Dubey et al. *Journal of Inequalities and Applications* (2020) 2020:30
https://doi.org/10.1186/s13660-019-2279-0

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

New class of G-Wolfe-type symmetric duality model and duality relations under G_f-bonvexity over arbitrary cones

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Abstract

This paper is devoted to theoretical aspects in nonlinear optimization, in particular, duality relations for some mathematical programming problems. In this paper, we introduce a new generalized class of second-order multiobjective symmetric G-Wolfe-type model over arbitrary cones and establish duality results under G_f-bonvexity/G_f-pseudobonvexity assumptions. We construct nontrivial numerical examples which are G_f-bonvex/G_f-pseudobonvex but neither η-bonvex/η-pseudobonvex nor η-invex/η-pseudoinvex.

MSC: 90C26; 90C30; 90C32; 90C46

Keywords: Second-order; Multiobjective; Efficient solution; Wolfe; G_f-Bonvexity/G_f-pseudobonvexity

1 Introduction

It is an undeniable fact that all of us are optimizers as we all make decisions for the sole purpose of maximizing our quality of life, productivity in time, and our welfare in some way or another. Since this is an ongoing struggle for creating the best possible among many inferior designs and is always the core requirement of human life, this fact yields the development of a massive number of techniques in this area, starting from the early ages of civilization until now. The efforts and lives behind this aim dedicated by many brilliant philosophers, mathematicians, scientists, and engineers have brought a high level of civilization we enjoy today. The decision process is relatively easier when there is a single criterion or object in mind. The process gets complicated when we have to make decisions in the presence of more than one criteria to judge the decisions. In such circumstances a single decision that optimizes all the criteria simultaneously may not exist. For handling such type of situations, we use multiobjective programming, also known as multiattribute optimization, which is the process of simultaneously optimizing two or more conflicting objectives subject to certain constraints. Multiobjective optimization problems can be found in various fields such as product and process design, finance, aircraft design, the oil and gas industry, automobile design, and other where optimal decisions need to be taken in the presence of trade-offs between two or more conflicting objectives.

Mangasarian [1] was the first who introduced the concept of second-order duality for nonlinear programming. Gulati and Gupta [2] introduced the concept of η_1-bonvexity/ η_2-boncavity and derived duality results for a Wolfe-type model. The concept of G-invex function is given by Antczak [3] and derived some duality results for a constrained optimization problem. Later on, generalizing his earlier work, Antzcak [4] introduced G_f-invex functions for multivariate models and obtained optimality results for multiobjective programming problems. Liang et al. [5] discussed conditions for optimality and duality in a multiobjective programming problem. Bhatia and Garg [6] discussed the concept of (V,p) invexity for nonsmooth vector functions and established duality results for multiobjective programs. Jayswal et al. [7] discussed multiobjective fractional programming problem involving an invex function. Stefaneseu and Ferrara [8] studied new invexities for multiobjective programming problem. Several researchers [9–21] have studied related areas.

This paper is organized as follows. In Sect. 2, we give some preliminaries and definitions used in this paper and also a nontrivial example for such type functions. In Sect. 3, we formulate second-order multiobjective symmetric G-Wolfe-type dual programs over arbitrary cones. We prove weak, strong, and converse duality theorems by using G_f-bonvexity/G_f-pseudobonvexity assumptions over arbitrary cones. Finally, we construct nontrivial numerical examples that are G_f-bonvex/G_f-pseudobonvex but neither η-bonvex/η-pseudobonvex nor η-invex/η-pseudoinvex functions.

2 Preliminaries and definitions

Let $f = (f_1,f_2,f_3,\dots,f_k) : X \to R^k$ be a vector-valued differentiable function defined on a nonempty open set $X \subseteq R^n$, and let $I_{f_i}(X)$, $i = 1,\dots,k$, be the range of f_i, that is, the image of X under f_i. Let $G_f = (G_{f_1}, G_{f_2},\dots,G_{f_k}) : R \to R^k$ be a differentiable function such that every component $G_{f_i} : I_{f_i}(X) \to R$ is strictly increasing on the range of I_{f_i}, $i = 1,2,3,\dots,k$.

Definition 2.1 The positive polar cone $S^*$of a cone $S \subseteq R^s$ is defined by

$$S^* = \{y \in R^s : x^T y \geq 0\}.$$

Consider the following vector minimization problem:

$$\begin{aligned}&\text{Min.}\quad f(y) = \{f_1(y),f_2(y),\dots,f_k(y)\}^T\\ &\quad\text{subject to}\quad S^0 = \{y \in S \subset R^n : h_j(y) \leq 0, j = 1,2,3,\dots,m\},\end{aligned}\tag{MP}$$

where $f = \{f_1,f_2,\dots,f_k\} : S \to R^k$ and $h = \{h_1,h_2,\dots,h_m\} : S \to R^m$ are differentiable functions on S.

Definition 2.2 $\bar{y} \in S^0$ is an efficient solution of (MP) if there exists no other $y \in S^0$ such that $f_r(y) < f_r(\bar{y})$ for some $r = 1,2,3,\dots,k$ and $f_i(y) \leq f_i(\bar{y})$ for all $i = 1,2,3,\dots,k$.

Definition 2.3 If there exists a function $\eta : S \times S \to R^n$ such that for all $y \in S$,

$$f_i(y) - f_i(v) \geq \eta^T(y,v)\nabla_y f_i(v) \quad \text{for all } i = 1,2,3,\dots,k,$$

then f is called invex at $v \in S$ with respect to η.

Definition 2.4 If there exist $G_{f_i}: I_{f_i}(S) \to R$ and $\eta: S \times S \to R^n$ such that for all $y \in S$,

$$\eta^T(y,\nu)G'_{f_i}\big(f_i(\nu)\big)\nabla_y f_i(\nu) \geq 0 \quad \Rightarrow \quad G_{f_i}\big(f_i(x)\big) - G_{f_i}\big(f_i(\nu)\big) \geq 0 \quad \text{for all } i = 1,2,3,\ldots,k,$$

then f_i is called G_{f_i}-pseudoinvex at $u \in S$ with respect to η.

Definition 2.5 If there exist $G_{f_i}: I_{f_i}(S) \to R$ and $\eta: S \times S \to R^n$ such that for all $y \in S$ and $p \in R^n$,

$$\begin{aligned} G_{f_i}\big(f_i(y)\big) - G_{f_i}\big(f_i(\nu)\big) \geq \eta^T(y,\nu)\big[&G'_{f_i}\big(f_i(\nu)\big)\nabla_y f_i(\nu) \\ &+ \big\{G''_{f_i}\big(f_i(\nu)\big)\nabla_y f_i(\nu)\big(\nabla_y f_i(\nu)\big)^T + G'_{f_i}\big(f_i(\nu)\big)\nabla_{yy} f_i(\nu)\big\}p\big] \\ &- \frac{1}{2}p^T\big[G''_{f_i}\big(f_i(\nu)\big)\nabla_y f_i(\nu)\big(\nabla_y f_i(\nu)\big)^T + G'_{f_i}\big(f_i(\nu)\big)\nabla_{yy} f_i(\nu)\big]p \\ &\text{for all } i = 1,2,3,\ldots,k, \end{aligned}$$

then f_i is called G_{f_i}-bonvex at $\nu \in S$ with respect to η.

Definition 2.6 If there exist functions G_f and $\eta: S \times S \to R^n$ such that for all $y \in S$ and $p \in R^n$,

$$\begin{aligned} &\eta^T(y,\nu)\big[G'_{f_i}\big(f_i(\nu)\big)\nabla_y f_i(\nu) \\ &\quad + \big\{G''_{f_i}\big(f_i(\nu)\big)\nabla_y f_i(\nu)\big(\nabla_y f_i(\nu)\big)^T + G'_{f_i}\big(f_i(\nu)\big)\nabla_{yy} f_i(\nu)\big\}p\big] \geq 0 \\ &\quad \Rightarrow \quad G_{f_i}\big(f_i(y)\big) - G_{f_i}\big(f_i(\nu)\big) \\ &\qquad\qquad + \frac{1}{2}p^T\big[G''_{f_i}\big(f_i(\nu)\big)\nabla_y f_i(\nu)\big(\nabla_y f_i(\nu)\big)^T + G'_{f_i}\big(f_i(\nu)\big)\nabla_{yy} f_i(\nu)\big]p \geq 0 \\ &\qquad\qquad \text{for all } i = 1,2,3,\ldots,k, \end{aligned}$$

then f_i is called G_{f_i}-pseudobonvex at $\nu \in S$ with respect to η.

Now, we discuss nontrivial numerical examples that are G_f-bonvex/G_f-pseudobonvex but neither η-bonvex/η-pseudobonvex nor η-invex/η-pseudoinvex functions.

Example 2.1 Let $f: [-1,1] \to R^4$ be defined as

$$f(y) = \big\{f_1(y), f_2(y), f_3(y), f_4(y)\big\},$$

where $f_1(y) = y^{10}$, $f_2(y) = \arcsin y$, $f_3(y) = \arctan y$, $f_4(y) = \operatorname{arccot} y$, and let $G_f = \{G_{f_1}, G_{f_2}, G_{f_3}, G_{f_4}\}: R \to R^4$ be defined as

$$G_{f_1}(t) = t^5 + 5, \qquad G_{f_2}(t) = \sin t + 2, \qquad G_{f_3}(t) = \tan t + 9, \qquad G_{f_4}(t) = \cot t + 2.$$

Let $\eta: [-1,1] \times [-1,1] \to R$ be given as

$$\eta(y,\nu) = -\frac{1}{9}y^{14} + y + \frac{1}{99}y^{17}\nu^5 - \frac{1}{7}y^4\nu^3 + \nu^3.$$

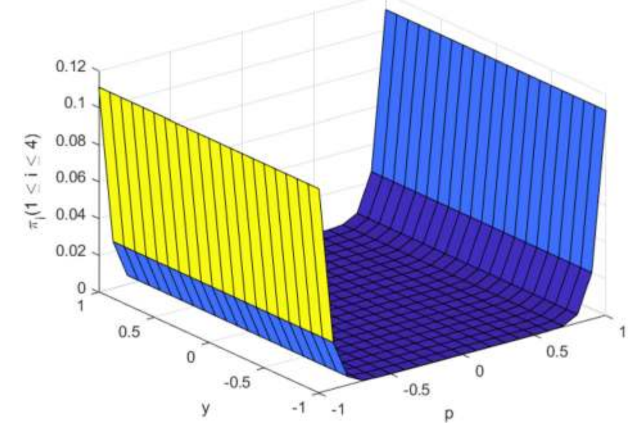

Figure 1 $\pi_i \geq 0$ ($i = 1,2,3,4$), $\forall p, \forall y \in [-1,1]$

To show that f is G_f-bonvex at $\nu = 0$ with respect to η, we have to claim that

$$\begin{aligned}\pi_i &= G_{f_i}\big(f_i(y)\big) - G_{f_i}\big(f_i(\nu)\big) \\ &\quad - \eta^T(y,\nu)\big[G'_{f_i}\big(f_i(\nu)\big)\nabla_y f_i(\nu) + \big\{G''_{f_i}\big(f_i(\nu)\big)\nabla_y f_i(\nu)\big(\nabla_y f_i(\nu)\big)^T + G'_{f_i}\big(f_i(\nu)\big)\nabla_{yy} f_i(\nu)\big\}p\big] \\ &\quad + \frac{1}{2}p^T\big[G''_{f_i}\big(f_i(\nu)\big)\nabla_y f_i(\nu)\big(\nabla_y f_i(\nu)\big)^T + G'_{f_i}\big(f_i(\nu)\big)\nabla_{yy} f_i(\nu)\big]p \\ &\geq 0, \quad i = 1,2,3,4.\end{aligned}$$

Putting the values of f_i, G_{f_i}, $i = 1,2,3,4$, into the last expression, after simplifying at the point $\nu = 0 \in [-1,1]$, we clearly see from Fig. 1 that $\pi_i \geq 0$, $i = 1,2,3,4$, for all $y \in [-1,1]$. Therefore f is G_f-bonvex at $\nu = 0 \in [-1,1]$ with respect to η and p.

Now, suppose

$$\xi = f_3(y) - f_3(\nu) - \eta^T(y,\nu)\big[\nabla_y f_3(\nu) - \nabla_{yy} f_3(\nu)p\big] + \frac{1}{2}p^T\big[\nabla_{yy} f_3(\nu)\big]p$$

or

$$\begin{aligned}\xi &= \arctan y - \arctan \nu \\ &\quad - \left(-\frac{1}{9}y^{14} + y + \frac{1}{99}y^{17}\nu^5 - \frac{1}{7}y^4\nu^3 + \nu^3\right)\left[\frac{1}{1+\nu^2} - \frac{2\nu p}{(1+\nu^2)^2}\right] - \frac{\nu p^2}{(1+\nu^2)^2}, \\ \xi &= \arctan y + \frac{1}{9}y^{14} - y \quad \text{at } \nu = 0, \\ \xi &\ngeq 0 \quad (\text{from Fig. 2}).\end{aligned}$$

Therefore f_3 is not η-bonvex at $\nu = 0$ with respect to p. Hence f is not η-bonvex at $\nu = 0$ with respect to p.

Next,

$$\delta = f_3(y) - f_3(\nu) - \eta^T(y,\nu)\nabla_y f_3(\nu)$$

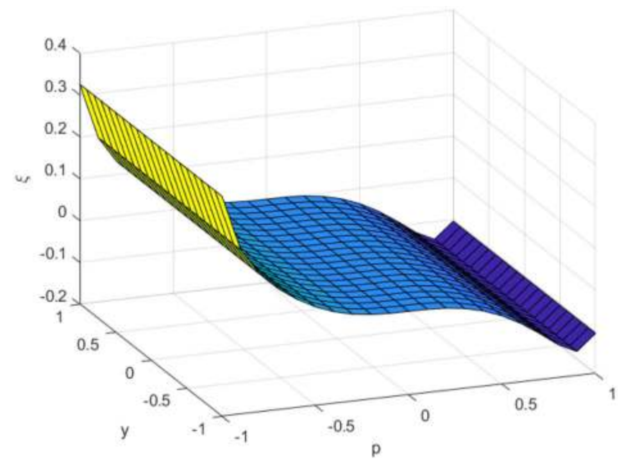

Figure 2 $\xi = \arctan y + \frac{1}{9}y^{14} - y \ngeq 0$, at $v = 0$, $\forall p, \forall y \in [-1, 1]$

or

$$\begin{aligned}
\delta &= \arctan y - \arctan v - \left(-\frac{1}{9}y^{14} + y + \frac{1}{99}y^{17}v^5 - \frac{1}{7}y^4v^3 + v^3\right)\frac{1}{1+v^2}, \\
\delta &= \arctan y + \frac{1}{9}y^{14} - y \quad \text{at } v = 0, \\
\delta &= \frac{\pi}{4} + \frac{1}{9} - 1 < 0 \quad \text{at } y = 1 \in [-1, 1].
\end{aligned}$$

Therefore f_3 is not η-invex at $v = 0$. Hence f is not η-invex at $v = 0$.

Example 2.2 Let $f : [-2, 2] \to R^2$ be defined as

$$f(y) = \{f_1(y), f_2(y)\},$$

where $f_1(y) = (\frac{e^{2y}-1}{e^y})$, $f_2(y) = y^3$, and $G_f = \{G_{f_1}, G_{f_2}\} : R \to R^2$ is defined as

$$G_{f_1}(t) = t^2 + 1, \qquad G_{f_2}(t) = t^2 + 3.$$

Let $\eta : [-2, 2] \times [-2, 2] \to R$ be given as

$$\eta(y, v) = y^6 + v^9y^4 + v^5y + v + 3.$$

To show that f is G_f-pseudobonvex at $v = 0$ with respect to η, we have to claim that, for $i = 1, 2$,

$$\begin{aligned}
\zeta_i &= \eta^T(y, v)\left[G'_{f_i}(f_i(v))\nabla_y f_i(v) + \left\{G''_{f_i}(f_i(v))\nabla_y f_i(v)\left(\nabla_y f_i(v)\right)^T + G'_{f_i}(f_i(v))\nabla_{yy} f_i(v)\right\}p\right] \geq 0 \\
&\Rightarrow \quad G_{f_i}(f_i(y)) - G_{f_i}(f_i(v)) \\
&\qquad + \frac{1}{2}p^T\left[G''_{f_i}(f_i(v))\nabla_y f_i(v)\left(\nabla_y f_i(v)\right)^T + G'_{f_i}(f_i(v))\nabla_{yy} f_i(v)\right]p \geq 0.
\end{aligned}$$

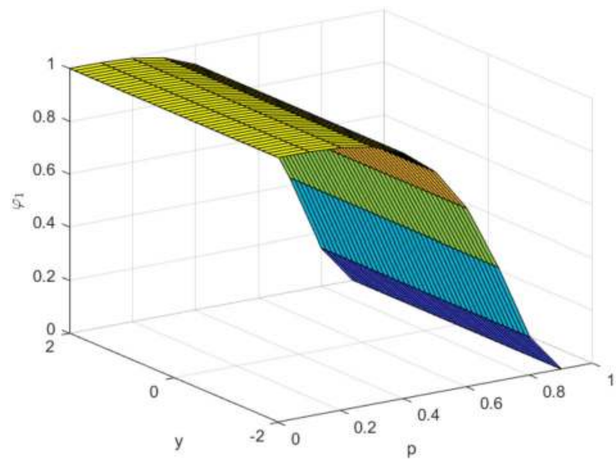

Figure 3 $\varphi_1 = (\frac{e^{2y}-1}{e^y})^2 \geq 0$ for all $y \in [-2,2]$ and p

Let

$$\phi_1 = \eta^T(y,v)\big[G'_{f_1}\big(f_1(v)\big)\nabla_y f_1(v) + \big\{G''_{f_1}\big(f_1(v)\big)\nabla_y f_1(v)\big(\nabla_y f_1(v)\big)^T + G'_{f_1}\big(f_1(v)\big)\nabla_{xx} f_1(u)\big\}p\big].$$

Substituting the values of η and f_1 at the point $v = 0$, we get

$$\phi_1 \geq 0 \quad \text{for all } y \in [-2,2] \text{ and } p.$$

Next, consider

$$\varphi_1 = G_{f_1}\big(f_1(y)\big) - G_{f_1}\big(f_1(v)\big) + \frac{1}{2}p^T\big[G''_{f_1}\big(f_1(v)\big)\nabla_y f_1(v)\big(\nabla_y f_1(v)\big)^T + G'_{f_1}\big(f_1(v)\big)\nabla_{xx} f_1(u)\big]p_1.$$

At $v = 0$, we get $\varphi_1 \geq 0$ for all $y \in [-1,1]$ and p (from Fig. 3);

$$\phi_2 = \eta^T(y,v)\big[G'_{f_2}\big(f_2(v)\big)\nabla_y f_2(v) + \big\{G''_{f_2}\big(f_2(v)\big)\nabla_y f_2(v)\big(\nabla_y f_2(v)\big)^T + G'_{f_2}\big(f_2(v)\big)\nabla_{yy} f_2(v)\big\}p\big],$$
$$\phi_2 = \big(y^6 + v^9y^4 + v^5y + v + 3\big)\big(6v^5 + 30v^5p\big).$$

At the point $v = 0$, we have

$$\phi_2 \geq 0 \quad \text{for all } y \in [-2,2] \text{ and } p.$$

Also,

$$\varphi_2 = G_{f_2}\big(f_2(y)\big) - G_{f_2}\big(f_2(v)\big) + \frac{1}{2}p^T\big[G''_{f_2}\big(f_2(v)\big)\nabla_y f_2(v)\big(\nabla_y f_2(v)\big)^T + G'_{f_2}\big(f_2(v)\big)\nabla_{yy} f_2(v)\big]p,$$
$$\varphi_2 = y^6 - v^6 + 15p^2v^4.$$

At the point $v = 0$, we obtain

$$\varphi_2 \geq 0 \quad \text{for all } y \in [-2,2] \text{ and } p.$$

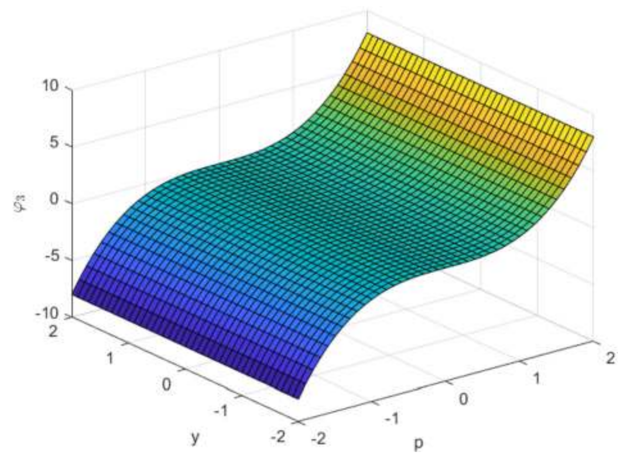

Figure 4 $\varphi_3 \geq 0$ for all $y \in [-2, 2]$ and p

Hence from the expressions ϕ_i and φ_i, $i = 1, 2$, we get that f is G_f-pseudobonvex at $v = 0$ with respect to η.

Next, let

$$\phi_3 = \eta^T(y, v)\big[\nabla_y f_2(v) + \nabla_{yy} f_2(v)p\big],$$
$$\phi_3 = \big(y^6 + v^9y^4 + v^5y + v + 3\big)\big[3v^2 + 6vp\big].$$

At the point $v = 0$, we have

$$\phi_3 \geq 0 \quad \text{for all } y \in [-2, 2] \text{ and } p.$$

Further, consider

$$\varphi_3 = f_2(y) - f_2(v) + \frac{1}{2}p^2\nabla_{yy} f_2(v),$$
$$\varphi_3 = y^3 - v^3 + 3p^2v.$$

At the point $v = 0$, we obtain

$$\varphi_3 \ngeq 0 \quad \text{for all } y \in [-2, 2] \text{ and } p \text{ (from Fig. 4)}.$$

Hence f_2 is not η-pseudobonvex at $v = 0 \in [-2, 2]$. Therefore $f = (f_1, f_2)$ is not η-pseudobonvex at $v = 0 \in [-2, 2]$.

Finally,

$$\phi_4 = \eta^T(y, v)\nabla_y f_2(v),$$
$$\phi_4 = 3\big(y^6 + v^9y^4 + v^5y + v + 3\big)v^2.$$

At the point $v = 0$, we have

$$\phi_4 \geq 0 \quad \text{for all } y \in [-2, 2] \text{ and } p.$$

Also,

$$\varphi_4 = f_2(y) - f_2(v),$$
$$\varphi_3 = y^3 - v^3.$$

At the point $v = 0$, we obtain

$$\varphi_4 \ngeq 0 \quad \text{for all } y \in [-2, 2].$$

Hence f_2 is not η-pseudoinvex at $v = 0 \in [-2, 2]$. Hence $f = (f_1, f_2)$ is not η-pseudoinvex at $v = 0 \in [-2, 2]$.

3 Second-order multiobjective *G*-Wolfe-type symmetric dual program

Consider the following pair of second-order multiobjective G-Wolfe-type dual programs over arbitrary cones.

Primal problem (GWP) Minimize

$$R(y, z, \lambda, p) = \big(R_1(y, z, \lambda_1, p), R_2(y, z, \lambda_2, p), \dots, R_k(y, z, \lambda_k, p)\big)^T$$

subject to

$$\begin{aligned} &-\sum_{i=1}^{k} \lambda_i \big[G'_{f_i}\big(f_i(y,z)\big)\nabla_y f_i(y,z) \\ &\quad + \big\{G''_{f_i}\big(f_i(y,z)\big)\nabla_y f_i(y,z)\big(\nabla_y f_i(y,z)\big)^T + G'_{f_i}\big(f_i(y,z)\big)\nabla_{yy} f_i(y,z)\big\}p\big] \in C_2^*, \quad (1) \\ &\lambda_i > 0, \qquad \lambda^T e_k = 1, \qquad x \in C_1, \qquad i = 1, 2, 3, \dots, k. \quad (2) \end{aligned}$$

Dual problem (GWD) Maximize

$$S(v, w, \lambda, q) = \big(S_1(v, w, \lambda_1, q), S_2(v, w, \lambda_2, q), \dots, S_k(v, w, \lambda_k, q)\big)^T$$

subject to

$$\begin{aligned} &\sum_{i=1}^{k} \lambda_i \big[G'_{f_i}\big(f_i(v,w)\big)\nabla_z f_i(v,w) \\ &\quad + \big\{G''_{f_i}\big(f_i(v,w)\big)\nabla_z f_i(v,w)\big(\nabla_z f_i(v,w)\big)^T + G'_{f_i}\big(f_i(v,w)\big)\nabla_{zz} f_i(v,w)\big\}q\big] \in C_1^*, \quad (3) \\ &\lambda_i > 0, \qquad \lambda^T e_k = 1, \qquad v \in C_2, \qquad i = 1, 2, 3, \dots, k, \quad (4) \end{aligned}$$

where for all $i = 1, 2, 3, \dots, k$,

$$\begin{aligned} R_i(y, z, \lambda, p) = {} & G_{f_i}\big(f_i(y,z)\big) - z^T \sum_{i=1}^{k} \lambda_i \big(G'_{f_i}\big(f_i(y,z)\big)\nabla_y f_i(y,z) \\ & + \big[G''_{f_i}\big(f_i(y,z)\big)\nabla_y f_i(y,z)\big(\nabla_y f_i(y,z)\big)^T + G'_{f_i}\big(f_i(y,z)\big)\nabla_{yy} f_i(y,z)\big]p\big) \\ & - \frac{1}{2}\sum_{i=1}^{k} \lambda_i p^T \big(G''_{f_i}\big(f_i(y,z)\big)\nabla_y f_i(y,z)\big(\nabla_y f_i(y,z)\big)^T + G'_{f_i}\big(f_i(y,z)\big)\nabla_{yy} f_i(y,z)\big)p, \end{aligned}$$

$$
\begin{aligned}
S_i(v,w,\lambda,q) = {} & G_{f_i}\big(f_i(v,w)\big) - v^T \sum_{i=1}^{k} \lambda_i \big(G'_{f_i}\big(f_i(v,w)\big)\nabla_z f_i(v,w) \\
& + \big[G''_{f_i}\big(f_i(v,w)\big)\nabla_z f_i(v,w)\big(\nabla_z f_i(v,w)\big)^T + G'_{f_i}\big(f_i(v,w)\big)\nabla_{zz} f_i(v,w)\big]q\big) \\
& - \frac{1}{2}\sum_{i=1}^{k} \lambda_i q^T \big(G''_{f_i}\big(f_i(v,w)\big)\nabla_z f_i(v,w)\big(\nabla_z f_i(v,w)\big)^T \\
& + G'_{f_i}\big(f_i(v,w)\big)\nabla_{zz} f_i(v,w)\big)q,
\end{aligned}
$$

and

(i) $e_k = (1,1,\ldots,1) \in R^k$ and $\lambda \in R^k$.
(ii) q and p are vectors in R^n and R^m, respectively.

Let Y^0 and Z^0 be the sets of feasible solutions of (GWP) and (GWD), respectively.

Theorem 3.1 (Weak duality) *Let $(y,z,\lambda,p) \in Y^0$ and $(v,w,\lambda,q) \in Z^0$. Suppose that for all $i = 1,2,3,\ldots,k$,*

(i) *$f_i(\cdot,v)$ is G_{f_i}-bonvex at v with respect η,*
(ii) *$f_i(x,\cdot)$ be G_{f_i}-boncave at y with respect η,*
(iii) *$\eta_1(y,v) + u \in C_1$ and $\eta_2(w,z) + y \in C_2$.*

Then the following inequalities cannot hold together:

$$
R_i(y,z,\lambda,p) \le S_i(v,w,\lambda,q) \quad \textit{for all } i = 1,2,3,\ldots,k, \tag{5}
$$

and

$$
R_r(y,z,\lambda,p) < S_r(v,w,\lambda,q) \quad \textit{for at least one } r \in K. \tag{6}
$$

Proof If possible, then suppose inequalities (5) and (6) hold. For $\lambda > 0$, we obtain

$$
\begin{aligned}
& \sum_{i=1}^{k} \lambda_i \Bigg[G_{f_i}\big(f_i(y,z)\big) - z^T \sum_{i=1}^{k} \lambda_i \big(G'_{f_i}\big(f_i(y,z)\big)\nabla_y f_i(y,z) \\
& \quad + G''_{f_i}\big(f_i(y,z)\big)\nabla_y f_i(y,z)\big(\nabla_y f_i(y,z)\big)^T + G'_{f_i}\big(f_i(y,z)\big)\nabla_{yy} f_i(y,z)\big)p \Bigg] \\
& \quad - \frac{1}{2}\sum_{i=1}^{k} \lambda_i \big(p^T \big[G''_{f_i}\big(f_i(y,z)\big)\nabla_y f_i(y,z)\big(\nabla_y f_i(y,z)\big)^T + G'_{f_i}\big(f_i(y,z)\big)\nabla_{yy} f_i(y,z)\big]p\big) \\
& < \sum_{i=1}^{k} \lambda_i \Bigg[G_{f_i}\big(f_i(v,w)\big) - v^T \sum_{i=1}^{k} \lambda_i \big(G'_{f_i}\big(f_i(v,w)\big)\nabla_z f_i(v,w) + G''_{f_i}\big(f_i(v,w)\big)\nabla_z f_i(v,w) \\
& \quad + \big(\nabla_z f_i(v,w)\big)^T + G'_{f_i}\big(f_i(v,w)\big)\nabla_{zz} f_i(v,w)\big)q \Bigg] \\
& \quad - \frac{1}{2}\sum_{i=1}^{k} \lambda_i \big(q^T \big[G''_{f_i}\big(f_i(v,w)\big)\nabla_z f_i(v,w)\big(\nabla_z f_i(v,w)\big)^T + G'_{f_i}\big(f_i(v,w)\big)\nabla_{zz} f_i(v,w)\big]q\big)\big].
\end{aligned} \tag{7}
$$

From assumption (i) we get

$$\begin{aligned}
&G_{f_i}\big(f_i(y,w)\big) - G_{f_i}\big(f_i(v,w)\big)\\
&\quad\geq \eta_1 x, u^T\big[G'_{f_i}\big(f_i(v,w)\big)\nabla_z f_i(v,w)\\
&\qquad+ \big\{G''_{f_i}\big(f_i(v,w)\big)\nabla_z f_i(v,w)\big(\nabla_z f_i(v,w)\big)^T + G'_{f_i}\big(f_i(v,w)\big)\nabla_{zz} f_i(v,w)\big\}q\big]\\
&\qquad- \frac{1}{2}q^T\big[G''_{f_i}\big(f_i(v,w)\big)\nabla_z f_i(v,w)\big(\nabla_z f_i(v,w)\big)^T + G'_{f_i}\big(f_i(v,w)\big)\nabla_{zz} f_i(v,w)\big]q.
\end{aligned}$$

Since $\lambda > 0$, this inequality yields

$$\begin{aligned}
&\sum_{i=1}^{k}\lambda_i\big[G_{f_i}\big(f_i(y,w)\big) - G_{f_i}\big(f_i(v,w)\big)\big]\\
&\quad\geq \eta_1^T(x,u)\Bigg\{\sum_{i=1}^{k}\lambda_i\big[G'_{f_i}\big(f_i(v,w)\big)\nabla_z f_i(v,w)\\
&\qquad+ \big\{G''_{f_i}\big(f_i(v,w)\big)\nabla_z f_i(v,w)\big(\nabla_z f_i(v,w)\big)^T + G'_{f_i}\big(f_i(v,w)\big)\nabla_{zz} f_i(v,w)\big\}q\big]\Bigg\}\\
&\qquad- \frac{1}{2}\sum_{i=1}^{k}\lambda_i q^T\big[G''_{f_i}\big(f_i(v,w)\big)\nabla_z f_i(v,w)\big(\nabla_z f_i(v,w)\big)^T + G'_{f_i}\big(f_i(v,w)\big)\nabla_{zz} f_i(v,w)\big]q. \qquad (8)
\end{aligned}$$

From the dual constraint (3) and assumption (iii) it follows that

$$\begin{aligned}
&\big[\eta_1(y,v) + v\big]^T\Bigg\{\sum_{i=1}^{k}\lambda_i\big[G'_{f_i}\big(f_i(v,w)\big)\nabla_z f_i(v,w)\\
&\quad+ \big\{G''_{f_i}\big(f_i(v,w)\big)\nabla_z f_i(v,w)\big(\nabla_z f_i(v,w)\big)^T + G'_{f_i}\big(f_i(v,w)\big)\nabla_{zz} f_i(v,w)\big\}q\big]\Bigg\} \geq 0,
\end{aligned}$$

which implies

$$\begin{aligned}
&\eta_1(y,v)^T\Bigg\{\sum_{i=1}^{k}\lambda_i\big[G'_{f_i}\big(f_i(v,w)\big)\nabla_z f_i(v,w)\\
&\qquad+ \big\{G''_{f_i}\big(f_i(v,w)\big)\nabla_z f_i(v,w)\big(\nabla_z f_i(v,w)\big)^T + G'_{f_i}\big(f_i(v,w)\big)\nabla_{zz} f_i(v,w)\big\}q\big]\Bigg\}\\
&\quad\geq -v^T\Bigg\{\sum_{i=1}^{k}\lambda_i\big[G'_{f_i}\big(f_i(v,w)\big)\nabla_z f_i(v,w)\\
&\qquad+ \big\{G''_{f_i}\big(f_i(v,w)\big)\nabla_z f_i(v,w)\big(\nabla_z f_i(v,w)\big)^T + G'_{f_i}\big(f_i(v,w)\big)\nabla_{zz} f_i(v,w)\big\}q\big]\Bigg\}.
\end{aligned}$$

Using inequalities (3) and (8), we obtain

$$\begin{aligned}
&\sum_{i=1}^{k}\lambda_i\Bigg[G_{f_i}\big(f_i(y,w)\big) - G_{f_i}\big(f_i(v,w)\big)\\
&\qquad+ \frac{1}{2}\sum_{i=1}^{k}\lambda_i q^T\big[G''_{f_i}\big(f_i(v,w)\big)\nabla_z f_i(v,w)\big(\nabla_z f_i(v,w)\big)^T + G'_{f_i}\big(f_i(v,w)\big)\nabla_{zz} f_i(v,w)\big]q\Bigg]
\end{aligned}$$

$$\geq -v^T\left\{\sum_{i=1}^{k}\lambda_i\big[G'_{f_i}\big(f_i(v,w)\big)\nabla_z f_i(v,w)\right.$$
$$\left.+\big\{G''_{f_i}\big(f_i(v,w)\big)\nabla_z f_i(v,w)\big(\nabla_z f_i(v,w)\big)^T + G'_{f_i}\big(f_i(v,w)\big)\nabla_{zz} f_i(v,w)\big\}q\big]\right\}. \tag{9}$$

Using assumption (iv) and primal constraint (1), we get

$$\sum_{i=1}^{k}\lambda_i\Big[-G_{f_i}\big(f(y,w)\big) + G_{f_i}\big(f_i(y,z)\big)$$
$$-\frac{1}{2}p^T\big[G''_{f_i}\big(f_i(y,z)\big)\nabla_y f_i(y,z)\big(\nabla_y f_i(y,z)\big)^T + G'_{f_i}\big(f_i(y,z)\big)\nabla_{yy} f_i(y,z)\big]p\Big]$$
$$\geq z^T\sum_{i=1}^{k}\lambda_i\big[G'_{f_i}\big(f_i(y,z)\big)\nabla_y f_i(y,z)$$
$$+\big\{G''_{f_i}\big(f_i(y,z)\big)\nabla_y f_i(y,z)\big(\nabla_y f_i(y,z)\big)^T + G'_{f_i}\big(f_i(y,z)\big)\nabla_{yy} f_i(y,z)\big\}p\big]. \tag{10}$$

Finally, adding inequalities (9) and (10) and using $\lambda^T e_k = 1$, we obtain

$$\sum_{i=1}^{k}\lambda_i\Big[G_{f_i}\big(f_i(y,z)\big) - z^T\sum_{i=1}^{k}\lambda_i\big(G'_{f_i}\big(f_i(y,z)\big)\nabla_y f_i(y,z)$$
$$+G'_{f_i}\big(f_i(y,z)\big)\nabla_y f_i(y,z)\big(\nabla_y f_i(y,z)\big)^T + G'_{f_i}\big(f_i(y,z)\big)\nabla_{yy} f_i(y,z)\big)p\Big]$$
$$-\frac{1}{2}\sum_{i=1}^{k}\lambda_i\big(p^T\big[G''_{f_i}\big(f_i(y,z)\big)\nabla_y f_i(y,z)\big(\nabla_y f_i(y,z)\big)^T + G'_{f_i}\big(f_i(y,z)\big)\nabla_{yy} f_i(y,z)\big]p\big)$$
$$\geq \sum_{i=1}^{k}\lambda_i\Big[G_{f_i}\big(f_i(v,w)\big) - v^T\sum_{i=1}^{k}\lambda_i\big(G'_{f_i}\big(f_i(v,w)\big)\nabla_z f_i(v,w)$$
$$+G'_{f_i}\big(f_i(v,w)\big)\nabla_z f_i(v,w)\big(\nabla_z f_i(v,w)\big)^T + G'_{f_i}\big(f_i(v,w)\big)\nabla_{zz} f_i(v,w)\big)q\Big]$$
$$-\frac{1}{2}\sum_{i=1}^{k}\lambda_i\big(q^T\big[G''_{f_i}\big(f_i(v,w)\big)\nabla_z f_i(v,w)\big(\nabla_z f_i(v,w)\big)^T + G'_{f_i}\big(f_i(v,w)\big)\nabla_{yy} f_i(v,w)\big]q\big).$$

This contradicts (7). Hence the result. □

Remark 3.1 Since every G_f-bonvex function is G_f-pseudobonvex, Theorem 3.1 can also be obtained under G_f-pseudobonvexity assumptions.

Remark 3.2 A vector space V over field K, the span of a set S, may be defined as the set of all finite linear combinations of elements (vectors) of S:

$$\operatorname{span}(S) = \left\{\sum_{i=1}^{k}\lambda_i v_i : k \in N, u_i \in S, \lambda_i = 1,2,3,\dots,k\right\}.$$

Theorem 3.2 (Strong duality) *Let* $(\bar{y},\bar{z},\bar{\lambda},\bar{p})$ *be an efficient solution of* (GWP); *fix* $\lambda=\bar{\lambda}$ *in* (GWD) *such that*

(i) *for all* $i=1,2,3,\ldots,k$, $[G''_{f_i}(f_i(\bar{y},\bar{z}))\nabla_z f_i(\bar{y},\bar{z})(\nabla_z f_i(\bar{y},\bar{z}))^T+G'_{f_i}(f_i(\bar{y},\bar{z}))\nabla_{zz}f_i(\bar{y},\bar{z})]$ *is nonsingular,*

(ii) $\sum_{i=1}^k\bar{\lambda}_i\nabla_z(\{G''_{f_i}(f_i(\bar{y},\bar{z}))\nabla_z f_i(\bar{y},\bar{z})(\nabla_z f_i(\bar{y},\bar{z}))^T+G'_{f_i}(f_i(\bar{y},\bar{z}))\nabla_{zz}f_i(\bar{y},\bar{z})\}\bar{p})\bar{p}\notin \operatorname{span}\{G'_{f_1}(f_1(\bar{y},\bar{z}))\nabla_z f_1(\bar{z},\bar{x}),\ldots,G'_{f_k}(f_k(\bar{y},\bar{z}))\nabla_z f_k(\bar{y},\bar{z})\}\setminus\{0\}$,

(iii) *the vectors* $\{G'_{f_1}(f_1(\bar{y},\bar{z}))\nabla_z f_1(\bar{z},\bar{x}),G'_{f_2}(f_2(\bar{y},\bar{z}))\nabla_z f_2(\bar{y},\bar{z}),\ldots,G'_{f_k}(f_k(\bar{y},\bar{z}))\nabla_z f_k(\bar{y},\bar{z})\}$ *are linearly independent,*

(iv) $\sum_{i=1}^k\bar{\lambda}_i\nabla_y(\{G''_{f_i}(f_i(\bar{y},\bar{z}))\nabla_z f_i(\bar{y},\bar{z})(\nabla_z f_i(\bar{y},\bar{z}))^T+G'_{f_i}(f_i(\bar{y},\bar{z}))\nabla_{zz}f_i(\bar{y},\bar{z})\}\bar{p})\bar{p}=0$ *implies that* $\bar{p}=0$.

Then for $\bar{q}=0$, *we have* $(\bar{v},\bar{w},\bar{\lambda},\bar{p}=0)\in Z^0$ *and* $R(\bar{y},\bar{z},\bar{\lambda},\bar{q})=S(\bar{y},\bar{z},\bar{\lambda},\bar{q})$. *Also, from Theorem* 3.1 *it follows that* $(\bar{v},\bar{w},\bar{\lambda},\bar{p}=0)$ *is an efficient solution for* (GWD).

Proof By the Fritz–John necessary conditions [22] there exist $\alpha\in R^k$, $\beta\in R^m$, and $\eta\in R$ such that

$$
\begin{aligned}
&(y-\bar{y})^T\Bigg\{\sum_{i=1}^k\alpha_i\big[G'_{f_i}\big(f_i(\bar{y},\bar{z})\big)\nabla_y f_i(\bar{y},\bar{z})\big]\\
&\quad+\sum_{i=1}^k\bar{\lambda}_i\big[G''_{f_i}\big(f_i(\bar{y},\bar{z})\big)\nabla_y f_i(\bar{y},\bar{z})\nabla_y f_i(\bar{y},\bar{z})+G'_{f_i}\big(f_i(\bar{y},\bar{z})\big)\nabla_{xy}f_i(\bar{y},\bar{z})\big]\big(\beta-\big(\alpha^Te_k\big)\bar{y}\big)\\
&\quad+\sum_{i=1}^k\bar{\lambda}_i\nabla_y\big[\big(G''_{f_i}\big(f_i(\bar{y},\bar{z})\big)\nabla_y f_i(\bar{y},\bar{z})\big(\nabla_y f_i(\bar{y},\bar{z})\big)^T+G'_{f_i}\big(f_i(\bar{y},\bar{z})\big)\nabla_{yy}f_i(\bar{y},\bar{z})\big)\bar{p}\big]\\
&\quad\times\left(\beta-\big(\alpha^Te_k\big)\left(\bar{y}+\frac{1}{2}\bar{p}\right)\right)\Bigg\}=0\quad\text{for all }y\in C_1,
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
&\sum_{i=1}^k\big(\alpha_i-\big(\alpha^Te_k\big)\bar{\lambda}_i\big)\big[G'_{f_i}\big(f_i(\bar{y},\bar{z})\big)\nabla_z f_i(\bar{y},\bar{z})\big]\\
&\quad+\sum_{i=1}^k\bar{\lambda}_i\big[\big\{G''_{f_i}\big(f_i(\bar{y},\bar{z})\big)\nabla_z f_i(\bar{y},\bar{z})\big(\nabla_z f_i(\bar{y},\bar{z})\big)^T+G'_{f_i}\big(f_i(\bar{y},\bar{z})\big)\nabla_{zz}f_i(\bar{y},\bar{z})\big\}\\
&\quad\times\big(\beta-\big(\alpha^Te_k\big)(\bar{y}+\bar{p})\big)\nabla_z f_i(\bar{y},\bar{z})\big(\nabla_z f_i(\bar{y},\bar{z})\big)^T\\
&\quad+G'_{f_i}\big(f_i(\bar{y},\bar{z})\big)\nabla_{zz}f_i(\bar{z},\bar{x})\big)\bar{p}\}\big]\left(\beta-\big(\alpha^Te_k\big)\left(\bar{y}+\frac{1}{2}\bar{p}\right)\right)=0,
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
&\big[G''_{f_i}\big(f_i(\bar{y},\bar{z})\big)\nabla_z f_i(\bar{y},\bar{z})\big(\nabla_z f_i(\bar{y},\bar{z})\big)^T+G'_{f_i}\big(f_i(\bar{y},\bar{z})\big)\nabla_{zz}f_i(\bar{y},\bar{z})\big]\\
&\quad\times\big[\big(\beta-\big(\alpha^Te_k\big)(\bar{p}+\bar{y})\big)\bar{\lambda}_i\big]=0,\quad i=1,2,3,\ldots,k,
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
&G'_{f_i}\big(f_i(\bar{y},\bar{z})\big)\nabla_z f_i(\bar{y},\bar{z})\big(\beta-\big(\alpha^Te_k\bar{y}\big)\big)+\eta e_k\\
&\quad+\left\{\left(\beta-\big(\alpha^Te_k\big)\left(\bar{y}+\frac{1}{2}\bar{p}_1\right)\right)^T\big(G''_{f_1}\big(f_1(\bar{y},\bar{z})\big)\nabla_y f_1(\bar{z},\bar{x})\big(\nabla_y f_1(\bar{z},\bar{x})\big)^T\right.\\
&\quad+G'_{f_1}\big(f_1(\bar{y},\bar{z})\big)\nabla_{zz}f_1(\bar{z},\bar{x})\big)\bar{p}_1\big),\ldots,\\
&\left(\beta-\big(\alpha^Te_k\big)\left(\bar{y}+\frac{1}{2}\bar{p}_k\right)\right)^T\big(G''_{f_k}\big(f_k(\bar{y},\bar{z})\big)\nabla_z f_k(\bar{y},\bar{z})\big(\nabla_z f_k(\bar{y},\bar{z})\big)^T
\end{aligned}
$$

$$+ G'_{f_k}\big(f_k(\bar{y},\bar{z})\big)\nabla_{zz}f_k(\bar{y},\bar{z})\big)\bar{p}\big)\Big\} = 0, \tag{14}$$

$$\beta^T \sum_{i=1}^{k} \bar{\lambda}_i \big[G'_{f_i}\big(f_i(\bar{y},\bar{z})\big)\nabla_z f_i(\bar{y},\bar{z})$$

$$+ \big\{G''_{f_i}\big(f_i(\bar{y},\bar{z})\big)\nabla_z f_i(\bar{y},\bar{z})\big(\nabla_z f_i(\bar{y},\bar{z})\big)^T + G'_{f_i}\big(f_i(\bar{y},\bar{z})\big)\nabla_{zz}f_i(\bar{y},\bar{z})\big\}\bar{p}\big] = 0, \tag{15}$$

$$\eta^T\big[\bar{\lambda}^T e_k - 1\big] = 0, \tag{16}$$

$$(\alpha,\beta) \geq 0, \qquad (\alpha,\beta,\eta) \neq 0. \tag{17}$$

Equation (14) can be rewritten as

$$G'_{f_i}\big(f_i(\bar{y},\bar{z})\big)\nabla_z f_i(\bar{y},\bar{z})\big(\beta - \big(\alpha^T e_k\big)\bar{y}\big)$$

$$+ \left(\beta - \big(\alpha^T e_k\big)\left(\bar{y} + \frac{1}{2}\bar{p}\right)\right)^T \big(\big(G''_{f_i}\big(f_i(\bar{y},\bar{z})\big)\nabla_z f_i(\bar{y},\bar{z})\big(\nabla_z f_i(\bar{y},\bar{z})\big)^T$$

$$+ G'_{f_i}\big(f_i(\bar{y},\bar{z})\big)\nabla_{zz}f_i(\bar{z},\bar{x})\big)\bar{p}\big) + \eta = 0, \quad i = 1,2,3,\ldots,k. \tag{18}$$

By assumption (i), since $\bar{\lambda}_i > 0$ for $i = 1,2,3,\ldots,k$, (18) gives

$$\beta = \big(\alpha^T e_k\big)(\bar{p} + \bar{y}), \quad i = 1,2,3,\ldots,k. \tag{19}$$

If $\alpha = 0$, then (19) implies that $\beta = 0$. Further, equation (18) gives $\eta = 0$. Consequently, $(\alpha,\beta,\eta) = 0$, which contradicts (17). Hence $\alpha \neq 0$, or $\alpha^T e_k > 0$.

Using (19) and $\alpha^T e_k > 0$ in (12), we get

$$\sum_{i=1}^{k} \bar{\lambda}_i \big[\big(\nabla_z\big\{\big(G''_{f_i}\big(f_i(\bar{y},\bar{z})\big)\nabla_z f_i(\bar{y},\bar{z})\big(\nabla_z f_i(\bar{y},\bar{z})\big)^T + G'_{f_i}\big(f_i(\bar{y},\bar{z})\big)\nabla_{zz}f_i(\bar{z},\bar{x})\big)\bar{p}\big\}\bar{p}\big)\big]$$

$$= -\frac{2}{\alpha^T e_k}\sum_{i=1}^{k}\big[G'_{f_i}\big(f_i(\bar{y},\bar{z})\big)\nabla_z f_i(\bar{y},\bar{z})\big]\big(\alpha_i - \big(\alpha^T e_k\big)\bar{\lambda}_i\big). \tag{20}$$

It follows from assumption (ii) that

$$\sum_{i=1}^{k} \bar{\lambda}_i \big[\big(\nabla_y\big\{\big(G''_{f_i}\big(f_i(\bar{y},\bar{z})\big)\nabla_z f_i(\bar{y},\bar{z})\big(\nabla_z f_i(\bar{y},\bar{z})\big)^T + G'_{f_i}\big(f_i(\bar{y},\bar{z})\big)\nabla_z f_i(\bar{y},\bar{z})\big)\bar{p}\big\}\bar{p}\big)\big] = 0. \tag{21}$$

Hence by assumption (iv) we get $\bar{p} = 0$, and therefore inequality (19) implies

$$\beta = \big(\alpha^T e_k\big)\bar{y}. \tag{22}$$

Now, using $\bar{p} = 0$ and (20), we obtain

$$\sum_{i=1}^{k}\big(\alpha_i - \big(\alpha^T e_k\big)\bar{\lambda}_i\big)\big[G'_{f_i}\big(f_i(\bar{y},\bar{z})\big)\nabla_z f_i(\bar{y},\bar{z})\big] = 0.$$

Assumption (iii) yields

$$\alpha_i = \big(\alpha^T e_k\big)\bar{\lambda}_i, \quad i = 1,2,3,\ldots,k. \tag{23}$$

Using $\alpha^T e_k > 0$ and (21)–(23) in (11), we get

$$(y-\bar{y})^T \sum_{i=1}^{k} \bar{\lambda}_i \big[G'_{f_i}\big(f_i(\bar{y},\bar{z})\big)\nabla_y f_i(\bar{y},\bar{z})\big] \geq 0 \quad \text{for all } y \in C_1. \tag{24}$$

Let $y \in C_1$. Then, $y+\bar{y} \in C_1$, and it follows that

$$y^T \sum_{i=1}^{k} \bar{\lambda}_i \big[G'_{f_i}\big(f_i(\bar{y},\bar{z})\big)\nabla_y f_i(\bar{y},\bar{z})\big] \geq 0 \quad \text{for all } y \in C_1. \tag{25}$$

Therefore

$$\sum_{i=1}^{k} \bar{\lambda}_i \big[G'_{f_i}\big(f_i(\bar{y},\bar{z})\big)\nabla_y f_i(\bar{y},\bar{z})\big] \in C_1^*. \tag{26}$$

Also, from (22) we have

$$\bar{y} = \frac{\bar{\beta}}{\bar{\alpha}^T e_k} \in C_2. \tag{27}$$

Hence $(\bar{v},\bar{w},\bar{\lambda},\bar{p}=0)$ satisfies the dual constraints and Z^0.

Now, letting $y=0$ and $y=2\bar{y}$ in (24), we get

$$\bar{y}^T \sum_{i=1}^{k} \bar{\lambda}_i \big[G'_{f_i}\big(f_i(\bar{y},\bar{z})\big)\nabla_y f_i(\bar{y},\bar{z})\big] = 0. \tag{28}$$

Using (28) and $\bar{q}=\bar{p}=0$ completes the proof. □

Theorem 3.3 (Converse duality) *Let $(\bar{v},\bar{w},\bar{\lambda},\bar{q})$ be an efficient solution of* (GWD). *Fix $\lambda=\bar{\lambda}$ in* (GWP) *such that*

(i) *for all $i=1,2,3,\ldots,k$, $[G''_{f_i}(f_i(\bar{v},\bar{w}))\nabla_z f_i(\bar{v},\bar{w})(\nabla_z f_i(\bar{v},\bar{w}))^T + G'_{f_i}(f_i(\bar{v},\bar{w}))\nabla_{zz} f_i(\bar{v},\bar{w})]$ is nonsingular,*

(ii) $\sum_{i=1}^{k} \bar{\lambda}_i \nabla_z(\{G''_{f_i}(f_i(\bar{v},\bar{w}))\nabla_z f_i(\bar{v},\bar{w})(\nabla_z f_i(\bar{v},\bar{w}))^T + G'_{f_i}(f_i(\bar{v},\bar{w}))\nabla_{zz} f_i(\bar{v},\bar{w})\}\bar{q})\bar{q} \notin \operatorname{span}\{G'_{f_1}(f_1(\bar{v},\bar{w}))\nabla_z f_1(\bar{v},\bar{w}),\ldots,G'_{f_k}(f_k(\bar{u},\bar{v}))\nabla_z f_k(\bar{u},\bar{v})\} \setminus \{0\}$,

(iii) *the vectors $\{G'_{f_1}(f_1(\bar{v},\bar{w}))\nabla_z f_1(\bar{v},\bar{w}), G'_{f_2}(f_2(\bar{v},\bar{w}))\nabla_z f_2(\bar{v},\bar{w}),\ldots,G'_{f_k}(f_k(\bar{v},\bar{w}))\nabla_z f_k(\bar{v},\bar{w})\}$ are linearly independent,*

(iv) $\sum_{i=1}^{k} \bar{\lambda}_i \nabla_z(\{G''_{f_i}(f_i(\bar{v},\bar{w}))\nabla_z f_i(\bar{v},\bar{w})(\nabla_z f_i(\bar{v},\bar{w}))^T + G'_{f_i}(f_i(\bar{v},\bar{w}))\nabla_{zz} f_i(\bar{v},\bar{w})\}\bar{q})\bar{q} = 0 \Rightarrow \bar{q}=0$.

Then, taking $\bar{p}=0$, we have that $(\bar{v},\bar{w},\bar{\lambda},\bar{p}=0) \in Y^0$ and $R(\bar{v},\bar{w},\bar{\lambda},\bar{p}) = S(\bar{u},\bar{v},\bar{\lambda},\bar{p})$. Also, by Theorem 3.1 $(\bar{v},\bar{w},\bar{\lambda},\bar{p}=0)$ *is an efficient solution for* (GWP).

Proof Proof follows the lines of Theorem 3.2. □

4 Concluding remarks

In this paper, we have formulated a second-order symmetric G-Wolfe-type dual problem for a nonlinear multiobjective optimization problem with cone constraints. A number of duality relations are further established under G_f-bonvexity/G_f-pseudobonvexity assumptions on the function f. We have discussed various numerical examples to show the existence of G_f-bonvex/G_f-pseudobonvex functions. The question arises whether the duality results developed in this paper hold for G-Wolfe- or mixed-type higher-order multiobjective optimization problems. This may be the future direction for the researchers working in this area.

Acknowledgements
The authors are thankful to the anonymous referees and editor for their valuable suggestions, which have substantially improved the presentation of the paper.

Funding
The authors extend their appreciation to the "Deanship of Scientific Research" at King Khalid University for funding this work through research groups program under grant R.G.P.1/152/40.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Publisher's Note

Received: 16 August 2019 Accepted: 29 December 2019 Published online: 07 February 2020

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