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Neutrosophic Quadruple Vector Spaces and Their Properties

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Abstract: In this paper authors for the first time introduce the concept of Neutrosophic Quadruple (NQ) vector spaces and Neutrosophic Quadruple linear algebras and study their properties. Most of the properties of vector spaces are true in case of Neutrosophic Quadruple vector spaces. Two vital observations are, all quadruple vector spaces are of dimension four, be it defined over the field of reals R or the field of complex numbers C or the finite field of characteristic p, Z_p; p a prime. Secondly all of them are distinct and none of them satisfy the classical property of finite dimensional vector spaces. So this problem is proposed as a conjecture in the final section.

Keywords: Neutrosophic Quadruple (NQ); Neutrosophic Quadruple set; NQ vector spaces; NQ linear algebras; NQ basis; NQ vector spaces; orthogonal or dual NQ vector subspaces

1. Introduction

In this section we just give a brief literature survey of this new field of Neutrosophic Quadruples [1]. Neutrosophic triplet groups, modal logic Hedge algebras were introduced in [2,3]. Duplet semigroup, neutrosophic homomorphism theorem and triplet loops and strong AG(1, 1) loops are defined and described in [4–6]. Neutrosophic triplet neutrosophic rings application to mathematical modelling, classical group of neutrosophic triplets on $\{Z_{2p}, \times\}$ and neutrosophic duplets in neutrosophic rings are developed and analyzed in [7–11]. Study of Algebraic structures of neutrosophic triplets and duplets, quasi neutrosophic triplet loops, extended triplet groups, AG-groupoids, NT-subgroups are carried out in [6,12–17]. Refined neutrosophic sets were developed by [18–21]. Neutrosophic algebraic structures in general were studied in [22–25]. The new notion of Neutrosophic Quadruples which assigns a known part happens to be very interesting and innovative, and was introduced by Smarandache [1,26] in 2015. Several research papers on the algebraic structure of Neutrosophic Quadruples, such as groups, monoids, ideals, BCI-algebras, BCI-positive implicative ideals, hyperstructures, BCK/BCI algebras [27–32] have been recently studied and analyzed. However in this paper authors have defined the new notion of Neutrosophic Quadruple vector spaces (NQ vector spaces) and Neutrosophic Quadruple linear algebras (NQ linear algebras) and have studied a few related properties. This work can later be used to propose neutrosophic based dynamical systems in particular in the area of hyperchoaos from cellular neural networks [33].

This paper is organized into five sections. Basic concepts needed to make this paper a self contained one is given in Section 2. NQ vector spaces are introduced in Section 3, further NQ subspaces are introduced and the notion of direct sum and NQ bases are analysed. It is shown all NQ vector spaces are of dimension 4 be it defined over R or C or Z_p, p a prime. Section 4 defines and develops the properties of NQ linear algebras. The final section proposes a conjecture which is related with the finite dimensional vector spaces, which are always isomorphic to finite direct product of fields over which the vector space is defined. Finally we give the future direction of research on this topic.

2. Basic Concepts

In this section basic concepts on vector spaces and a few of its properties and some NQ algebraic structures and their properties needed for this paper are given.

Through out this paper R denotes the field of reals, C denotes the field of complex numbers and Z_p denotes the finite field of characteristic p, p a prime. $NQ = \{(a, bT, cI, dF)$ denotes the Neutrosophic Quadruple; with a, b, c, d in R or C or Z_p, where T, I and F has the usual neutrosophic logic meaning of Truth, Indeterminate and False respectively and a denotes the known part [26].

For basic properties of vector spaces and linear algebras please refer [22].

Definition 1 ([22])**.** *A vector space or a linear space V consists of the following;*

1. *A field of R or C or Z_p of scalars.*
2. *A set V of objects called vectors.*
3. *A rule (or operation) called vector addition; which associates with each pair of vectors x, y in V; $x + y$ is in V, called sum of the vectors x and y in such a way that ;*
   (a) *$x + y = y + x$ (addition is commutative).*
   (b) *$x + (y + z) = (x + y) + z$ (addition is associative).*
   (c) *There is a unique vector 0 in V such that $x + 0 = x$ for all $x \in V$.*
   (d) *For each vector $x \in V$ there is a unique vector $-x \in V$ such that $x + -x = 0$.*
   (e) *A rule or operation called scalar multiplication that associates with each scalar $c \in R$ or C or Z_p and for a vector $x \in V$, called product denoted by '.' of c and x in such a way that for $x \in V$ and $c.x \in V$ and ;*
      i. *$c.x = x.c$ for every $x \in V$.*
      ii. *$(c + d).x = c.x + d.x$*
      iii. *$c.(x + y) = c.x + c.y$*
      iv. *$c.(d.x) = (c.d)x$;*

      for all $x, y \in V$ and c, d in R or C or Z_p.

We can just say $(V, +)$ is a vector space over a field R or C or Z_p if $(V, +)$ is an additive abelian group and V is compatible with the product by the scalars. If on V is defined a product such that $(V, \times)$ is a monoid and $c(x \times y) = (cx) \times y$ then V is a linear algebra over R or C or Z_p [22].

Definition 2 ([22])**.** *Let V be a vector space over R (or C or Z_p). A subspace of V is a subset W of V which is itself a vector space over R (or C or Z_p) with the operations of addition and scalar multiplication as in V.*

Definition 3. *Let V be a vector space over R (or C or Z_p). A subset B of V is said to be linearly dependent or simply dependent if there exist distinct vectors, $x_1, x_2, x_3, \ldots, x_t \in B$ and scalars $a_1, a_2, a_3, \ldots, a_t \in R$ or C or Z_p not all of which are zero such that $a_1x_1 + a_2x_2 + a_3x_3 + \ldots + a_tx_t = 0$. A set which is not linearly dependent is called independent or linearly independent. If B contains only finitely many vectors $x_1, x_2, x_3, \ldots, x_k$ we sometimes say $x_1, x_2, x_3, \ldots, x_k$ are dependent instead of saying B is dependent.*

The following facts are true [22].

1. A subset of a linearly independent set is linearly independent.
2. Any set which contains a linearly dependent subset is linearly dependent.
3. Any set which contains the zero vector (0 vector) is linearly dependent for 1.0 = 0.
4. A set B is linearly independent if and only if each finite subset of B is linearly independent; that is if and only if there exist distinct vectors $x_1, x_2, x_3, \ldots, x_k$ of B such that $a_1x_1 + a_2x_2 + a_3x_3 + \ldots + a_kx_k = 0$ implies each $a_i = 0; i = 1, 2, \ldots, k$.

For a vector space V over a field R or C or Z_p , the basis for V is a linearly independent set of vectors in V which spans the space V. We say the vector space V over R or C or Z_p is a direct sum of subspaces $W_1, W_2, \ldots, W_t$ if and only if $V = W_1 + W_2 + \ldots + W_t$ and $W_i \cap W_j$ is the zero vector for $i \neq j$ and $1 \leq i, j \leq t$.

The other properties of vector spaces are given in book [22].

Now we proceed on to recall some essential definitions and properties of Neutrosophic Quadruples [26].

Definition 4 ([26]). *The quadruple (a, bT, cI, dF) where $a, b, c, d \in R$ or C or Z_p, with T, I, F as in classical Neutrosophic logic with a the known part and (bT, cI, dF) defined as the unknown part, denoted by $NQ = \{(a, bT, cI, dF) | a, b, c, d \in R$ or C or $Z_n\}$ in called the Neutrosophic set of quadruple numbers.*

The following operations are defined on NQ, for more refer [26].

For $x = (a, bT, cI, dF)$ and $y = (e, fT, gI, hF)$ in NQ [26] have defined

$$x + y = (a, bT, cI, dF) + (e, fT, gI, hF) = (a + e, (b + f)T, (c + g)I, (d + h)F)$$

$$\text{and} \quad x - y = (a - e, (b - f)T, (c - g)I, (d - h)F)$$

are in NQ. For $x = (a, bT, cI, dF)$ in NQ and s in R or C or Z_p where s is a scalar and x is a vector in V. $s.x = s.(a, bT, cI, dF) = (sa, sbT, scI, sdF) \in V$.

If $x = 0 = (0, 0, 0, 0)$ in V usually termed as zero Neutrosophic Quadruple vector and for any scalar s in R or C or Z_p we have $s.0 = 0$.

Further $(s + t)x = sx + tx, s(tx) = (st)x, s(x + y) = sx + sy$ for all $s, t \in R$ or C or Z_p and $x, y \in NQ$. $-x = (-a, -bT, -cI, -dF)$ which is in NQ.

The main results proved in [26] and which is used in this paper are mentioned below;

Theorem 1 ([26]). *$(NQ, +)$ is an abelian group.*

Theorem 2 ([26]). *$(NQ, .)$ is a monoid which is commutative.*

We mainly use only these two results in this paper, for more literature about Neutrosophic Quadruples refer [26].

3. Neutrosophic Quadruple Vector Spaces and Their Properties

In this section we proceed on to define for the first time the new notion of Neutrosophic Quadruple vector spaces (NQ -vector spaces) their NQ vector subspaces, NQ bases and direct sum of NQ vector subspaces. All these NQ vector spaces are defined over R, the field of reals or C, the field of complex numbers and finite field of characteristic p, Z_p, p a prime. All these three NQ vector spaces are different in their properties and we prove all three NQ vector spaces defined over R or C or Z_P are of dimension 4.

We mostly use the notations from [26]. They have proved $(NQ, +) = \{(a, bT, cI, dF) | a, b, c, d \in R$ or C or Z_p, p a prime; $+\}$ is an infinite abelian group under addition.

We prove the following theorem.

Theorem 3. *$(NQ, +) = \{(a, bT, cI, dF) | a, b, c, d \in R$ or C or Z_p; p a prime, $+\}$ be the Neutrosophic quadruple group. Then $V = (NQ, +, \circ)$ is a Neutrosophic Quadruple vector space (NQ-vector space) over R or C or Z_p, where '$\circ$' is the special type of operation between V and R (or C or Z_p) defined as scalar multiplication.*

Proof. To prove V is a Neutrosophic quadruple vector space over R (or C or Z_p, p is a prime), we have to show all the conditions given in Section two (Definition 1) of this paper is satisfied. In the first place we have R or C or Z_p are field of scalars, and elements of V we call as vectors. It has been proved by [26] that $V = (NQ, +)$ is an additive abelian group, which is the basic property on V to be a vector space. Further the quadruple is defined using R or C or Z_p, p a prime, or used in the mutually exclusive sense. Now we see if $x = (a, bT, cI, dF)$ is in V and $n \in R$ (or C or Z_p) then the scalar multiplication '$\circ$' which associates with each scalar $n \in R$ and the NQ vector $x \in V$,

$n \circ x = n \circ (a, bT, cI, dF) = (n \circ a, n \circ bT, n \circ cI, n \circ dF)$ which is in V, called the product of n with x in such a way that

1. $1 \circ x = x \circ 1 \quad \forall x \in V$
2. $(nm) \circ v = n \circ (mv)$
3. $n \circ (v + w) = n \circ v + n \circ w$
4. $(m + n) \circ v = m \circ v + n \circ v$

for all $m, n \in R$ or C or Z_p and $v, w \in V$.

$0 = (0, 0, 0, 0)$ is the zero vector of V and for 0 in R or C or Z_p; we have $0 \circ x = 0 \circ (a, bT, cI, dF) = (0, 0, 0, 0); \forall x \in V$.

Clearly $V = (NQ, +, \circ)$ is a vector space known as the NQ vector space over R or C or Z_p. □

However we can as in case of vector spaces say in case of NQ-vector spaces also $(NQ, +)$ is a NQ vector space with special scalar multiplication $\circ$.

We now proceed on to define the concept of linear dependence, linear independence and basis of NQ vector spaces.

Definition 5. *Let $V = (NQ, +)$ be a NQ vector space over R (or C or Z_p). A subset L of V is said to be NQ linearly dependent or simply dependent, if there exists distinct vectors $a_1, a_2, \ldots, a_k \in L$ and scalars $d_1, d_2, \ldots, d_k \in R$ (or C or Z_p) not all zero such that $d_1 \circ a_1 + d_2 \circ a_2 + \ldots + d_k \circ a_k = 0$. We say the set of vectors $a_1, a_2, \ldots, a_k$ is NQ linearly independent if it is not NQ linearly dependent.*

We provide an example of this situation.

Example 4. *Let $V = (NQ, +)$ vector space over R. Let $x = (3, -4T, 5I, 2F), y = (-2, 3T, -2I, -2F)$ and $z = (-1, T, -3I, 0)$ be in V. We see $1 \circ x + 1 \circ y + 1 \circ z = (0, 0, 0, 0)$, so x, y and z are NQ linearly dependent. Let $x = (5, 0, 0, 2F)$ and $y = (0, 5T, -3I, 0)$ be in V. We cannot find a $a, b \in R$ such that $a \circ x + b \circ y = (0, 0, 0, 0)$. If possible $a \circ x + b \circ y = (0, 0, 0, 0)$; this implies $a \circ 5 + b \circ 0 = 0$, forcing $a = 0$; $a \circ 0 + b \circ 5 = 0$, forcing $b = 0$; $a \circ 0 + b \circ -3 = 0$, forcing $b = 0$ and $a \circ 2 + b \circ 0 = 0$ forcing $a = 0$. Thus the equations are consistent and $a = b = 0$. So x and y are NQ linearly independent over R.*

The following properties are true in case of all vector spaces hence true in case of NQ vector spaces also.

1. A subset of a NQ linearly independent set is NQ linearly independent.
2. A set L of vectors in NQ is linearly independent if and only if for any distinct vectors $a_1, a_2, \ldots, a_k$ of L; $d_1 \circ a_1 + d_2 \circ a_2 + \ldots + d_k \circ a_k = 0$ implies each $d_i = 0$, for $i = 1, 2, \ldots, k$.

We now proceed on to define Neutrosophic Quadruple basis (NQ basis) for $V = (NQ, +)$, Neutrosophic Quadruple vector space over R or C or Z_p (or used in the mutually exclusive sense).

Definition 6. *Let $V = (NQ, +)$ vector space over R (or C or Z_p). We say a subset L of V spans V if and only if every vector in V can be got as a linear combination of elements from L and scalars from R (or C or Z_p). That is if $a_1, a_2, \ldots, a_n$ are n elements in L; then $v = d_1 \circ a_1 + d_2 \circ a_2 + \ldots + d_n \circ a_n$, is the NQ linear combination of vectors of L; where $d_1, d_2, \ldots, d_n$ are in R or C or Z_p and not all these scalars are zero.*

The Neutrosophic Quadruple basis for $V = (NQ, +)$ is a set of vectors in V which spans V. We say a set of vectors B in V is a basis of V if B is a linearly independent set and spans V over R or C or Z_p.

We say V is finite dimensional if the number of elements in basic of V is a finite set; otherwise V is infinite dimensional.

Theorem 5. *Let $V = (NQ, +)$ be the Neutrosophic Quadruple vector space over R (or C or Z_p). V is a finite dimensional NQ vector space over R (or C or Z_p) and dimension of these NQ vector spaces over R(or C or Z_p) are always four.*

Proof. Let $V = (NQ, +) = \{(a, bT, cI, dF)|a, b, c, d \in R$ (or C or Z_p), $+\}$, be the collection of all neutrosophic quadruples of the Neutrosophic Quadruple vector space over R (or C or Z_p). To prove dimension of V over R is four it is sufficient to prove that V has four linearly independent vectors which can span V, which will prove the result. Take the set $B = \{(1, 0, 0, 0), (0, T, 0, 0), (0, 0, I, 0), (0, 0, 0, F)\}$ contained in V; to show B is independent and spans V it enough if we prove for any $v = (a, bT, cI, dF) \in V$, v can be represented uniquely as a linear combination of elements from B and scalars from R (or C or Z_p). Now $v = (a, bT, cI, dF) = a \circ (1, 0, 0, 0) + b \circ (0, T, 0, 0) + c \circ (0, 0, I, 0) + d \circ (0, 0, 0, F)$ for the scalars $a, b, c, d \in R$ (or C or Z_p). Hence we see the elements of V are uniquely represented as a linear combination of vectors using only B, further B is a set of linearly independent elements, hence B is a basis of V and B is finite, so V is finite dimensional over R (or C or Z_p). As order of B is four, dimension of all NQ vector spaces V over R (or C or Z_p) is four. Hence the theorem. □

We call the NQ basis B as the special standard NQ basis of V.

Definition 7. *Let $V = (NQ, +)$ be a NQ vector space over R (or C or Z_p). A subset W of V is said to be Neutrosophic Quadruple vector subspace of V if W itself is a Neutrosophic Quadruple vector space over R (or C or Z_p).*

We will illustrate this situation by examples.

Example 6. *Let $V = \{NQ, +\}$ be a NQ vector space over R. $W = \{(a, bT, 0, 0)|a, b \in R\}$ is a subset of V which is a NQ vector subspace of V over R. $U = \{(0, 0, cI, dF)|c, d \in R\}$ is again a vector subspace of V and is different from W.*

We observe that the only common element between W and U is the zero quadruple vector $(0, 0, 0, 0)$.

Further it is observed if we define the dot product or inner product on elements in V. For $x = (a, bT, cI, dF)$ and $y = (e, fT, gI, hF) \in V$, $x \bullet y$ denoted as $x \bullet y = (a \bullet e, bT \bullet fT, cI \bullet gI, dF \bullet hF)$; and $x \bullet y$ is in V. If $x \bullet y = (0, 0, 0, 0)$ for some $x, y \in V$ then we say x is orthogonal (or dual) with y and vice versa. In fact $x \bullet y = y \bullet x$; $\forall x, y \in V$. We say two NQ vector subspaces W and U are orthogonal (or dual subspaces) if for every $x \in W$ and for every $y \in U$; $x \bullet y = (0, 0, 0, 0)$, that is two NQ vector subspaces are orthogonal if and only if the dot product of every vector in W with every vector in U is the zero vector.

$\{(0, 0, 0, 0)\}$ is the zero vector subspace of V. Every NQ vector subspace of V trivial or nontrivial is orthogonal with the zero vector subspace $\{(0, 0, 0, 0)\}$ of V. V the NQ vector space is orthogonal with only the zero vector subspace of V, and with no other vector subspace of V. W orthogonal U = $W \bullet U$ = $\{w \bullet u|w \in W$ and $u \in U\}$ = $\{(0, 0, 0, 0)\}$; we call the pair of NQ subspaces as orthogonal or dual NQ subspaces of V.

Definition 8. *Let $V = (NQ, +)$ be a Neutrosophic Quadruple vector space over R (or C or Z_p); $W_1, W_2, \ldots, W_n$ be n distinct NQ vector subspaces of V. We say $V = W_1 \oplus W_2 \oplus \ldots \oplus W_n$ is a direct sum of NQ vector subspaces if and only if the following conditions are true;*

1. *Every vector $v \in V$ can be written in the form $v = d_1 \circ w_1 + d_2 \circ w_2 + \ldots + d_n \circ w_n$, where $d_1, d_2, \ldots, d_n$ are in R (or C or Z_p) not all zero with $w_i \in W_i$, $i = 1, 2, \ldots, n$.*
2. *$W_i \bullet W_j = \{(0, 0, 0, 0)\}$ for $i \neq j$ and true for all i, j varying in the set $\{1, 2, \ldots, n\}$.*

First we record that in case of all NQ vector spaces over R (or C or Z_p) we can have the value of n given in definition to be only four, we cannot have more than four as dimension of all NQ vector spaces are only four. Secondly the minimum of n can be two which is true in case of all vector spaces of any finite dimension. Finally we wish to prove not all NQ vector subspaces are orthogonal and there are only finitely many nontrivial NQ vector subspaces for any NQ vector space over R (or C or Z_p).

We prove as theorem a few of the properties.

Theorem 7. *Let $V = (NQ, +)$ be a NQ vector space over R (or C or Z_p). V has only finite number of NQ vector subspaces.*

Proof. We see in case of NQ vector spaces over R (or C or Z_p) the dimension is four and the special standard NQ basis for V is $B = \{(1,0,0,0),(0,T,0,0),(0,0,I,0),(0,0,0,F)\}$. So any non trivial subspace of V can be of dimension less than four; so it can be 1 or 2 or 3. Clearly there are some vector subspaces of dimension one given by, $W_1 = \langle(1,0,0,0)\rangle$, $W_2 = \langle(0,T,0,0)\rangle$, $W_3 = \langle(0,0,I,0)\rangle$, $W_4 = \langle(0,0,0,F)\rangle$, $W_5 = \langle(1,T,0,0)\rangle$, $W_6 = \langle(1,0,I,0)\rangle$, $W_7 = \langle(1,0,0,F)\rangle$, $W_8 = \langle(0,T,I,0)\rangle$, $W_9 = \langle(0,T,0,F)\rangle$, $W_{10} = \langle(0,0,I,F)\rangle$, $W_{11} = \langle(1,T,I,0)\rangle$, $W_{12} = \langle(1,T,0,F)\rangle$, $W_{13} = \langle(1,0,I,F)\rangle$, $W_{14} = \langle(0,T,I,F)\rangle$ and $W_{15} = \langle(1,T,I,F)\rangle$. Some the two dimensional vector spaces are $U_1 = \langle(1,0,0,0),(0,T,0,0)\rangle$, $U_2 = \langle(1,0,0,0),(0,0,I,0)\rangle, \ldots, U_{105} = \langle(0,T,I,F),(1,T,I,F)\rangle$;

in fact there are 105 NQ vector subspaces of dimension two. Further there are 1365 NQ vector subspaces of dimension three. Thus there are 1485 non trivial NQ vector subspaces in any NQ vector space $V = (NQ,+)$ over R (or C or Z_p). We have shown that there are four NQ vector subspaces of dimension three all of them are hyper subspaces of V, of course we are not enumerating other types of dimension three subspaces generated by vectors of the form $M_1 = \{\langle(1,T,0,0),(0,0,I,0),(0,0,0,F)\rangle\}$, or $M_2 = \{\langle(1,0,0,F),(0,0,I,0),(0,T,0,0)\rangle\}$ are spaces of dimension three which we do not take into account as hyper subspaces. □

We define the three dimensional NQ vector subspace generated only by $\{\langle(0,T,0,0),(0,0,I,0),(0,0,0,F)\rangle\}$ is defined as the special pseudo Singled Valued Neutrosophic hyper NQ vector subspace of V [22,24].

4. Neutrosophic Quadruple Linear Algebras over R or C or Z_p

In this section we take the basic concepts defined in [26] $(NQ,+)$ for the Neutrosophic Quadruple additive abelian group and $(NQ,.)$ as the commutative monoid with $(1,0,0,0)$ as the identity with respect to '.' and for any $(a,bT,cI,dF) = x$, and $y = (e,fT,gI,hF)$ in NQ [26] have defined $x.y = (ae,(af+be+bf)T,(ag+bg+ce+cf+cg)I,(ah+bh+ch+de+df+dg+dh)F)$.

Theorem 8. *$V = (NQ,+,.)$ is a Neutrosophic Quadruple linear algebra (NQ linear algebra) over R (or C or Z_p).*

Proof. To prove V is a NQ linear algebra we have to prove the following; $(NQ,+)$ is an abelian group under addition given in [26] and it is proved that $(NQ,+)$ is a vector space (Theorem 3). To prove V is a NQ linear algebra it is sufficient if we prove $(NQ,.)$ is a monoid under product '.' which is proved in [26], further $d \circ (x.y) = (d \circ x).y$ for $d \in R$ (or C or Z_p) and $x, y \in V$ which is true as $x.y$ is in V. Thus $(V,+,.)$ is a NQ linear algebra over R (or C or Z_p). □

Definition 9. *Let $V = (NQ,+,.)$ be a NQ linear algebra over R (or C or Z_p). Let W be a nonempty proper subset of V, we say W is a NQ sublinear algebra of V over R (or C or Z_p), if W itself is a linear algebra over R (or C or Z_p).*

We provide some examples of them.

Example 9. *Let $V = (NQ,+.)$ be a linear algebra over the field Z_7. $W = \{\langle(1,0,0,0)\rangle\}$ generated under +, . and '$\circ$' multiplication by scalar from elements of Z_7 is a sublinear algebra and of order 7 and dimension of W over Z_7 is one. Similarly $U = \{\langle(1,t,0,0),(0,0,I,0)\rangle\}$ generated by these two vectors is a sublinear algebra of dimension two. Just we show how the product of $x = (3,4T,I,5F)$ and $y = (2,3T,4I,F)$ in V is carried out; $x.y = (6,2T,I,2F)$ which is in V.*

We can as in case of NQ vector spaces derive all properties of NQ linear algebras , further as in case of NQ vector spaces dimension of all these NQ-linear algebras is four.

We in the following section propose some open conjectures and the future work to be carried out in this direction.

5. Conclusions and Open Conjectures

In this paper for the first time we define the notion of NQ vector spaces and NQ linear algebras. All the three NQ vector spaces are of dimension four only. The NQ vector space V over R, is different from the NQ vector space W over C, and both has infinite number of vectors; but is of dimension four and U the NQ vector space over Z_p has only p^4 elements and is of dimension four.

We know the classical result on vector spaces states "A vector space V of say dimension n (n a finite integer) defined over the field F is isomorphic to $F \times F \times \ldots \times F$ n-times"; in view of this we propose the following conjectures:

1. Is the NQ vector space V defined over R isomorphic to $R \times R \times R \times R$?
2. Is the NQ vector space W defined over C isomorphic to $C \times C \times C \times C$?
3. Is the NQ vector space U defined over Z_p isomorphic to $Z_p \times Z_p \times Z_p \times Z_p$?

Finally we would be developing the new notion of NQ algebraic codes and analyse them for future research. In our opinion a new type of NQ algebraic codes can certainly be defined with appropriate modifications. Also we would develop the notion of Neutrosophic quadruples in which the unknown part would be these neutrosophic triplets or modified form of neutrosophic duplets which would be taken for further study.

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Abbreviations

The following abbreviations are used in this manuscript:

NQ Neutrosophic Quadruple

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