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Neutrosophic Duplets of $\{Z_{p^n}, \times\}$ and $\{Z_{pq}, \times\}$ and Their Properties

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Received: 31 July 2018; Accepted: 15 August 2018; Published: 17 August 2018

Abstract: The notions of neutrosophy, neutrosophic algebraic structures, neutrosophic duplet and neutrosophic triplet were introduced by Florentin Smarandache. In this paper, the neutrosophic duplets of Z_{p^n}, Z_{pq} and $Z_{p_1p_2\ldots p_n}$ are studied. In the case of Z_{p^n} and Z_{pq}, the complete characterization of neutrosophic duplets are given. In the case of $Z_{p_1\ldots p_n}$, only the neutrosophic duplets associated with p_is are provided; $i = 1, 2, \ldots, n$. Some open problems related to neutrosophic duplets are proposed.

Keywords: neutrosophic duplets; semigroup; neutrosophic triplet groups

1. Introduction

Real world data, which are predominately uncertain, indeterminate and inconsistent, were represented as neutrosophic set by Smarandache [1]. Neutrosophy deals with the existing neutralities and indeterminacies of the problems. Neutralities in neutrosophic algebraic structures have been studied by several researchers [1–8]. Wang et al. [9] proposed Single-Valued Neutrosophic Set (SVNS) to overcome the difficulty faced in relating neutrosophy to engineering discipline and real world problems. Neutrosophic sets have evolved further as Double Valued Neutrosophic Set (DVNS) [10] and Triple Refined Indeterminate Neutrosophic Set (TRINS) [11]. Neutrosophic sets are useful in dealing with real-world indeterminate data, which Intuitionistic Fuzzy Set (IFS) [12] and Fuzzy sets [13] are incapable of handling accurately [1].

The current trends in neutrosophy and related theories of neutrosophic triplet, related triplet group, neutrosophic duplet, and duplet set was presented by Smarandache [14]. Neutrosophic duplets and neutrosophic triplets have been of interest and many have studied them [15–24]. Neutrosophic duplet semigroup were studied in [19] and the neutrosophic triplet group was introduced in [8]. Neutrosophic duplets and neutrosophic duplet algebraic structures were introduced by Smarandache.

In the case of neutrosophic duplets, we see $ax = a$ and $x = neut(a)$, where, as in L-fuzzy sets [25] as per definition is a mapping from $A : X \to L$, L may be semigroup or a poset or a lattice or a Boolean σ-ring; however, neutrosophic duplets are not mapping, more so in our paper algebraic properties of them are studied for Z_n for specific values of n. However, in the case of all structures, the semigroup or lattice or Boolean σ-ring or a poset, there are elements which are neutrosophic duplets. Here, we mainly analyze neutrosophic duplets in the case of Z_n only number theoretically.

In this paper, we investigate the neutrosophic duplets of $\{Z_{p^n}, \times\}$, where p is a prime (odd or even) and $n \geq 2$. Similarly, neutrosophic duplets in the case of Z_{pq} and $Z_{p_1p_2\ldots p_n}$ are studied. It is noted that the major difference between the neutrals of neutrosophic triplets and that of neutrosophic duplets is that in the former case they are idempotents and in the latter case they are units. Idempotents in the neutrosophic duplets are called trivial neutrosophic duplets.

This paper is organized as five sections, Section 1 is introductory in nature and Section 2 provides the important results of this paper. Neutrosophic duplets in the case of Z_{p^n}; p an odd prime are studied in Section 3. In Section 4, neutrosophic duplets of Z_{pq} and $Z_{p_1p_2\ldots p_n}$, and their properties are analyzed. Section 5 discusses the conclusions, probable applications and proposes some open problems.

2. Results

The basic definition of neutrosophic duplet is recalled from [8].

Consider U to be the universe of discourse, and D a set in U, which has a well-defined law #.

Definition 1. *Consider $\langle a, neut(a)\rangle$, where a, and $neut(a)$ belong to D. It is said to be a neutrosophic duplet if it satisfies the following conditions:*

1. *$neut(a)$ is not the same as the unitary element of D in relation with the law # (if any);*
2. *a# $neut(a)$ = neut(a) # a = a; and*
3. *$anti(a) \notin D$ for which a # anti(a) = anti(a) # a = neut(a).*

Here, the neutrosophic duplets of $\{Z_{p^n}, \times\}$, p is a prime (odd or even) and $n \geq 2$ are analyzed number theoretically. Similarly, neutrosophic duplets in the case of Z_{pq} and $Z_{p_1p_2\ldots p_n}$ are studied in this paper.

The results proved by this study are:

1. The neutrals of all nontrivial neutrosophic duplets are units of $\{Z_{p^n}, \times\}$, $\{Z_{pq}, \times\}$ and $\{Z_{p_1p_2\ldots p_n}, \times\}$.
2. If p is a prime in anyone of the semigroups ($\{Z_{p^n}, \times\}$ or $\{Z_{pq}, \times\}$ or $\{Z_{p_1p_2\ldots p_n}, \times\}$) as mentioned in 1, then mp has only p number of neutrals, for the appropriate m.
3. The neutrals of any mp^t for a prime p; $(m, p) = 1$ are obtained and they form a special collection.

3. Neutrosophic Duplets of $\{Z_{p^n}, \times\}$ and its Properties

Neutrosophic duplets and neutrosophic duplet algebraic structures were introduced by Florentin Smarandache in 2016. Here, we investigate neutrosophic duplets of $\{Z_{p^n}, \times\}$, where p is a prime (odd or even) and $n \geq 2$. First, neutrosophic duplets in the case of Z_{2^4} and Z_{3^3} and their associated number theoretic properties are explored to provide a better understanding of the theorems proved. Then, several number theoretical properties are derived.

Example 1. *Let $S = \{Z_{16}, \times\}$ be the semigroup under $\times$ modulo 16. Z_{16} has no idempotents. The units of Z_{16} are $\{1, 3, 5, 7, 9, 11, 13, 15\}$. The elements which contribute to the neutrosophic duplets are $\{2, 4, 6, 8, 10, 12, 14\}$. The neutrosophic duplet sets under usual product modulo 16 are:*

$$\{\{2,1\},\{2,9\}\},\{\{4,1\},\{4,5\},\{4,9\},\{4,13\}\},$$
$$\{\{6,1\},\{6,9\}\},\{\{8,1\},\{8,3\},\{8,5\},\{8,7\},\{8,9\},\{8,11\},\{8,13\},\{8,15\}\},$$
$$\{\{10,1\},\{10,9\}\},\{\{12,1\},\{12,5\},\{12,9\},\{12,13\}\},\{\{14,1\},\{14,9\}\}$$

The observations made from this example are:

1. *Every non-unit of Z_{16} is a neutrosophic duplet.*
2. *Every non-unit divisible by 2, viz. $\{2, 6, 10, 14\}$, has only $\{1, 9\}$ as their neutrals.*
3. *Every non-unit divisible by 4 are 4 and 12, which has $\{1, 5, 9, 13\}$ as neutrals.*

The biggest number which divides 16 is 8 and all units act as neutrals in forming neutrosophic duplets. Thus, $A = \{1, 3, 5, 7, 9, 11, 13, 15\}$, which forms a group of order 8, yields the 8 neutrosophic duplets; $8 \times i = 8$ for all $i \in A$ and A forms a group under multiplication modulo 16; and $\{1, 9\}$ and $\{1, 5, 9, 13\}$ are subgroups of A.

In view of this, we have the following theorem.

Theorem 1. *Let $S = \{Z_{2^n}, \times\}$, be the semigroup under product modulo 2^n, $n \geq 2$.*

(i) *The set of units of S are $A = \{1, 3, 5, \ldots, 2^n - 1\}$, forms a group under $\times$ and $|A| = 2^{n-1}$.*
(ii) *The set of all neutrosophic duplets with 2^{n-1} is A; neutrals of 2^{n-1} are A.*
(iii) *All elements of the form $2m \in Z_{2^n}$ (m an odd number) has only the elements $\{1, 2^{n-1} + 1\}$ to contribute to neutrosophic duplets (neutrals are $1, 2^{n-1} + 1$).*
(iv) *All elements of the form $m2^t \in Z_{2^n}; 1 < t < n - 1$; m odd has its neutrals from $B = \{1, 2^{n-t} + 1,\ 2^{n-t+1} + 1,\ 2^{n-t+2} + 1,\ \ldots, 2^{n-1} + 1, 2^{n-t} + 2^{n-t+1} + 1, \ldots, 2^{n-t} + 2^{n-1} + 1, \ldots,\ 1 + 2^{n-t} + 2^{n-t+1} + \ldots + 2^{n-1}\}$.*

Proof.

(i) Given $S = \{Z_{2^n}, \times\}$ where $n \geq 2$ and S is a semigroup under product modulo 2^n. $A = \{1, 3, 5, 7, \ldots, 2^n - 1\}$ is a group under product as every element is a unit in S and closure axiom is true by property of modulo integers and $|A| = 2^{n-1}$. Hence, Claim (i) is true.
(ii) Now, consider the element 2^{n-1}; the set of duplets for 2^{n-1} is A for $2^{n-1} \times 1 = 2^{n-1}$; $2^{n-1} \times 3 = 2^{n-1}[2 + 1] = 2^n + 2^{n-1} = 2^{n-1}, \ldots, 2^{n-1}(m)$; ($m$ is odd) will give only $m2^{n-1}$. Hence, this proves Claim (ii).
(iii) Consider $2m \in Z_{2^n}$; we see $2m \times 1 = 2m$ and $2m(2^{n-1} + 1) = m2^n + 2m = 2m$. $(2m, 2^{n-1} + 1)$ is a neutrosophic duplet pair; hence, the claim.
(iv) Let $m2^t \in Z_{2^n}$; clearly, $m2^t \times x = m2^t$ for all $x \in B$.

□

Next, we proceed onto describe the duplet pairs in $S = \{Z_{3^3}, \times\}$.

Example 2. *Let $S = \{Z_{3^3}, \times\}$ be a semigroup under product modulo 3^3. The units of S are $A = \{1, 2, 4, 5, 7, 8, 10, 11, 13, 14, 16, 17, 19, 20, 22, 23, 25, 26\}$. Clearly, A forms a group under a product. The non-units of S are $\{3, 6, 9, 12, 15, 18, 21, 24\}$. Zero can be included for $0 \times x = 0$ for all $x \in S$, in particular for $x \in A$. The duplet pairs related to 3 are $B_1 = \{\{3, 1\}, \{3, 10\}, \{3, 19\}\}$. The duplet pairs related to 6 are $B_2 = \{\{6, 1\}, \{6, 10\}, \{6, 19\}\}$. The duplet pairs related to 9 are*

$$B_3 = \{\{9, 1\}, \{9, 4\}, \{9, 7\}, \{9, 13\}, \{9, 10\}, \{9, 16\}, \{9, 19\}, \{9, 22\}, \{9, 25\}\}.$$

The neutrosophic duplets of 12 are $B_4 = \{\{12, 1\}, \{12, 10\}, \{12, 19\}\}$. The neutrosophic duplets of 15 are $B_5 = \{\{15, 1\}, \{15, 10\}, \{15, 19\}\}$. Finally, the neutrosophic duplets of 18 are

$$B_6 = \{\{18, 1\}, \{18, 4\}, \{18, 7\}, \{18, 13\}, \{18, 10\}, \{18, 16\}, \{18, 19\}, \{18, 22\}, \{18, 25\}\}.$$

The neutrosophic duplets associated with 21 are $B_7 = \{\{21, 1\}, \{21, 10\}, \{21, 19\}\}$ and 24 are $B_8 = \{\{24, 1\}, \{24, 10\}, \{24, 19\}\}$. Now, the trivial duplet of 0, which we take is

$$B_0 = \{\{0, 1\}, \{0, 4\}, \{0, 7\}, \{0, 13\}, \{0, 10\}, \{0, 16\}, \{0, 19\}, \{0, 22\}, \{0, 25\}\}.$$

We see $L = \{B_0 \cup B_1 \cup B_2 \cup \ldots \cup B_8\}$ forms a semigroup under product modulo 27 and $o(L) = 45$.

We have the following result.

Theorem 2. *Let $S = \{Z_{p^n}, \times\}$, where p is an odd prime, $n \geq 2$ is a semigroup under $\times$, and product modulo is p^n. The units of S are denoted by A and non-units of S are denoted by B. The neutrosophic duplets of S associated with B are groups under product and are subgroups of A. The neutrals of $tp^s = b \in B$ are of the form $D = \{1, 1 + p^{n-s}, 1 + p^{n-s+1}, 1 + p^{n-s+2}, \ldots, 1 + p^{n-1}, 1 + p^{n-s} + p^{n-s+1}, 1 + p^{n-s} + p^{n-s+2}, \ldots, 1 + p^{n-1} + p^{n-s}, \ldots 1 + p^{n-s} + \ldots + p^{n-1}\}; 1 \leq t < m, p/m; 1 < s < n$.*

Proof. Let $tp^s \in Z_{p^n}$ all elements which act as neutrosophic duplets for tp^s are from the set D. For any $x \in D$ and $tp^s \in Z_{p^s}$, we see $xtp^s = tp^s$; hence, the claim. □

It is important to note that $S = \{Z_{p^n}, \times\}$ has no non-trivial neutrosophic triplets as Z_{p^n} has no non-trivial idempotents.

Next, we proceed to finding the neutrosophic duplets of Z_{pq}; p and q are distinct primes.

4. Neutrosophic Duplets of Z_{pq} and $Z_{p_1 p_2 \ldots p_n}$

In this section, we study the neutrosophic duplets of Z_{pq} where p and q are primes. Further, we see Z_{pq} also has neutrosophic triplets. The neutrosophic triplets in the case of Z_{pq} have already been characterized in [23]. We find the neutrosophic duplets of Z_{2p}, p a prime. We find the neutrosophic duplets and neutrosophic triplets groups of Z_{26} in the following.

Example 3. *Let $S = \{Z_{26}, \times\}$ be the semigroup under product modulo 26. The idempotents of S are 13 and 14. We see 13 is just a trivial neutrosophic triplet, however only 14 contributes to non-trivial neutrosophic triplets. We now find the neutrosophic duplets of Z_{26}. The units of Z_{26} are $A = \{1, 3, 5, 7, 9, 11, 15, 17, 19, 21, 23, 25\}$ and they act as neutrals of the duplets. The non-units which contribute for neutrosophic duplets are $B = \{2, 4, 6, 8, 10, 12, 13, 14, 16, 18, 20, 22, 24\}$. 0 is the trivial duplet as $0 \times x = 0$ for all $x \in A$. Consider $2 \in B$ the pairs of duplets are $\{2, 1\}, 2 \times 14 = 2$ but 14 cannot be taken as $anti(2) = 20$ and $anti(2)$ exists so 2 is not a neutrosophic duplet for $(2, 14, 20)$ is a neutrosophic triplet group.*

Consider $4 \in B; \{4, 1\}$ is a trivial neutrosophic duplet. Then, $4 \times 14 = 4$ and $(4, 14, 16)$ are again a neutrosophic triplet as $anti(4) = 16$ so 4 is not a neutrosophic duplet. Thus, 16 and 20 are also not neutrosophic duplets. Consider $6 \in B$; we see $\{6, 1\}$ is a non-trivial neutrosophic duplet. In addition, $(6, 14, 10)$ are neutrosophic triplet groups so 6 and 10 are not non-trivial neutrosophic duplets. Consider $8 \in B, (8, 14, 18)$ is a neutrosophic triplet group. hence 8 and 18 are not neutrosophic duplets. Then, $(12, 14, 12)$ is also a neutrosophic triplet group. Thus, 12 is not a neutrosophic duplet. Let $22 \in B$ be such that $(22, 14, 24)$ is a neutrosophic triplet group, hence 22 and 24 are not neutrosophic duplets.

Consider $13 \in B$; we see the neutrals are $\{1, 3, 5, 7, 9, 11, 15, 17, 19, 21, 23, 25\}$. We see the collection of neutrosophic duplets associated with $13 \in Z_{26}$ happens to yield a semigroup under product if 13 is taken as the trivial neutrosophic duplets, as it is an idempotent in Z_{26}, and, in all pairs, it is treated as semigroup of order 13, where $(13, 1)$ and $(13, 13)$ are trivial neutrosophic duplets.

In view of this, we have the following theorem.

Theorem 3. *Let $S = \{Z_{2p}, \times\}$ be a semigroup under product modulo $2p$; p an odd prime. This S has only p and $p + 1$ to be the idempotents and only p contributes for a neutrosophic duplet collection with all units of Z_{2p} and the collection $B = \{(p, x) | x \in Z_{2p}\}$, x is a unit in Z_{2p} forms a commutative semigroup of order p which includes 1 and p which result in the trivial duplets pair $(p, 1)$ and (p, p).*

Proof. Given $S = \{Z_{2p}, \times\}$ is a semigroup under $\times$ and p is an odd prime. We see from [23] p and $p + 1$ are idempotents of Z_{2p}. It is proven in [23] that $p + 1$ acts for the neutrosophic triplet group of Z_{2p} (formed by elements $2, 4, 6, \ldots, 2p - 2$) as the only neutral. (p, p, p) is a trivial neutrosophic triplet. However, Z_{2p} has no neutrosophic duplet other than those related with p alone and $p \times x = p$ for all x belonging to the collection of all units of Z_{2p} including 1. If x is a unit in Z_{2p}, two things are essential: x is odd and $x \neq p$. Since x is odd, we see $x = 2y + 1$ and $p(x) = p(2y + 1) = 2yp + p = p$, hence (p, x) is a neutrosophic duplet. The units of Z_{2p} are $(p - 1)$ in number. Further, (p, p) and $(p, 1)$ form trivial neutrosophic duplets. Thus, the collection of all neutrosophic duplets $B = \{(p, x)\}$, x is a unit and $x = p$ is also taken to form the semigroup of order p and is commutative as the collection of all odd numbers forms a semigroup under product modulo $2p$; hence, the claim. □

It is important and interesting to note that, unlike Z_{p^n}, p is a prime and $n \geq 2$. We see Z_{2p} has both non-trivial neutrosophic triplet groups which forms a classical group [23] as well as has a neutrosophic duplet which forms a semigroup of order p.

Next, we study the case when Z_{pq} is taken where both p and q are odd primes first by an example.

Example 4. *Let $S = \{Z_{15}, \times\}$ be a semigroup under product. The idempotents of Z_{15} are 10 and 6. However, 10 does not contribute to non-trivial neutrosophic triplet groups other than $\{5, 10, 5\}$, $\{10, 10, 10\}$. The neutrosophic triplet groups associated with 6 are $(3, 6, 12)$, $(12, 6, 3)$, $(9, 6, 9)$ and $(6, 6, 6)$. The neutrosophic duplets of Z_{15} are contributed by $\{5\}$, $\{10\}$ and $\{3, 12, 6, 9\}$ in a unique way.*

$$D_1 = \{\{5,1\},\{5,4\},\{5,7\},\{5,13\},\{5,10\}\},$$
$$D_2 = \{\{10,13\},\{10,7\},\{10,1\},\{10,4\},\{10,10\}\},$$
$$D_3 = \{\{3,11\},\{3,1\},\{3,6\},\{12,11\},\{12,1\},\{12,6\},\{6,11\},\{6,1\},\{6,6\},\{9,11\},\{9,1\},\{9,6\}\}$$

All three collections of duplets put together is not closed under $\times$; however, D_2 and D_3 form a semigroup under product modulo 15. If we want to make D_1 a semigroup, we should adjoin the trivial duplets $\{0, 4\}$, $\{0, 7\}$, $\{0, 13\}$, $\{0, 1\}$, $\{0, 6\}$, $\{0, 10\}$ as well as D_2. Further, we see $D_1 \cup D_2 \cup D_3$ is not closed under product.

Thus, the study of Z_{pq} where p and q are odd primes happens to be a challenging problem. We give the following examples in the case when $p = 5$ and $q = 7$.

Example 5. *Let $S = \{Z_{35}, \times\}$ be a semigroup of order 35. The idempotents of Z_{35} are 15 and 21. The neutrosophic triplets associated with 15 are $\{(15, 15, 15), (5, 15, 10), (25, 15, 30), (20, 15, 20), (30, 15, 25), (10, 15, 5)\}$, a cyclic group of order six. The cyclic group contributed by the neutrosophic triplet groups associated with 21 is as follows: $\{(21, 21, 21), (7, 21, 28), (28, 21, 7), (14, 21, 14)\}$, which is of order four. The neutrosophic duplets are tabulated in Table 1. Similarly, the neutrosophic duplets associated with $S = \{Z_{105}, \times\}$ are tabulated in Table 2.*

Table 1. Neutrosophic Duplets of $\{Z_{35}, \times\}$.

Neutrals for duplets 5, 10, 15, 20, 25, 30	**Neutrals for duplets** 7, 14, 21, 28
1, 8, 15, 22, 24	1, 6, 11, 16, 21, 26, 31

Table 2. Neutrosophic Duplets of $\{Z_{105}, \times\}$.

Neutrals for duplets 3, 6, 9, 12, 18, 21, 24, 27, 33, 36, 39, 48, 51, 54, 57, 66, 69, 78, 81, 87, 93, 96, 99, 102	**Neutrals for duplets** 5, 10, 20, 25, 40, 50, 55, 65, 80, 85, 95, 100
1, 36, 71	1, 22, 43, 64, 84

Neutrals for duplets 7, 14, 28, 49, 56, 77, 91, 98	**Neutrals for duplets** 15, 30, 45, 60, 75, 90
1, 16, 31, 46, 61, 76, 91	1, 8, 15, 22, 29, 36, 43, 50, 57, 64, 71, 78, 85, 92, 99

Neutrals for duplets 21, 42, 63, 84	**Neutrals for duplets** 35, 70
1, 6, 11 16, 21, 26, 31, 36, 41, 46, 51, 56, 61, 66, 71, 76, 81, 86, 91, 96, 101	1, 4, 7, 10, 13, 16, 19, 22, 25, 28, 31, 34, 37, 40, 43, 46, 49, 52, 55, 58, 61, 64, 67, 70, 73, 76, 79, 82, 85, 88, 91, 94, 97, 100, 103

Theorem 4. *Let $\{Z_n, \times\}$ be a semigroup under product modulo n; $x \in Z_n \setminus \{0\}$ has a neutral $y \in Z_n \setminus \{0\}$ or is a non-trivial neutrosophic duplet if and only if x is not unit in Z_n.*

Proof. $x \in Z_n \setminus \{0\}$ is a neutrosophic duplet if $x \times y = x(mod\, n)$ and y is called the neutral of x. If $x^2 = x$, then we call the pair (x, x) as trivial neutrosophic duplet pair. We see $x \times y = x$, if x is a unit in Z_n, then there exists a $z \in Z_n$ such that $z \times (x \times y) = z \times x$, so that $y = 1$ as $z \times x = 1(mod\, n)$; so $y = 1$ gives trivial neutrosophic duplets. Thus, x is not a unit if it has to form a non-trivial neutrosophic duplet pair; $x \times y = x$ and $y \neq 1$ then if x is a unit we arrive at contradiction; hence, the theorem. □

Theorem 5. *Let* $S = \{Z_{pq}, \times\}$ *be a semigroup under product modulo pq, p and q distinct odd primes. There is p number of neutrosophic duplets for every* $p, 2p, 3p, \ldots, (q-1)p$. *Similarly, there is q number of neutrosophic duplets associated with every* $q, 2q, \ldots (p-1)q$. *The neutrals of sq and tp is given by* $1 + nq$ *for* $1 \leq t \leq q-1, 0 \leq n \leq p-1$ *and that of sq is given by* $1 + mp$; $1 \leq s \leq p-1, 0 \leq m \leq q-1$.

Proof. Given $\{Z_{pq}, \times\}$ is a semigroup under product modulo pq (p and q two distinct odd primes). The neutrals associated with any $tp; 1 \leq t \leq q-1$ is given by the sequence $\{1+q, 2q+1, 3q+1, \ldots, (p-1)q+1\}$ for every $tp \in \{p, 2p, \ldots, (q-1)p\}$. We see, if $tp \in Z_{pq}$,

$$\begin{aligned} tp \times (1+nq) &= tp + tpnq \\ &= tp + tnpq = tp(mod\, pq). \end{aligned}$$

A similar argument for sq completes the proof; hence, the claim. □

Theorem 6. *Let* $S = \{Z_{p_1p_2\ldots p_n}, \times\}$ *be the semigroup under product modulo* $p_1p_2\ldots p_n$, *where* $p_1, p_2, \ldots, p_n$ *are n distinct primes. The duplets are contributed by the non-units of S. The neutrosophic duplets associated with* $A_i = \{p_i, 2p_i, \ldots, (p_1p_2\ldots p_{i-1}p_{i+1}\ldots p_n - 1)p_i\}$ *are* $\{1 + (p_1p_2\ldots p_{i-1}p_{i+1}\ldots p_n)t\}$ *where* $t = 1, 2, \ldots, p_i - 1$; *and* $i = 1, 2, \ldots, n$. *Thus, every element* x_i *of* A_i *has only* $p_i - 1$ *number of elements which neutralizes* x_i; *thus, using each* x_i, *we have* $p_i - 1$ *neutrosophic duplets.*

Proof. Given $S = \{Z_{p_1p_2\ldots p_n}, \times\}$ is a semigroup under product modulo $p_1\ldots p_n$, where p_is are distinct primes, $i = 1, 2, \ldots, n$. Considering $A_i = \{p_i, 2p_i, \ldots, (p_1p_2\ldots p_{i-1}p_{i+1}\ldots p_n - 1)p_i\}$, we have to prove that, for any sp_i, $sp_i \times [1 + (p_1p_2\ldots p_{i-1}p_{i+1}\ldots p_n)t] = sp_i; 1 \leq t \leq p_{i-1}$.

Clearly,

$$sp_i \times [1 + (p_1p_2\ldots p_{i-1}p_{i+1}\ldots p_n)t] = sp_i + sp_i[(p_1p_2\ldots p_{i-1}p_{i+1}\ldots p_n)t]$$

$$= sp_i + st[(p_1p_2\ldots p_{i-1}p_ip_{i+1}\ldots p_n)] = sp_i$$

as $p_1p_2\ldots p_n = 0(mod\,(p_1p_2\ldots p_n))$. Hence, the claim. □

Thus, for varying t and varying s given in the theorem, we see

$$\{sp_i, (1 + (p_1p_2\ldots p_{i-1}p_{i+1}\ldots p_n)t)\}$$

is a neutrosophic duplet pair $1 \leq t \leq p_i - 1; 1 \leq s \leq p_1p_2\ldots p_{i-1}p_{i+1}\ldots p_n$ and $i = 1, 2, \ldots, n$.

5. Discussions and Conclusions

This paper studies the neutrosophic duplets in the case Z_{p^n}, Z_{pq} and $Z_{p_1p_2\ldots p_n}$. In the case of Z_{p^n} and Z_{pq}, a complete characterization of them is given; however, in the case $Z_{p_1\ldots p_n}$, only the neutrosophic duplets associated with p_is are provided; $i = 1, 2, \ldots n$. Further, the following problems are left open:

1. For Z_{pq}, p and q odd primes, how many neutrosophic duplet pairs are there?
2. For $Z_{p_1\ldots p_n}$, what are the neutrals of $p_ip_j, p_ip_jp_k, \ldots, p_1p_2\ldots p_{i-1}p_{i+1}\ldots p_n$?
3. The study of neutrosophic duplets of $Z_{p_1^{t_1}p_2^{t_2}\ldots p_n^{t_n}}$; $p_1, \ldots, p_n$ are distinct primes and $t_i \geq 1; 1 \leq i \leq n$ is left open.

For future research, one can apply the proposed neutrosophic duplets to SVNS, DVNS or TRINS. These neutrosophic duplets can be applied in problems where neutral elements for a given a in Z_{p^n} or Z_{pq} happens to be many. However, the concept of $anti(a)$ does not exist in the case of neutrosophic duplets. Finally, these neutrosophic duplet collections form a semigroup only when all the trivial neutrosophic duplet pairs $(0, a)$ for all appropriate a are taken. These neutrosophic duplets from Z_{p^n} and Z_{pq} can be used to model suitable problems where the $anti(a)$ under study does not exist and many neutrals are needed. This study can be taken up for further development.

Acknowledgments: The authors would like to thank the reviewers for their reading of the manuscript and many insightful comments and suggestions.

Funding: This research received no external funding.

Author Contributions: The contributions of the authors are roughly equal.

Conflicts of Interest: The authors declare no conflict of interest.

Abbreviations

The following abbreviations are used in this manuscript:

SVNS	Single Valued Neutrosophic Sets
DVNS	Double Valued Neutrosophic Sets
TRINS	Triple Refined Indeterminate Neutrosophic Sets
IFS	Intuitionistic Fuzzy Sets

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