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Research Paper

Multivalently meromorphic functions with two fixed points defined by Srivastava-Attiya operator

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Abstract

In the present investigation we define a new class of meromorphic functions on the punctured unit disk $\mathcal{U}^* := \{z \in \mathbb{C} : 0 < |z| < 1\}$ by making use of the Srivastava-Attiya operator. Coefficient inequalities, growth and distortion inequalities, as well as radii of meromorphically starlikeness are obtained. We also establish some results concerning the convolution products and inclusion results.

Keywords: *Convolution(Hadamard product),Hurwitz-Learch zeta function, Integral operator, Meromorphic functions,Starlike function,Convex function,.*

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1 Introduction

Let $\mathcal{M}_p$ denote the class of functions of the form

$$f(z) = \frac{a_p}{z^p} + \sum_{n=1}^{\infty} a_n z^{n-p},\ (a_n \geq 0; p \in \mathbb{N} = \{1, 2, 3 \ldots\}), \tag{1}$$

which are meromorphic and p-valent in the punctured unit disc

$$\mathcal{U}^* = \{z \in \mathbb{C} : 0 < |z| < 1\}.$$

Let $g(z) = \frac{b_p}{z^p} + \sum_{n=1}^{\infty} b_n z^{n-p},\ (b_n \geq 0; p \in \mathbb{N} = \{1, 2, 3 \ldots\})$, then the Convolution (or Hadamard) product of $f(z)$ and $g(z)$ is defined as

$$f(z) * g(z) = (f * g)(z) = \frac{a_p b_p}{z^p} + \sum_{n=1}^{\infty} a_n b_n z^{n-p},\ (a_n, b_n \geq 0; p \in \mathbb{N} = \{1, 2, 3 \ldots\}).$$

Now, we recall a general Hurwitz-Lerch Zeta function $\Phi(z, s, a)$ defined by

$$\Phi(z, s, a) := \sum_{k=0}^{\infty} \frac{z^k}{(k + a)^s}, \tag{2}$$

where $z \in \mathcal{U} = z : |z| < 1$ and $a \in \mathbb{C} \setminus \{\mathbb{Z}_0^-\}; s \in \mathbb{C}, \Re(s) > 1$ and $|z| = 1$ as usual $\mathbb{Z}_0^- := \mathbb{Z} \setminus \{\mathbb{N}\}; (\mathbb{Z} := \{0, \pm1, \pm2, \pm3, ...\}; \mathbb{N} := \{1, 2, 3, ...\})$. Several interesting properties and characteristics of the Hurwitz-Lerch Zeta function $\Phi(z, s, a)$ can be found in the recent investigations by Choi and Srivastava [8], Ferreira and Lopez [10], Garg et al. [12], Lin and Srivastava [13], Lin et al. [14], and others.

For the class of analytic functions denote by $\mathcal{A}$ consisting of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathcal{U}.$$

Srivastava and Attiya [34] introduced and investigated the linear operator

$$\mathcal{J}_b^s : \mathcal{A} \to \mathcal{A}$$

defined in terms of the Hadamard product (or convolution) by

$$\mathcal{J}_b^s f(z) = G_{b,s} * f(z) \tag{3}$$

where $z \in \mathcal{U}; b \in \mathbb{C} \setminus \{\mathbb{Z}_0^-\}; s \in \mathbb{C}$, for convenience,we write

$$G_{s,b}(z) := (1 + b)^s [\Phi(z, s, b) - b^{-s}] \qquad (z \in \mathcal{U}). \tag{4}$$

It is easy to observe from (1), (3) and (4)that

$$\mathcal{J}_b^s f(z) = z + \sum_{n=2}^{\infty} \left(\frac{1 + b}{n + b}\right)^s a_n z^n. \tag{5}$$

It is well known that the Srivastava-Attiya operator $\mathcal{J}_b^s$ contains, among its special cases, the integral operators introduced and investigated earlier by (for example) Alexander [1], Libera [19], Bernardi [6], and Jung et al. [17].

Motivated essentially by the above-mentioned Srivastava-Attiya operator, in this paper for functions $f \in \mathcal{M}_p$ we define the operator

$$\mathcal{J}_b^s : \mathcal{M}_p \to \mathcal{M}_p$$

by the convolution

$$\mathcal{J}_b^s f(z) = G_{b,p}^s * f(z) \quad (z \in \mathcal{U}^*; b \in \mathbb{C} \setminus \{\mathbb{Z}_0^-\}; s \in \mathbb{C}) \tag{6}$$

for convenience,

$$G_{b,p}^s(z) := (1 + b)^s [\Phi_p(z, s, b) - b^{-s}] \quad (z \in \mathcal{U}^*) \tag{7}$$

where

$$\Phi_p(z, s, b) = \frac{1}{b^s} + \frac{z^{-p}}{(1 + b)^s} + \frac{z^{-p+1}}{(2 + b)^s} + ...$$

For $f(z) \in \mathcal{M}_p$ it is easy to observe from the above equations that

$$\mathcal{J}_b^s f(z) = z^{-p} + \sum_{n=1}^{\infty} C_b^s(n) a_n z^{n-p} \tag{8}$$

where

$$C_b^s(n) = \left| \left(\frac{1 + b}{n + 1 + b}\right)^s \right| \tag{9}$$

and (throughout this paper unless otherwise mentioned) the parameters s, b are constrained as

$$b \in \mathbb{C} \setminus \{\mathbb{Z}_0^-\}; s \in \mathbb{C} \quad \text{and} \quad p \in \mathbb{N}.$$

For multivalently meromorphic functions $f \in \mathcal{M}_p$, the normalization

$$z^{1+p} f(z)|_{z=0} = 0 \quad \text{and} \quad z^p f(z)|_{z=0} = 1 \tag{10}$$

is classical. One can obtain interesting results by applying Montel's normalization [25] of the form

$$z^{1+p}f(z)|_{z=0}=0 \quad \text{and} \quad z^{p}f(z)|_{z=\rho}=1 \tag{11}$$

where ρ is a fixed point from the unit disk $\mathcal{U}^*$. Note that if $\rho=0$ the normalization 11 is the classical normalization 10.

Meromorphic multivalent functions have been extensively studied by (for example) Mogra [23],Uralegaddi and Ganigi[36], Uralegaddi and Somanatha [37], Aouf [2], Aouf and Hossen[3],
Srivastava et al.[35] , Owa et al. [27], Joshi and Aouf [15], Joshi and Srivastava [16],Aouf et al.[4],Raina and Srivastava [30], Yang [39], Kulkarni et al.[18],Liu [22]and Liu and Srivastava [20]and [21]). Motivated by the works of Vijaya et. al. [38],we define the following new subclass $\mathcal{M}_b^s(\alpha,\beta)$ of meromorphic starlike functions in the parabolic region of functions in $\mathcal{M}_p$ by making use of the generalized operator $\mathcal{J}_b^s$ with Montel's normalization, to study its characteristic properties (for example coefficient inequalities, growth and distortion inequalities, radii of starlikeness. We also establish some results concerning the convolution products).

For fixed parameters $\alpha \geq \frac{1}{2+\beta}; 0\leq\beta<1$, we let $\mathcal{M}_b^s(\alpha,\beta)$ the meromorphically p-valent function $f\in\mathcal{M}_p$ with two fixed points (or classical normalization) if it satisfies the following

$$\left|\frac{z(\mathcal{J}_b^s f(z))'}{p(\mathcal{J}_b^s f(z))}+\alpha+\alpha\beta\right| \leq \Re\left\{\frac{-z(\mathcal{J}_b^s f(z))'}{p(\mathcal{J}_b^s f(z))}+\alpha-\alpha\beta\right\},(n\in\mathbb{N}_0) \tag{12}$$

where $\mathcal{J}_b^s f(z)$ given by (8).

Further ,we let the subclass $\mathcal{M}_b^s(\alpha,\beta,\rho)$ satisfying the condition (12) with Montel's normalization (11).
In the following section we discuss certain characterization properties for $f\in\mathcal{M}_b^s(\alpha,\beta)$.

2 Properties of the class $\mathcal{M}_b^s(\alpha,\beta)$

Theorem 2.1. *Let $f\in\mathcal{M}_p$, then f is in the class $\mathcal{M}_b^s(\alpha,\beta)$ if and only if*

$$\sum_{n=1}^{\infty} d_n|a_n| \leq p(1-\alpha\beta)a_p \tag{13}$$

where

$$d_n=[n-p(1-\alpha\beta)]C_b^s(n) \tag{14}$$

and

$$\alpha>\frac{1}{2+\beta};\quad 0\leq\beta<1;\ p\in\mathbb{N},\ n\in\mathbb{N}_0.$$

Proof. Suppose that $f\in\mathcal{M}_b^s(\alpha,\beta)$, then by the inequality(12), we have

$$\left|\frac{z(\mathcal{J}_b^s f(z))'}{p(\mathcal{J}_b^s f(z))}+\alpha+\alpha\beta\right| \leq \Re\left\{\frac{-z(\mathcal{J}_b^s f(z))'}{p(\mathcal{J}_b^s f(z))}+\alpha-\alpha\beta\right\},(n\in\mathbb{N}_0)$$

that is,

$$\begin{aligned}\Re\left\{\frac{z(\mathcal{J}_b^s f(z))'}{p(\mathcal{J}_b^s f(z))}+\alpha+\alpha\beta\right\} &\leq \left|\frac{z(\mathcal{J}_b^s f(z))'}{p(\mathcal{J}_b^s f(z))}+\alpha+\alpha\beta\right| \\ &\leq \Re\left\{\frac{-z(\mathcal{J}_b^s f(z))'}{p(\mathcal{J}_b^s f(z))}+\alpha-\alpha\beta\right\}.\end{aligned}$$

Hence,

$$\Re\left\{\frac{z(\mathcal{J}_b^s f(z))'}{p(\mathcal{J}_b^s f(z))}+\alpha\beta\right\} \leq 0.$$

Substituting for $\mathcal{J}_b^s f(z)$ and $(\mathcal{J}_b^s f(z))'$, we get

$$\Re\left(\frac{\frac{-pa_p}{z^p}+\sum_{n=1}^{\infty}(n-p)C_b^s(n)a_nz^{n-p}}{\frac{pa_p}{z^p}+\sum_{n=1}^{\infty}pC_b^s(n)a_nz^{n-p}}+\alpha\beta\right)\le 0.$$

Since $\Re(z)\le|z|$, we have

$$|-pa_p+\sum_{n=1}^{\infty}(n-p)C_b^s(n)a_nz^n+pa_p\alpha\beta+p\alpha\beta\sum_{n=1}^{\infty}C_b^s(n)a_nz^n|\le 0$$

and by letting $|z|\to 1^-$, we get

$$\sum_{n=1}^{\infty}[n-p(1-\alpha\beta)]C_b^s(n)|a_n|\le p(1-\alpha\beta)a_p.$$

In order to prove the converse, we assume that the inequality holds true.Then, if we let $z\in\partial\mathcal{U}$, we find from 1 and 13, that

$$\Re\left\{\frac{z(\mathcal{J}_b^s f(z))'}{p(\mathcal{J}_b^s f(z))}+\alpha\beta\right\}\quad\le\quad 0.$$

$$\Re\left(\frac{\frac{-pa_p}{z^p}+\sum_{n=1}^{\infty}(n-p)C_b^s(n)a_nz^{n-p}}{\frac{pa_p}{z^p}+\sum_{n=1}^{\infty}pC_b^s(n)a_nz^{n-p}}+\alpha\beta\right)\le 0.$$

Since $\Re(z)\le|z|$, we have

$$\sum_{n=1}^{\infty}\frac{[n-p(1-\alpha\beta)]C_b^s(n)|a_n|}{p(1-\alpha\beta)a_p}\le 1.$$

which completes the proof.

For the sake of brevity throughout this paper, we let

$$d_n=[n-p(1-\alpha\beta)]C_b^s(n)\tag{15}$$

and

$$\alpha>\frac{1}{2+\beta};\quad 0\le\beta<1;\ p\in\mathbb{N},\ n\in\mathbb{N}_0,$$

unless otherwise state.

Theorem 2.2. (Coefficient Estimate) *Let* $f\in\mathcal{M}_b^s(\alpha,\beta)$, *then*

$$\sum_{n=1}^{\infty}|a_n|\le\frac{p(1-\alpha\beta)a_p}{d_n}.\tag{16}$$

Theorem 2.3. *Let* $f\in\mathcal{M}_b^s(\alpha,\beta,\rho)$, *then*

$$\sum_{n=1}^{\infty}|a_n|\le\frac{p(1-\alpha\beta)}{d_n+p(1-\alpha\beta)\rho^n}.\tag{17}$$

Proof. Let $f\in\mathcal{M}_b^s(\alpha,\beta,\rho)$. Since $f\in\mathcal{M}_b^s(\alpha,\beta)$ by Theorem 2.1, we have

$$\sum_{n=1}^{\infty}d_n|a_n|\le p(1-\alpha\beta)a_p.\tag{18}$$

For $f\in\mathcal{M}_p$, by Montel's normalization (11),we have

$$z^p(a_pz^{-p}+\sum_{n=1}^{\infty}a_n^{n-p})|_{z=\rho}=1.$$

$$a_p = 1 - \sum_{n=1}^{\infty} a_n \rho^n.$$

Therefore from (13),we have

$$\sum_{n=1}^{\infty} d_n|a_n| \leq p(1-\alpha\beta)(1-\sum_{n=1}^{\infty} a_n\rho^n).$$

$$\sum_{n=1}^{\infty}[d_n + p(1-\alpha\beta)\rho^n]|a_n| \leq p(1-\alpha\beta).$$

Hence

$$\sum_{n=1}^{\infty}|a_n| \leq \frac{p(1-\alpha\beta)}{d_n + p(1-\alpha\beta)\rho^n}.$$

Theorem 2.4. (**Distortion Bounds**) *If* $f \in \mathcal{M}_b^s(\alpha,\beta,\rho)$, *then*

$$\left(\frac{d_1 - p(1-\alpha\beta)r}{d_1 + p(1-\alpha\beta)\rho}\right) r^{-p} \leq |f(z)| \leq \left(\frac{d_1 + p(1+\alpha\beta)r}{d_1 + p(1-\alpha\beta)\rho}\right) r^{-p}, (0 < |z| = r < 1).$$

Proof. Let $f \in \mathcal{M}_b^s(\alpha,\beta)$. Then we find from Theorem 2.3 , that

$$\sum_{n=1}^{\infty}[d_n + p(1-\alpha\beta)\rho^n]|a_n| \leq p(1-\alpha\beta)$$

which yields,

$$\sum_{n=1}^{\infty}|a_n| \leq \frac{p(1-\alpha\beta)}{[d_1 + p(1-\alpha\beta)\rho]}.$$

We have

$$\begin{aligned}
|f(z)| &= |a_p z^{-p} + \sum_{n=1}^{\infty} a_n z^{n-p}| \\
&\leq \left(1 - \sum_{n=1}^{\infty}|a_n|\rho^n + \sum_{n=1}^{\infty}|a_n| r^n\right) r^{-p} \\
&\leq \left(1 - (\rho - r)\sum_{n=1}^{\infty}|a_n|\right) r^{-p} \\
&\leq \left(\frac{d_1 + p(1-\alpha\beta)r}{d_1 + p(1-\alpha\beta)\rho}\right) r^{-p}.
\end{aligned}$$

On the other hand we have,

$$|f(z)| \geq \left(\frac{d_1 - p(1-\alpha\beta)r}{d_1 + p(1-\alpha\beta)\rho}\right) r^{-p}.$$

Hence the proof.

Using classical normalization,(that is by taking $\rho = 0$) we state the following distortion result without proof.

Theorem 2.5. *If* $f \in \mathcal{M}_b^s(\alpha,\beta)$, *then*

$$\left(1 - \frac{p(1-\alpha\beta)r}{d_1}\right) r^{-p} \leq |f(z)| \leq \left(1 + \frac{p(1+\alpha\beta)r}{d_1}\right) r^{-p}, (0 < |z| = r < 1).$$

3 The Radii of Meromorphically Starlikeness

Theorem 3.1. *Let the function $f(z)$ defined by (1) be in the class $\mathcal{M}_b^s(\alpha,\beta)$, then we have $f(z)$ is meromorphically p-valent starlike of order $\mu(0\le\mu<p)$ in the disc $|z|<r_1$, that is,*

$$\Re\left(-\frac{zf'(z)}{f(z)}\right)>\mu,\quad |z|<r_1; 0\le\mu<p; p\in\mathbb{N},$$

where

$$|z|\le\left(\frac{d_n(p-\mu)}{p(n-p+\mu)(1-\alpha\beta)}\right)^{\frac{1}{n}}. \tag{19}$$

Proof. Let $f(z)=a_pz^{-p}+\sum_{n=1}^{\infty}a_nz^{n-p}$. Then we easily get

$$\left|\frac{\frac{zf'(z)}{f(z)}+p}{\frac{zf'(z)}{f(z)}-p+2\mu}\right| \quad\le\quad \frac{\sum_{n=1}^{\infty}na_n|z|^n}{2(p-\mu)a_p+\sum_{n=1}^{\infty}(n-2p+2\mu)a_n|z|^n}.$$

Thus, we have the desired inequality

$$\left|\frac{\frac{zf'(z)}{f(z)}+p}{\frac{zf'(z)}{f(z)}-p+2\mu}\right|\le 1,\quad\text{if,}\quad \sum_{n=1}^{\infty}\frac{(n-p+\mu)}{|p-\mu|a_p}|a_n||z|^n\le 1. \tag{20}$$

Since $f\in\mathcal{M}_b^s(\alpha,\beta)$ from Theorem 2.1 , we have

$$\sum_{n=1}^{\infty}\frac{d_n|a_n|}{p(1-\alpha\beta)a_p}\le 1. \tag{21}$$

From (20) and (21)

$$\frac{n-p+\mu}{|p-\mu|a_p}|z|^n \quad\le\quad \left\{\frac{d_n}{p(1-\alpha\beta)a_p}\right\}$$

$$|z|\le\left\{\frac{d_n(p-\mu)}{p(n-p+\mu)(1-\alpha\beta)}\right\}^{\frac{1}{n}}.$$

which completes proof .

4 Convolution properties

For the function

$$f_j(z)=a_{p,j}z^{-p}+\sum_{n=1}^{\infty}|a_{n,j}|z^{n-p},\quad (j=1,2;p\in\mathbb{N}) \tag{22}$$

we denote by $(f_1*f_2)(z)$ the Hadamard product(or Convolution) of the functions $f_1(z)$ and $f_2(z)$, that is,

$$(f_1*f_2)(z)=a_{p,1}a_{p,2}z^{-p}+\sum_{n=1}^{\infty}|a_{n,1}||a_{n,2}|z^{n-p}. \tag{23}$$

Theorem 4.1. *For the function $f_j(z)\quad(j=1,2)$ defined by (22)be in the class $\mathcal{M}_b^s(\alpha,\beta)$.Then $(f_1*f_2)(z)\in\mathcal{M}_b^s(\alpha,\delta)$ where*

$$\delta\le\frac{1}{\alpha}\left(1-\frac{p(1-\alpha\beta)^2C_b^s(1)}{p^2(1-\alpha\beta)^2C_b^s(1)+d_1^2}\right)$$

where $d_1=[1-p(1-\alpha\beta)]C_b^s(1)$ and $C_b^s(1)=\left|\left(\frac{1+b}{2+b}\right)^s\right|$.

Proof. Let $f_1(z) = a_{p,1}z^{-p} + \sum_{n=1}^{\infty} |a_{n,1}|z^{n-p}$ and $f_2(z) = a_{p,2}z^{-p} + \sum_{n=1}^{\infty} |a_{n,2}|z^{n-p}$ be in the class $\mathcal{M}_b^s(\alpha, \beta)$. Then by Theorem 2.1 , we have

$$\sum_{n=1}^{\infty} \frac{d_n}{p(1-\alpha\beta)a_{p,1}}|a_{n,1}| \leq 1.$$
$$\sum_{n=1}^{\infty} \frac{d_n}{p(1-\alpha\beta)a_{p,2}}|a_{n,2}| \leq 1.$$

Employing the technique used earlier by Schild and Silverman [32],we need to find smallest δ such that

$$\sum_{n=1}^{\infty} \frac{(n-p+p\alpha\delta)C_b^s(n)}{p(1-\alpha\delta)a_{p,1}a_{p,2}}|a_{n,1}||a_{n,2}| \leq 1. \tag{24}$$

By Cauchy-Schwarz inequality, we have

$$\sum_{n=1}^{\infty} \frac{d_n}{p(1-\alpha\beta)\sqrt{a_{p,1}a_{p,2}}}\sqrt{|a_{n,1}||a_{n,2}|} \leq 1 \tag{25}$$

then

$$\frac{(n-p+p\alpha\delta)C_b^s(n)|a_{n,1}||a_{n,2}|}{p(1-\alpha\delta)a_{p,1}a_{p,2}} \leq \frac{d_n}{p(1-\alpha\beta)\sqrt{a_{p,1}a_{p,2}}}\sqrt{|a_{n,1}||a_{n,2}|}. \tag{26}$$

Hence that,

$$\sqrt{|a_{n,1}||a_{n,2}|} \leq \frac{d_n(1-\alpha\delta)\sqrt{a_{p,1}a_{p,2}}}{(n-p+p\alpha\delta)C_b^s(n)(1-\alpha\beta)}. \tag{27}$$

we know that

$$\sqrt{|a_{n,1}||a_{n,2}|} \leq \frac{p(1-\alpha\beta)\sqrt{a_{p,1}a_{p,2}}}{d_n}. \tag{28}$$

from (27) and (28), we have

$$\frac{p(1-\alpha\beta)}{d_n} \leq \frac{d_n(1-\alpha\delta)}{(n-p+p\alpha\delta)C_b^s(n)(1-\alpha\beta)}.$$

It follows that

$$\delta = \frac{1}{\alpha}\left(1 - \frac{np(1-\alpha\beta)^2C_b^s(n)}{p^2(1-\alpha\beta)^2C_b^s(n) + d_n^2}\right).$$

Now defining a function $\Psi(n)$ by

$$\Psi(n) = \frac{1}{\alpha}\left(1 - \frac{np(1-\alpha\beta)^2C_b^s(n)}{p^2(1-\alpha\beta)^2C_b^s(n) + d_n^2}\right), (n \geq 1),$$

we observe that $\Psi(n)$ is an increasing function of n. We thus, conclude that

$$\delta = \Psi(1) = \frac{1}{\alpha}\left(1 - \frac{p(1-\alpha\beta)^2C_b^s(1)}{p^2(1-\alpha\beta)^2C_b^s(1) + d_1^2}\right)$$

which completes the proof.

Theorem 4.2. *For functions $f_1(z) \in \mathcal{M}_b^s(\alpha, \beta)$ and $f_2(z) \in \mathcal{M}_b^s(\alpha, \gamma)$, then $(f_1 * f_2)(z) \in \mathcal{M}_b^s(\alpha, \zeta)$ where*

$$\zeta \leq \frac{(1+\alpha\beta)(1+\alpha\gamma)C_b^s(1) - (1-\alpha\beta)(1-\alpha\gamma)}{\alpha[(1-\alpha\beta)(1-\alpha\gamma) + (1+\alpha\beta)(1+\alpha\gamma)C_b^s(1)]}$$

where

$$d_n(p,\alpha,\beta)=[n-p(1-\alpha\beta)],$$

$$d_n(p,\alpha,\gamma)=[n-p(1-\alpha\gamma)],$$

and

$$d_n(p,\alpha,\zeta)=[n-p(1-\alpha\zeta)].$$

Proof. For the function

$$f_1(z)=a_{p,1}z^{-p}+\sum_{n=1}^{\infty}|a_{n,1}|z^{n-p}\in\mathcal{M}_b^s(\alpha,\beta)$$

and

$$f_2(z)=a_{p,2}z^{-p}+\sum_{n=1}^{\infty}|a_{n,2}|z^{n-p}\in\mathcal{M}_b^s(\alpha,\gamma),$$

then by Theorem 2.1 , we have

$$\sum_{n=1}^{\infty}\frac{d_n(p,\alpha,\beta)C_b^s(n)}{p(1-\alpha\beta)a_{p,1}}|a_{n,1}| \quad\leq\quad 1 \tag{29}$$

and

$$\sum_{n=1}^{\infty}\frac{d_n(p,\alpha,\gamma)C_b^s(n)}{p(1-\alpha\gamma)a_{p,2}}|a_{n,2}| \quad\leq\quad 1. \tag{30}$$

where

$$d_n(p,\alpha,\beta)=[n-p(1-\alpha\beta)] \quad\text{and}\quad d_n(p,\alpha,\gamma)=[n-p(1-\alpha\gamma)].$$

Since $(f_1*f_2)(z)\in\mathcal{M}_b^s(\lambda,\alpha,\zeta)$, and again by Theorem 2.1 , we have

$$\sum_{n=1}^{\infty}\frac{d_n(p,\alpha,\zeta)C_b^s(n)}{p(1-\alpha\zeta)a_{p,1}a_{p,2}}|a_{n,1}||a_{n,2}|\leq 1. \tag{31}$$

Applying Cauchy-Schwarz inequality, we have

$$\sum_{n=1}^{\infty}\frac{C_b^s(n)\sqrt{d_n(p,\alpha,\beta)d_n(p,\alpha,\gamma)}}{p\sqrt{(1-\alpha\beta)(1-\alpha\gamma)a_{p,1}a_{p,2}}}\sqrt{|a_{n,1}||a_{n,2}|}\leq 1. \tag{32}$$

From(31)and (32), we have

$$\frac{d_n(p,\alpha,\zeta)C_b^s(n)}{p(1-\alpha\zeta)a_{p,1}a_{p,2}}|a_{n,1}||a_{n,2}|\leq\frac{C_b^s(n)}{p}\frac{\sqrt{d_n(p,\alpha,\beta)d_n(p,\alpha,\gamma)}}{\sqrt{(1-\alpha\beta)(1-\alpha\gamma)a_{p,1}a_{p,2}}}\sqrt{|a_{n,1}||a_{n,2}|}$$

$$\sqrt{|a_{n,1}||a_{n,2}|}\leq\frac{\sqrt{d_n(p,\alpha,\beta)d_n(p,\alpha,\gamma)}}{\sqrt{(1-\alpha\beta)(1-\alpha\gamma)}}\frac{(1-\alpha\zeta)\sqrt{a_{p,1}a_{p,2}}}{d_n(p,\alpha,\zeta)}. \tag{33}$$

We know that

$$\sqrt{|a_{n,1}||a_{n,2}|}\leq\frac{p\sqrt{(1-\alpha\beta)(1-\alpha\gamma)a_{p,1}a_{p,2}}}{C_b^s(n)\sqrt{d_n(p,\alpha,\beta)d_n(p,\alpha,\gamma)}}, \tag{34}$$

from equation (33) and (34), we have

$$\frac{p\sqrt{(1-\alpha\beta)(1-\alpha\gamma)}}{C_b^s(n)\sqrt{d_n(p,\alpha,\beta)d_n(p,\alpha,\gamma)}}\leq\frac{\sqrt{d_n(p,\alpha,\beta)d_n(p,\alpha,\gamma)}}{\sqrt{(1-\alpha\beta)(1-\alpha\gamma)}}\frac{(1-\alpha\zeta)}{d_n(p,\alpha,\zeta)} \tag{35}$$

$$\zeta\leq\frac{1}{\alpha}\left[\frac{C_b^s(n)d_n(p,\alpha,\beta)d_n(p,\alpha,\gamma)-p(n-p)(1-\alpha\beta)(1-\alpha\gamma)}{d_n(p,\alpha,\beta)d_n(p,\alpha,\gamma)C_b^s(n)+p^2(1-\alpha\beta)(1-\alpha\gamma)}\right]. \tag{36}$$

Now defining a function $\Psi(n)$ by

$$\Psi(n) = \frac{1}{\alpha}\left[1 - \frac{np(1-\alpha\beta)(1-\alpha\gamma)}{d_n(p,\alpha,\beta)d_n(p,\alpha,\gamma)C_b^s(n) + p^2(1-\alpha\beta)(1-\alpha\gamma)}\right], (n \geq 1),$$

we observe that $\Psi(n)$ is an increasing function of n. We thus, conclude that

$$\zeta \leq \frac{(1+\alpha\beta)(1+\alpha\gamma)C_b^s(1) - (1-\alpha\beta)(1-\alpha\gamma)}{\alpha[(1-\alpha\beta)(1-\alpha\gamma) + (1+\alpha\beta)(1+\alpha\gamma)C_b^s(1)]}$$

which completes the proof.

Theorem 4.3. *Let the functions $f_j(z)(j = 1, 2)$ defined by*

$$f_j(z) = a_{p,i}z^{-p} + \sum_{n=1}^{\infty} |a_{n,j}|z^{n-p} \quad (j = 1, 2)$$

be in the class $\mathcal{M}_b^s(\alpha, \beta, \rho)$ then the function $h(z)$ defined by

$$h(z) = (a_{p,1} + a_{p,2})z^{-p} + \sum_{n=1}^{\infty}(|a_{n,1}|^2 + |a_{n,2}|^2)z^{n-p} \tag{37}$$

belongs to the class $\mathcal{M}_b^s(\alpha, \gamma, \rho)$ where

$$\gamma \leq \frac{1}{\alpha}\left(\frac{c_1^2 - 2p(1-p)(1-\alpha\beta)^2C_b^s(1) + p\rho}{c_1^2 + 2p^2(1-\alpha\beta)^2(C_b^s(1) - \rho)}\right). \tag{38}$$

Proof. Noting that

$$\sum_{n=1}^{\infty}\left[\frac{c_n}{p(1-\alpha\beta)}\right]^2 |a_{n,j}|^2 \leq \left[\sum_{n=1}^{\infty}\frac{c_n}{p(1-\alpha\beta)}|a_{n,j}|\right]^2 \leq 1, (j = 1, 2). \tag{39}$$

where

$$c_n = [n - p(1-\alpha\beta)]C_b^s(n) + p(1-\alpha\beta)\rho^n$$

For $f_j(z) \in \mathcal{M}_b^s(\alpha, \beta, \rho)(j = 1, 2)$, we have

$$\sum_{n=1}^{\infty}\frac{1}{2}\left[\frac{c_n}{p(1-\alpha\beta)}\right]^2 (|a_{n,1}|^2 + |a_{n,2}|^2) \leq 1. \tag{40}$$

Therefore we have to find the largest γ such that

$$\sum_{n=1}^{\infty}\left[\frac{[n - p(1-\alpha\gamma)]C_b^s(n) + p(1-\alpha\gamma)\rho^n}{p(1-\alpha\gamma)}\right](|a_{n,1}|^2 + |a_{n,2}|^2) \leq 1, (n \geq 1). \tag{41}$$

From equation (41) and (40) we have

$$\left[\frac{[n - p(1-\alpha\gamma)]C_b^s(n) + p(1-\alpha\gamma)\rho^n}{p(1-\alpha\gamma)}\right] \leq \frac{1}{2}\left[\frac{c_n}{p(1-\alpha\beta)}\right]^2, (n \geq 1). \tag{42}$$

$$\gamma \leq \frac{1}{\alpha}\left(\frac{c_n^2 - 2p(n-p)(1-\alpha\beta)^2C_b^s(n) + p\rho^n}{c_n^2 + 2p^2(1-\alpha\beta)^2(C_b^s(n) - \rho^n)}\right), (n \geq 1). \tag{43}$$

Now defining a function $\Psi(n)$ by

$$\Psi(n) = \frac{1}{\alpha}\left(\frac{c_n^2 - 2p(n-p)(1-\alpha\beta)^2C_b^s(n) + p\rho^n}{c_n^2 + 2p^2(1-\alpha\beta)^2(C_b^s(n) - \rho^n)}\right), (n \geq 1)$$

we observe that $\Psi(n)$ is an increasing function of n. We thus conclude that

$$\gamma \leq \frac{1}{\alpha}\left(\frac{c_1^2 - 2p(1-p)(1-\alpha\beta)^2C_b^s(1) + p\rho}{c_1^2 + 2p^2(1-\alpha\beta)^2(C_b^s(1) - \rho)}\right).$$

5 Closure properties

In this section, we consider integral transforms of functions in the class $\mathcal{M}_b^s(\alpha,\beta)$.

Theorem 5.1. *Let the function $f(z)$ given by (1) be in $\mathcal{M}_b^s(\alpha,\delta)$. Then the integral operator*

$$F(z) = c\int_0^1 u^{c+p-1} f(uz)du, \quad (0 < u \leq 1, \quad 0 < c < \infty),$$

is in $M_p(\alpha,\delta)$ where

$$\delta = \frac{1}{\alpha}\left(\frac{(c+n)(n-p+p\alpha\beta) - c(n-p)(1-\alpha\beta)}{(c+n)(n-p+p\alpha\beta) + cp(1-\alpha\beta)}\right). \tag{44}$$

Proof. Let $f(z) \in \mathcal{M}_b^s(\alpha,\beta)$. Then

$$\begin{aligned} F(z) &= c\int_0^1 u^{c+p-1} f(uz)du \\ &= c\int_0^1 \left(\frac{u^{c-1}a_p}{z^p} + \sum_{n=1}^{\infty} a_n u^{n+c-1} z^{n-p}\right) du \\ &= \frac{a_p}{z^p} + \sum_{n=1}^{\infty} (\frac{c}{c+n}) a_n z^{n-p}. \end{aligned}$$

It is sufficient to show that

$$\sum_{n=1}^{\infty} \frac{(n-p+p\alpha\delta)}{p(1-\alpha\delta)a_p}(\frac{c}{c+n})a_n \leq 1. \tag{45}$$

Since $f \in \mathcal{M}_b^s(\alpha,\beta)$, we have

$$\sum_{n=1}^{\infty} \frac{(n-p+p\alpha\beta)}{p(1-\alpha\beta)a_p} a_n \leq 1.$$

Note that (45) is satisfied if

$$\frac{(n-p+p\alpha\delta)}{p(1-\alpha\delta)a_p}(\frac{c}{c+n}) \leq \frac{(n-p+p\alpha\beta)}{p(1-\alpha\beta)a_p}.$$

Rewriting the inequality, we have

$$c(n-p+p\alpha\delta)(1-\alpha\beta) \leq (c+n)(n-p+p\alpha\beta)(1-\alpha\delta).$$

Solving for δ, we have

$$\delta \leq \frac{1}{\alpha}\left(\frac{(c+n)(n-p+p\alpha\beta) - c(n-p)(1-\alpha\beta)}{(c+n)(n-p+p\alpha\beta) + cp(1-\alpha\beta)}\right) = F(n).$$

$$F(n+1) - F(n) = \frac{1}{\alpha}\left(\frac{c(1-\alpha\beta)}{[c+n-p(1-\alpha\beta)][c+n+1-p(1-\alpha\beta)]}\right) > 0,$$

for each n. Hence, $F(n)$ is an increasing function of n,which yields the desired result(44). □

Theorem 5.2. *Let $f(z)$ given by (1), be in $\mathcal{M}_b^s(\alpha,\beta)$. Then*

$$F(z) = \frac{1}{c}[(c+p)f(z) + zf'(z)] = \frac{a_p}{z^p} + \sum_{n=1}^{\infty} \frac{c+n}{c} a_n z^{n-p}, \quad c > 0, \tag{46}$$

is in $\mathcal{M}_b^s(\alpha,\beta)$ for $|z| \leq r(\alpha,\beta,\delta)$ where

$$r(\alpha,\beta,\delta) = \inf_n \left(\frac{c(1-\alpha\delta)\{n-p+p\alpha\beta\}}{(1-\alpha\beta)(c+n)\{n-p+p\alpha\delta\}}\right)^{1/n}, \quad n = 1,2,3,\ldots. \tag{47}$$

Proof. Let $w = \left\{ \frac{-z(\mathcal{J}_b^s f(z))'}{p\alpha(\mathcal{J}_b^s f(z))} \right\}$. Then it is sufficient to show that

$$\left| \frac{w+p}{w-p+2\delta} \right| < 1.$$

A computation shows that this is satisfied if

$$\sum_{n=1}^{\infty} \frac{(n-p+p\alpha\delta)}{p(1-\alpha\delta)a_p} \frac{c+n}{c} a_n |z|^n \leq 1. \tag{48}$$

Since $f \in \mathcal{M}_b^s(\alpha, \beta)$, by Theorem 2.1, we have

$$\sum_{n=1}^{\infty} \frac{(n-p+p\alpha\beta)}{p(1-\alpha\beta)a_p} |a_n| \leq 1.$$

The equation (48) is satisfied if

$$\sum_{n=1}^{\infty} \frac{(n-p+p\alpha\delta)(c+n)}{c(1-\alpha\delta)} |z|^n \leq \sum_{n=1}^{\infty} \frac{(n-p+p\alpha\beta)}{(1-\alpha\beta)}.$$

A simple computation yields, the inequality asserted in equation (47). □

Theorem 5.3. (**Arithmetic Mean**) *Let the functions $f_i(z)(i = 1, 2, \dots \mu)$ defined by*

$$f_i(z) = \frac{a_{p,i}}{z^p} + \sum_{n=1}^{\infty} a_{n,i} z^{n-p}, \ \ (a_{n,i} \geq 0, i = 1, 2..., \mu, n \geq 1)$$

be in the class $\mathcal{M}_b^s(\alpha, \beta, \rho)$. Then the arithmetic mean of $f_i(z)(i = 1, 2, \dots \mu)$ defined by

$$h(z) = \frac{1}{\mu} \sum_{n=1}^{\mu} f_i(z)$$

is also in the class $\mathcal{M}_b^s(\alpha, \beta, \rho)$.

proof Since $f_i(z) \in \mathcal{M}_b^s(\alpha, \beta, \rho)$ $(i = 1, 2, \dots \mu)$, then by using Theorem 2.3,we have

$$\begin{aligned}
& \sum_{n=1}^{\infty} [d_n + p(1-\alpha\beta)\rho^n] \left(\frac{1}{\mu} \sum_{i=1}^{\mu} a_{n,i} \right) \\
= \ & \frac{1}{\mu} \sum_{i=1}^{\mu} \left(\sum_{n=1}^{\infty} [d_n + p(1-\alpha\beta)\rho^n] a_{n,i} \right) \\
\leq \ & \frac{1}{\mu} \sum_{i=1}^{\mu} p(1-\alpha\beta) \\
\leq \ & p(1-\alpha\beta)
\end{aligned}$$

which in view of Theorem 2.3, again implies that $h(z) \in \mathcal{M}_b^s(\alpha, \beta, \rho)$ and so the proof is complete.

Theorem 5.4. (**Weighted Mean**) *Let the functions $f_i(z)(i = 1, 2)$ defined by*

$$f_i(z) = \frac{a_{p,i}}{z^p} + \sum_{n=1}^{\infty} a_{n,i} z^{n-p}, \ \ (a_{n,i} \geq 0, i = 1, 2)$$

be in the class $\mathcal{M}_b^s(\alpha, \beta, \rho)$. Then the weighted mean of $f_i(z)(i = 1, 2)$ defined by

$$W_c(z) = \frac{1}{2} \left[(1-c) f_1(z) + (1+c) f_2(z) \right] \tag{49}$$

is also in the class $\mathcal{M}_b^s(\alpha, \beta, \rho)$.

proof Since

$$f_i(z) = \frac{a_{p,i}}{z^p} + \sum_{n=1}^{\infty} a_{n,i} z^{n-p} \in \mathcal{M}_b^s(\alpha, \beta, \rho)$$

for $(a_{n,i} \geq 0, i = 1, 2)$ and by (49) we have,

$$W_c(z) = (a_{p,1} + a_{p,2})z^{-p} + \sum_{n=1}^{\infty} \frac{1}{2}\left[(1-c)a_{n,1} + (1+c)a_{n,2}\right] z^{n-p}.$$

By using Theorem 2.3,

$$\sum_{n=1}^{\infty} \frac{[d_n + p(1-\alpha\beta)\rho^n]}{p(1-\alpha\beta)} |a_{n,1}| \leq 1 \tag{50}$$

and

$$\sum_{n=1}^{\infty} \frac{[d_n + p(1-\alpha\beta)\rho^n]}{p(1-\alpha\beta)} |a_{n,2}| \leq 1. \tag{51}$$

Using 50and 51 in 49, we get

$$\begin{aligned} W_c(z) &= \frac{1}{2}(1-c)p(1-\alpha\beta) + \frac{1}{2}(1+c)p(1-\alpha\beta) \\ &\leq p(1-\alpha\beta). \end{aligned}$$

Therefore $W_c(z) \in \mathcal{M}_b^s(\alpha, \beta, \rho)$, which completes the proof.

6 Concluding remarks

If $b = 1, s = \nu$ $(\nu > -1)$ the operator $\mathcal{J}_b^s$ turns into Libera-Bernardi integral operator $\mathcal{L}_\nu$ and if $b = \sigma(\sigma > 0),\ s = 1$ the operator $\mathcal{J}_b^s$ turns into Jung-Kim-Srivastava integral operator $\mathcal{I}_\sigma$. So, various other interesting corollaries and consequences of our main results can be derived similarly. The details involved may be left as an exercise for the interested reader.

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