$

$$\sum_{i=1}^{k}\lambda_i\left[(G'_{f_i}(f_i(u,v))\nabla_x f_i(u,v))-V_i\left(G'_{g_i}(g_i(u,v))\nabla_x g_i(u,v)\right)\right]\geq 0, \tag{6}$$

$$u^T \sum_{i=1}^{k} \lambda_i \left[(G'_{f_i}(f_i(u,v)) \nabla_x f_i(u,v)) - V_i (G'_{g_i}(g_i(u,v)) \nabla_x g_i(u,v)) \right] \leq 0, \tag{7}$$

$$\lambda > 0,\ \lambda^T e = 1. \tag{8}$$

Next, we prove duality theorems for $(MFP)_U$ and $(MFP)_V$, which one equally apply to (MFP) and (MFD), respectively.

Theorem 1. *(Weak duality). Let (x, y, U, λ) and (u, v, V, λ) be feasible for (MFP)$_U$ and (MFD)$_V$, respectively. Let*

(i) $f(.,v)$ *be* (C, G_f)*-invex at u for fixed v,*
(ii) $g(.,v)$ *be* (C, G_g)*-incave at u for fixed v,*
(iii) $f(x,.)$ *be* $(\bar{C}, G_f)$*-incave at y for fixed x,*
(iv) $g(x,.)$ *be* $(\bar{C}, G_f)$*-invex at y for fixed x,*
(v) $\sum_{i=1}^{k} \lambda_i [1 - U_i] > 0$ *and* $\sum_{i=1}^{k} \lambda_i [1 - V_i] > 0,$
(vi) $G_{g_i}(g_i(x,v)) > 0,\ \forall\ i = 1, 2, ..., k,$
(vii) $C_{x,u}(a) + a^T u \geq 0,\ \forall a \geq 0$ *and* $\bar{C}_{v,y}(b) + b^T y \geq 0,\ \forall b \geq 0,$

where $C : R^n \times R^n \times R^n \to R$ *and* $\bar{C} : R^m \times R^m \times R^m \to R.$

Then, $U \nleq V.$

Proof. By hypotheses (i) and (ii), we have

$$G_{f_i}(f_i(x,v)) - G_{f_i}(f_i(u,v)) \geq C_{x,u}\Big(G'_{f_i}(f_i(u,v)) \nabla_x f_i(u,v) \Big) \tag{9}$$

and

$$-G_{g_i}(g_i(x,v)) + G_{g_i}(g_i(u,v)) \geq -C_{x,u}\Big(G'_{g_i}(g_i(u,v)) \nabla_x g_i(u,v) \Big). \tag{10}$$

Using (v), $\lambda > 0$, $\frac{\lambda_i}{\tau}$, and $\frac{\lambda_i V_i}{\tau}$, where $\tau = \sum_{i=1}^{k} \lambda_i (1 - V_i)$ and (9)–(10), respectively, we obtain

$$\frac{\lambda_i}{\tau}\Big(G_{f_i}(f_i(x,v)) - G_{f_i}(f_i(u,v)) \Big) \geq \frac{\lambda_i}{\tau} C_{x,u}\Big(G'_{f_i}(f_i(u,v)) \nabla_x f_i(u,v) \Big).$$

and

$$\frac{\lambda_i V_i}{\tau}\Big[- G_{f_i}(g_i(x,v)) + G_{f_i}(g_i(u,v)) \Big] \geq -\frac{\lambda_i V_i}{\tau} C_{x,u}\Big(G'_{f_i}(g_i(u,v)) \nabla_x g_i(u,v) \Big).$$

Now, summing over i and adding the above two inequalities and using convexity of $C_{x,u}$, we have

$$\sum_{i=1}^{k} \frac{\lambda_i}{\tau}\Big[G_{f_i}(f_i(x,v)) - G_{f_i}(f_i(u,v)) \Big] + \sum_{i=1}^{k} \frac{\lambda_i V_i}{\tau}\Big[- G_{g_i}(g_i(x,v)) + G_{g_i}(g_i(u,v)) \Big]$$

$$\geq C_{x,u}\left[\sum_{i=1}^{k} \frac{\lambda_i}{\tau}\Big((G'_{f_i}(f_i(u,v)) \nabla_x f_i(u,v)) - V_i (G'_{g_i}(g_i(u,v)) \nabla_x g_i(u,v)) \Big) \right]. \tag{11}$$

Now, from (6), we have

$$a = \sum_{i=1}^{k} \frac{\lambda_i}{\tau}[(G'_{f_i}(f_i(u,v))\nabla_x f_i(u,v) - V_i(G'_{g_i}(g_i(u,v))\nabla_x g_i(u,v)] \geq 0.$$

Hence, for this a, $C_{x,u}(a) \geq -u^T a \geq 0$ $\big($from $(vii)\big)$. Using this in (11), we obtain

$$\sum_{i=1}^{k} \lambda_i \big(G_{f_i}(f_i(x,v)) - G_{f_i}(f_i(u,v))\big) + \sum_{i=1}^{k} \lambda_i V_i \big[- G_{g_i}(g_i(x,v)) + G_{g_i}(g_i(u,v))\big] \geq 0.$$

Using (5) in above inequality, we get

$$\sum_{i=1}^{k} \lambda_i [G_{f_i}(f_i(x,v)) - V_i G_{g_i} g_i(x,v)] \geq 0. \tag{12}$$

From hypotheses $(iii) - (v)$ and from the condition (vii), for

$$b = -\sum_{i=1}^{k} \frac{\lambda_i}{\tau}[G'_{f_i}(f_i(x,y))\nabla_y f_i(x,y) - U_i(G'_{g_i}(g_i(x,y))\nabla_y g_i(x,y))] \geq 0,$$

we get

$$\sum_{i=1}^{k} \lambda_i [-G_{f_i}(f_i(x,v)) + U_i(G_{g_i}(g_i(x,v))] \geq 0. \tag{13}$$

Adding the inequalities (12) and (13), we get

$$\sum_{i=1}^{k} \lambda_i (U_i - V_i)(G_{g_i}(g_i(x,v)) \geq 0. \tag{14}$$

Since $\lambda > 0$ and using (vi), it follows that $U \nleq V$. This completes the proof. □

Theorem 2. *(Weak duality). Let (x, y, U, λ) and (u, v, V, λ) be feasible for (MFP)$_U$ and (MFD)$_V$, respectively. Let*

- (i) *$f(.,v)$ be (C, G_f)-pseudoinvex at u for fixed v,*
- (ii) *$g(.,v)$ be (C, G_g)-pseudoincave at u for fixed v,*
- (iii) *$f(x,.)$ be $(\bar{C}, G_f)$-pseudoincave at y for fixed x,*
- (iv) *$g(x,.)$ be $(\bar{C}, G_f)$-pseudoinvex at y for fixed x,*
- (v) *$\sum_{i=1}^{k} \lambda_i [1 - U_i] > 0$ and $\sum_{i=1}^{k} \lambda_i [1 - V_i] > 0$,*
- (vi) *$G_{g_i}(g_i(x,v)) > 0, \ \forall \ i = 1, 2, ..., k$,*
- (vii) *$C_{x,u}(a) + a^T u \geq 0, \ \forall a \geq 0$ and $\bar{C}_{v,y}(b) + b^T y \geq 0, \ \forall b \geq 0$,*

where $C : R^n \times R^n \times R^n \to R$ and $\bar{C} : R^m \times R^m \times R^m \to R$.

Then, $U \nleq V$.

Proof. The proof follows on the lines of Theorem 2. □

Theorem 3. *(Strong duality). Let $(\bar{x}, \bar{y}, \bar{U}, \bar{\lambda})$ be an efficient solutions of (MFP)$_U$ and fix $\lambda = \bar{\lambda}$ in $(MFD)_V$. If the following conditions hold:*

(i) the matrix

$$\sum_{i=1}^{k} \bar{\lambda}_i [G''_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y}) \left(\nabla_y f_i(\bar{x},\bar{y})\right)^T + G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_{yy} f_i(\bar{x},\bar{y})$$
$$-\bar{U}_i(G''_{g_i}(g_i(\bar{x},\bar{y}))\nabla_y g_i(\bar{x},\bar{y}) \left(\nabla_y g_i(\bar{x},\bar{y})\right)^T + G'_{g_i}(g_i(\bar{x},\bar{y}))\nabla_{yy} g_i(\bar{x},\bar{y}))]$$

is positive definite or negative definite,

(ii) the vectors

$$\Big((G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y})) - \bar{U}_i(G'_{g_i}(g_i(\bar{x},\bar{y}))\nabla_y g_i(\bar{x},\bar{y}))\Big)_{i=1}^{k}$$

are linearly independent,

(iii) $\bar{U}_i > 0,\ i = 1,2,...,k,$

then, $(\bar{x},\bar{y},\bar{U},\bar{\lambda})$ *is feasible solution for* $(MFD)_V$*. Furthermore, if the hypotheses of Theorem 2 and 3 hold, then* $(\bar{x},\bar{y},\bar{U},\bar{\lambda})$ *is an efficient solution of* $(MFD)_V$ *and the objective functions have same values.*

Proof. Since $(\bar{x},\bar{y},\bar{U},\bar{\lambda})$ is an efficient solution of $(MFD)_U$, therefore by the Fritz John necessary optimality conditions [13], there exist $\alpha \in R^k, \beta \in R^k, \gamma \in R_+, \delta \in R, \xi \in R^k,\ i = 1,2,...,k$ such that

$$\sum_{i=1}^{k} \beta_i((G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_x f_i(\bar{x},\bar{y})) - \bar{U}_i(G'_{g_i}(g_i(\bar{x},\bar{y}))\nabla_x g_i(\bar{x},\bar{y})))$$
$$+ (y-\delta\bar{y})^T \sum_{i=1}^{k} \bar{\lambda}_i [G''_{f_i}(f_i(\bar{x},\bar{y}))\nabla_x f_i(\bar{x},\bar{y})(\nabla_y f_i(\bar{x},\bar{y}))^T + G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_{xy} f_i(\bar{x},\bar{y})]$$
$$-\bar{U}_i[G''_{g_i}(g_i(\bar{x},\bar{y}))\nabla_x g_i(\bar{x},\bar{y})(\nabla_y g_i(\bar{x},\bar{y}))^T + G'_{g_i}(g_i(\bar{x},\bar{y}))\nabla_{xy} g_i(\bar{x},\bar{y})] = 0, \tag{15}$$

$$\sum_{i=1}^{k} (\beta_i - \delta\bar{\lambda}_i)(G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y}) - \bar{U}_i(G'_{g_i}(g_i(\bar{x},\bar{y}))\nabla_y g_i(\bar{x},\bar{y}))$$
$$+ (\gamma-\delta\bar{y})^T \sum_{i=1}^{k} \bar{\lambda}_i [G''_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y})(\nabla_y f_i(\bar{x},\bar{y}))^T + G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_{yy} f_i(\bar{x},\bar{y})]$$
$$-\bar{U}_i[G''_{g_i}(g_i(\bar{x},\bar{y}))\nabla_y g_i(\bar{x},\bar{y})(\nabla_y g_i(\bar{x},\bar{y}))^T + G'_{g_i}(g_i(\bar{x},\bar{y}))\nabla_{yy} g_i(\bar{x},\bar{y})] = 0, \tag{16}$$

$$\alpha_i - \beta_i(G_{g_i}(g_i(\bar{x},\bar{y}))) - (\gamma-\delta\bar{y})\bar{\lambda}_i(G'_{g_i}(g_i(\bar{x},\bar{y}))\nabla_y g_i(\bar{x},\bar{y})) = 0,\ i = 1,2,...,k, \tag{17}$$

$$(\gamma-\delta\bar{y})^T[(G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y}))$$
$$-\bar{U}_i(G'_{g_i}(g_i(\bar{x},\bar{y}))\nabla_y g_i(\bar{x},\bar{y}))] - \xi_i = 0,\ i = 1,2,...,k, \tag{18}$$

$$\gamma^T \sum_{i=1}^{k} \bar{\lambda}_i [G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y}) - \bar{U}_i(G'_{g_i}(g_i(\bar{x},\bar{y}))\nabla_y g_i(\bar{x},\bar{y}))] = 0, \tag{19}$$

$$\delta\bar{y}^T\sum_{i=1}^{k}\bar{\lambda}_i[G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y})-\bar{U}_i(G'_{g_i}(g_i(\bar{x},\bar{y}))\nabla_y g_i(\bar{x},\bar{y}))]=0, \tag{20}$$

$$\bar{\lambda}^T\xi=0, \tag{21}$$

$$(\alpha,\delta,\xi)\geq 0,\ (\alpha,\beta,\gamma,\delta,\xi)\neq 0. \tag{22}$$

Since $\bar{\lambda}>0$ and $\bar{\xi}\geq 0$, (21) implies that $\bar{\xi}=0$.
Post-multiplication $(\gamma-\delta\bar{y})$ in (16) and using (18) and $\xi=0$, we get

$$(\gamma-\delta\bar{y})^T\sum_{i=1}^{k}\bar{\lambda}_i(G''_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y})(\nabla_y f_i(\bar{x},\bar{y}))^T+G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_{yy}f_i(\bar{x},\bar{y})$$

$$-\bar{U}_i(G''_{g_i}(g_i(\bar{x},\bar{y}))\nabla_y g_i(\bar{x},\bar{y})(\nabla_y g_i(\bar{x},\bar{y}))^T+G'_{g_i}(g_i(\bar{x},\bar{y}))\nabla_{yy}g_i(\bar{x},\bar{y}))(\gamma-\delta\bar{y})=0, \tag{23}$$

which from hypothesis (i) yields

$$\gamma=\delta\bar{y}. \tag{24}$$

Using (24) in (16), we have

$$\sum_{i=1}^{k}(\beta_i-\delta\bar{\lambda}_i)[G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y})-\bar{U}_i(G'_{g_i}(g_i(\bar{x},\bar{y}))\nabla_y g_i(\bar{x},\bar{y}))]=0.$$

It follows from hypothesis (ii) that

$$\beta_i=\delta\bar{\lambda}_i,\ i=1,2,...,k. \tag{25}$$

Now, we claim that $\beta_i\neq 0, \forall\, i$. Otherwise, if $\beta_{t_0}=0$, for some $i=t_0$, then from (25), since $\bar{\lambda}>0$, we have $\delta=0$. Again from (25), $\beta_i=0,\ \forall\, i$. Thus from (17), we get $\alpha_i=0, \forall\, i$. Also from (24), $\gamma=0$. This contradicts (22). Hence, $\beta_i\neq 0$, for all i. Further, if $\beta_i<0$, for any i, then from (25), $\delta<0$, which again contradicts (22). Hence, $\beta_i>0,\ \forall\, i$.

Further, using (22) and (25) in (15), we get

$$\sum_{i=1}^{k}\bar{\lambda}_i[(G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_x f_i(\bar{x},\bar{y}))-\bar{U}_i(G'_{g_i}(g_i(\bar{x},\bar{y}))\nabla_x g_i(\bar{x},\bar{y}))]=0, \tag{26}$$

and

$$\bar{x}^T\sum_{i=1}^{k}\bar{\lambda}_i[(G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_x f_i(\bar{x},\bar{y}))-\bar{U}_i(G'_{g_i}(g_i(\bar{x},\bar{y}))\nabla_x g_i(\bar{x},\bar{y}))]=0. \tag{27}$$

Next, it follows that

$$(G_{f_i}(f_i(\bar{x},\bar{y})))-\bar{U}_i(G_{g_i}(g_i(\bar{x},\bar{y})))=0,\ i=1,2,...,k. \tag{28}$$

This together with (26), (27) and (28) shows that $(\bar{x}, \bar{y}, \bar{U}, \bar{\lambda})$ is feasible solution of $(\text{MFD})_V$. Now, let $(\bar{x}, \bar{y}, \bar{U}, \bar{\lambda})$ be not an efficient solution of $(\text{MFD})_V$. Then, there exists other (u, v, V, λ) is feasible solution of $(\text{MFD})_V$ such that $\bar{U}_i \leq V_i$, $\forall\, i \in K$ and $\bar{U}_j < V_j$, for some $j \in K$. This contradicts the result of the Theorems 2 and 3. Hence, this completes the proof. □

Theorem 4. *(Converse duality). Let $(\bar{u}, \bar{v}, \bar{V}, \bar{\lambda})$ be an efficient solutions of $(MFD)_V$ and fix $\lambda = \bar{\lambda}$ in $(MFP)_U$. If the following conditions hold:*

(i) the matrix

$$\sum_{i=1}^{k} \bar{\lambda}_i [G''_{f_i}(f_i(\bar{u}, \bar{v}))\nabla_x f_i(\bar{u}, \bar{v})\,(\nabla_x f_i(\bar{u}, \bar{v}))^T + G'_{f_i}(f_i(\bar{u}, \bar{v}))\nabla_{xx} f_i(\bar{u}, \bar{v})$$

$$-\bar{V}_i(G''_{g_i}(g_i(\bar{u}, \bar{v}))\nabla_x g_i(\bar{u}, \bar{v})\,(\nabla_x g_i(\bar{u}, \bar{v}))^T + G'_{g_i}(g_i(\bar{u}, \bar{v}))\nabla_{xx} g_i(\bar{u}, \bar{v}))]$$

is positive definite or negative definite,

(ii) the vectors

$$\Big(G'_{f_i}(f_i(\bar{u}, \bar{v}))\nabla_x f_i(\bar{u}, \bar{v}) - \bar{V}_i(G'_{g_i}(g_i(\bar{u}, \bar{v}))\nabla_x g_i(\bar{u}, \bar{v}))\Big)_{i=1}^{k}$$

are linearly independent,

(iii) $\bar{V}_i > 0,\ i = 1, 2, ..., k$,
then, $(\bar{u}, \bar{v}, \bar{V}, \bar{\lambda})$ is feasible solution of $(MFP)_U$. Furthermore, if the assumptions of Theorems 2 and 3 hold, then $(\bar{u}, \bar{v}, \bar{V}, \bar{\lambda})$ is an efficient solution of $(MFP)_U$ and objective functions have equal values.

Proof. The results can be obtained on the lines of Theorem 3. □

4. Conclusions

In this paper, we have considered a new type of nondifferentiable multiobjective fractional symmetric programming problem and derived duality theorems under generalized assumptions. The present work can further be extended to nondifferentiable second order fractional symmetric programming problems over arbitrary cones. This will orient the future task of the authors.

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