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Original article

New coupled fixed point theorems in cone metric spaces with applications to integral equations and Markov process

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Abstract

In this paper, we define a generalized T-contraction and derive some new coupled fixed point theorems in cone metric spaces with total ordering condition. An illustrative example is provided to support our results. As an application, we utilize the results obtained to study the existence of common solution to a system of integral equations. We also present an application to Markov process.

MSC: 47H10; 54H25

Keywords: T-contraction; Coupled fixed point; Sequentially convergent; Integral equations; Markov process; Cone metric space

1. Introduction and preliminaries

In 1922, the Banach contraction principle [1] was introduced and it remains a powerful tool in nonlinear analysis which incites many authors to extend it, for instance [2–13]. In 2007, Huang and Zhang [14] generalized the notion of metric space by replacing the set of real numbers by ordered normed spaces, defined a cone metric space and extended the Banach contraction principle on these spaces over a normal solid cone. There are many fixed point results for generalized contractive conditions in metric spaces which were extended to cone metric spaces when the underlying cone is normal or not normal. Fixed point theory in cone metric spaces has been studied recently by many authors [2,14–26]. Bhaskar and Lakshmikantham [27] introduced the concept of coupled fixed point and applied their results to the study of existence and uniqueness of solution for a periodic boundary value problem in partially ordered metric spaces. Recently, Rahimi et al. [28] defined the concept of T-contraction in coupled fixed-point theory and obtained some coupled fixed point results on cone metric spaces without normality condition. For the detailed study on coupled fixed point results in ordered metric spaces and ordered cone metric spaces, we refer the reader to [29–34].

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After the study of T-contractions in a metric space by Beiranvand et al. [35], in [19], Filipović et al. obtained some fixed and periodic points satisfying T-Hardy-Rogers contraction in a cone metric space. Recently, Rahimi et al. [36] proved new fixed and periodic point results under T-contractions of two maps in cone metric spaces.

Motivated by the above work, we define generalized T-contraction and establish the existence and uniqueness of a coupled fixed point in cone metric spaces with a total ordering cone and dropping the normality condition which in turn will extend and generalize the results of [27,28,37]. We state some illustrative example to justify the obtained results. Also, we prove the existence of common solution to a system of integral equations. Further, we present an application to Markov process. The presented results improve and generalize many known results in cone metric spaces.

Now, we recall the definition of cone metric spaces and some of their properties.

Definition 1.1 (*[14]*). Let E be a real Banach space. A subset P of E is called a cone if the following conditions are satisfied:

(i) P is closed, nonempty and $P \neq \{\theta\}$;
(ii) $a, b \in \mathbb{R}, a, b \geq 0$ and $x, y \in P$ imply that $ax + by \in P$.
(iii) $P \cap (-P) = \{\theta\}$.

Given a cone P of E, we define a partial ordering $\preceq$ with respect to P by $x \preceq y$ if and only if $y - x \in P$. We shall write $x \prec y$ to indicate that $x \preceq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in int P$ (interior of P).

A cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E$,

$$\theta \preceq x \preceq y \text{ implies } \|x\| \leq K\|y\|. \tag{1}$$

or equivalently, if, for any n, $x_n \preceq y_n \preceq z_n$ and

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = x \text{ imply } \lim_{n\to\infty} y_n = x.$$

The least positive number K satisfying the inequality (1) is called the normal constant of P.

Recently, Küçük et al. studied the characterization of total ordering cones with some properties and optimality conditions in [38].

Proposition 1.2 (*[38]*). *Let E be a vector space and P be a partial ordering cone with partial order "$\preceq$" defined by $x \preceq y$ if and only if $y - x \in P$. Then "$\preceq$" is a total order on X if and only if $P \cup (-P) = E$.*

Definition 1.3 (*[14]*). Let X be a nonempty set and $d : X \times X \to E$ be a mapping such that the following conditions hold:

(i) $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(iii) $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

Example 1.4 (*[14]*). Let $X = R, E = R^2, P = \{(x, y) \in E : x, y \geq 0\} \subset R^2$ and $d : X \times X \to E$ such that $d(x, y) = (|x - y|, \delta|x - y|)$, where $\delta \geq 0$ is a constant. Then (X, d) is a cone metric space.

Definition 1.5 (*[14]*). Let (X, d) be a cone metric space. We say that $\{x_n\}$ is;

(i) a Cauchy sequence if for every $c \in E$ with $\theta \ll c$, there is N such that for all $m, n > N$, $d(x_n, x_m) \ll c$;
(ii) a convergent sequence if for every $c \in E$ with $\theta \ll c$, there is N such that for all $n > N$, $d(x_n, x) \ll c$, for some $x \in X$. We denote it by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$.

A cone metric space X is said to be complete if every Cauchy sequence in X is convergent in X.

Proposition 1.6 (*[19]*). *Let (X, d) be a cone metric space. Then the following properties are often used, particularly when dealing with cone metric spaces in which the cone need not be normal.*

(P1) *If* $u \preceq v$ *and* $v \ll w$*, then* $u \ll w$.
(P2) *If* $\theta \preceq u \ll c$ *for each* $c \in int P$*, then* $u = \theta$.
(P3) *If* E *is a real Banach space with a cone* P *and if* $a \preceq \lambda a$ *where* $a \in P$ *and* $0 \leq \lambda < 1$*, then* $a = \theta$.
(P4) *If* $c \in int P$, $a_n \in E$ *and* $a_n \to \theta$*, then there exists* n_0 *such that for all* $n > n_0$*, we have* $a_n \ll c$.

Definition 1.7 (*[19]*)**.** Let (X, d) be a cone metric space, P be a solid cone and $f : X \to X$. Then

(i) f is said to be continuous if $\lim_{n\to\infty} x_n = x$ implies that $\lim_{n\to\infty} f x_n = f x$, for all $\{x_n\}$ in X.
(ii) f is said to be sequentially convergent if, for every sequence $\{x_n\}$, such that $\{f x_n\}$ is convergent, then $\{x_n\}$ also is convergent.
(iii) f is said to be subsequentially convergent if, for every sequence $\{x_n\}$, such that $\{f x_n\}$ is convergent, then $\{x_n\}$ has a convergent subsequence.

Definition 1.8 (*[19]*)**.** Let (X, d) be a cone metric space and $T, f : X \to X$ two mappings. A mapping f is said to be a T-Hardy-Rogers contraction, if there exist $a_i \geq 0$, $i = 1, 2, \ldots, 5$ with $\sum_{i=1}^{5} a_i < 1$ such that for $x, y \in X$,

$$d(Tfx, Tfy) \preceq a_1 d(Tx, Ty) + a_2 d(Tx, Tfx) + a_3 d(Ty, Tfy) + a_4 d(Tx, Tfy) + a_5 d(Ty, Tfx). \tag{2}$$

Taking $a_1 = a_4 = a_5 = 0$, $a_2 = a_3 \neq 0$ (respectively $a_1 = a_2 = a_3 = 0$, $a_4 = a_5 \neq 0$) in (2), we obtain T-Kannan (respectively T-Chatterjea) contraction.

Definition 1.9 (*[28]*)**.** Let (X, d) be a cone metric space and $T : X \to X$ be a mapping. A mapping $S : X \times X \to X$ is called a T-Sabetghadam-contraction if there exist $a, b \geq 0$ with $a + b < 1$ such that for all $x, y \in X$,

$$d(TS(x, y), TS(\tilde{x}, \tilde{y})) \preceq a d(Tx, T\tilde{x}) + b d(Ty, T\tilde{y}).$$

Definition 1.10 (*[31]*)**.** Let (X, d) be a cone metric space. An element $(x, y) \in X \times X$ is called a coupled fixed point of the mapping $F : X \times X \to X$ if $F(x, y) = x$ and $F(y, x) = y$.

Note that if (x, y) is a coupled fixed point of F, then also (y, x) is a coupled fixed point of F.

2. Main results

Initially we define the following contraction condition which generalizes T-Sabetghadam-contraction.

Definition 2.1. Let (X, d) be a cone metric space with $P \cup (-P) = E$, (i.e. P is a total ordering cone) and $T : X \to X$ be a mapping. A mapping $S : X \times X \to X$ is called a generalized T-contraction if there exists λ with $0 \leq \lambda < 1$ such that

$$d(TS(x, y), TS(\tilde{x}, \tilde{y})) \preceq \lambda \max\{d(Tx, T\tilde{x}), d(Ty, T\tilde{y})\}. \tag{3}$$

for all $x, y, \tilde{x}, \tilde{y} \in X$.

The first main result in this paper is the following coupled fixed point result which generalizes Theorem 3 of Rahimi et al. [28].

Theorem 2.2. *Let* (X, d) *be a complete cone metric space,* P *be a solid cone with* $P \cup (-P) = E$ *and* $T : X \to X$ *be a continuous, one-to-one mapping and* $S : X \times X \to X$ *be a mapping such that* (3) *holds for all* $x, y, \tilde{x}, \tilde{y} \in X$*. Then*

(i) *there exist* $z_{x_0}, z_{y_0} \in X$ *such that*

$$\lim_{n\to\infty} TS^n(x_0, y_0) = z_{x_0} \quad \text{and} \quad \lim_{n\to\infty} TS^n(y_0, x_0) = z_{y_0};$$

where $S^n(x_0, y_0) = x_n$ *and* $S^n(y_0, x_0) = y_n$ *are the iterative sequences.*
(ii) *if* T *is subsequentially convergent, then* $\{S^n(x_0, y_0)\}$ *and* $\{S^n(y_0, x_0)\}$ *have a convergent subsequence;*
(iii) *there exist unique* $w_{x_0}, w_{y_0} \in X$ *such that* $S(w_{x_0}, w_{y_0}) = w_{x_0}$ *and* $S(w_{y_0}, w_{x_0}) = w_{y_0}$*;*

(iv) *if T is sequentially convergent, then for every $x_0, y_0 \in X$, the sequence $\{S^n(x_0, y_0)\}$ converges to $w_{x_0} \in X$ and the sequence $\{S^n(y_0, x_0)\}$ converges to $w_{y_0} \in X$.*

Proof. For $x_0, y_0 \in X$, we define a sequence as follows:

$$x_{n+1} = S(x_n, y_n) = S^{n+1}(x_0, y_0) \text{ and } y_{n+1} = S(y_n, x_n) = S^{n+1}(y_0, x_0),\ \forall n = 0, 1, 2, \ldots.$$

Now, using (3), we have

$$\begin{aligned} d(Tx_n, Tx_{n+1}) &= d(TS(x_{n-1}, y_{n-1}), TS(x_n, y_n)) \\ &\preceq \lambda \max\{d(Tx_{n-1}, Tx_n), d(Ty_{n-1}, Ty_n)\}, \end{aligned} \tag{4}$$

and

$$\begin{aligned} d(Ty_n, Ty_{n+1}) &= d(TS(y_{n-1}, x_{n-1}), TS(y_n, x_n)) \\ &\preceq \lambda \max\{d(Ty_{n-1}, Ty_n), d(Tx_{n-1}, Tx_n)\}. \end{aligned} \tag{5}$$

Let $D_n = \max\{d(Tx_n, Tx_{n+1}), d(Ty_n, Ty_{n+1})\}$. Applying (4) and (5), we get

$$D_n \preceq \lambda \max\{d(Ty_{n-1}, Ty_n), d(Tx_{n-1}, Tx_n)\} = \lambda D_{n-1},$$

where $0 \leq \lambda < 1$. Continuing in this fashion, we obtain

$$\theta \preceq D_n \preceq \lambda D_{n-1} \preceq \cdots \preceq \lambda^n D_0.$$

If we take $D_0 = \theta$, then (x_0, y_0) is a coupled fixed point of S. Suppose that $D_0 > \theta$ and for $n > m$, we have

$$d(Tx_m, Tx_n) \preceq d(Tx_m, Tx_{m+1}) + d(Tx_{m+1}, Tx_{m+2}) + \cdots + d(Tx_{n-1}, Tx_n) \tag{6}$$

and

$$d(Ty_m, Ty_n) \preceq d(Ty_m, Ty_{m+1}) + d(Ty_{m+1}, Ty_{m+2}) + \cdots + d(Ty_{n-1}, Ty_n). \tag{7}$$

From (6) and (7), we get

$$\begin{aligned} \max\{d(Tx_m, Tx_n), d(Ty_m, Ty_n)\} &\preceq \max\{d(Tx_m, Tx_{m+1}), d(Ty_m, Ty_{m+1})\} + \cdots \\ &\quad \max\{d(Tx_{n-1}, Tx_n), d(Ty_{n-1}, Ty_n)\} \\ &= D_m + D_{m+1} + \cdots + D_{n-1} \\ &\preceq (\lambda^m + \lambda^{m+1} + \cdots + \lambda^{n-1}) D_0 \\ &\preceq \frac{\lambda^m}{1-\lambda} D_0. \end{aligned}$$

Now applying (P1) and (P4), we have for every $c \in int P$, there exists a positive integer N such that $\max\{d(Tx_m, Tx_n), d(Ty_m, Ty_n)\} \ll c$ for every $n > m > N$ which implies that $\{Tx_n\}$ and $\{Ty_n\}$ are Cauchy sequences in X. By the completeness of X, we can find $z_{x_0}, z_{y_0} \in X$ such that

$$\lim_{n\to\infty} TS^n(x_0, y_0) = z_{x_0} \text{ and } \lim_{n\to\infty} TS^n(y_0, x_0) = z_{y_0}. \tag{8}$$

If T is subsequentially convergent, then $S^n(x_0, y_0)$ and $S^n(y_0, x_0)$ have convergent subsequences. Thus, there exist w_{x_0}, w_{y_0} in X and sequences $\{x_{n_j}\}$ and $\{y_{n_j}\}$ such that

$$\lim_{j\to\infty} S^{n_j}(x_0, y_0) = w_{x_0} \text{ and } \lim_{j\to\infty} S^{n_j}(y_0, x_0) = w_{y_0}.$$

Now, since T is continuous, we obtain

$$\lim_{j\to\infty} TS^{n_j}(x_0, y_0) = Tw_{x_0} \text{ and } \lim_{j\to\infty} TS^{n_j}(y_0, x_0) = Tw_{y_0}. \tag{9}$$

Hence, from (8) and (9), we have

$$Tw_{x_0} = z_{x_0},\ Tw_{y_0} = z_{y_0}.$$

On the other hand, using (3) we get

$$d(TS(w_{x_0}, w_{y_0}), Tw_{x_0}) \preceq \lambda \max\{d(Tw_{x_0}, Tx_{n_j}), d(Tw_{y_0}, Ty_{n_j})\} + d(Tx_{n_j+1}, Tw_{x_0}).$$

Applying Proposition 1.6, we obtain $d(TS(w_{x_0}, w_{y_0}), Tw_{x_0}) = \theta$, that is, $TS(w_{x_0}, w_{y_0}) = Tw_{x_0}$. As T is one-to-one, we have $S(w_{x_0}, w_{y_0}) = w_{x_0}$. Similarly, $S(w_{y_0}, w_{x_0}) = w_{y_0}$. Therefore, (w_{x_0}, w_{y_0}) is a coupled fixed point of S. Suppose that (v_{x_0}, v_{y_0}) is another coupled fixed point of S, then

$$d(Tw_{x_0}, Tv_{x_0}) = d(TS(w_{x_0}, w_{y_0}), TF(v_{x_0}, v_{y_0})) \preceq \lambda \max\{d(Tw_{x_0}, Tv_{x_0}), d(Tw_{y_0}, Tv_{y_0})\}$$

and

$$d(Tw_{y_0}, Tv_{y_0}) = d(TS(w_{y_0}, w_{x_0}), TF(v_{y_0}, v_{x_0})) \preceq \lambda \max\{d(Tw_{y_0}, Tv_{y_0}), d(Tw_{x_0}, Tv_{x_0})\},$$

which implies that

$$\max\{d(Tw_{y_0}, Tv_{y_0}), d(Tw_{x_0}, Tv_{x_0})\} \leq \lambda \max\{d(Tw_{y_0}, Tv_{y_0}), d(Tw_{x_0}, Tv_{x_0})\}$$

which yields

$$d(Tw_{x_0}, Tv_{x_0}) = d(Tw_{y_0}, Tv_{y_0}) = \theta,$$

as $\lambda < 1$. Thus, $Tw_{x_0} = Tv_{x_0}$, $Tw_{y_0} = Tv_{y_0}$. Since T is one-to-one, we have $(w_{x_0}, w_{y_0}) = (v_{x_0}, v_{y_0})$. Further, if T is sequentially convergent, by replacing n by n_j, we obtain

$$\lim_{n\to\infty} S^n(x_0, y_0) = w_{x_0} \quad \text{and} \quad \lim_{n\to\infty} S^n(y_0, x_0) = w_{y_0}.$$

This completes the proof. □

Example 2.3. Let $X = [0, 1]$, $E = \mathbb{R}$ with $P = \{x \in E : x \geq 0\}$ and define $d(x, y) = |x - y|$. Then (X, d) is a cone metric space. Consider the mappings $T : X \to X$ defined by $Tx = \frac{x^2}{2}$ and $S : X \times X \to X$ defined by $S(x, y) = \frac{\sqrt{x^8+y^8}}{5}$. Clearly, T is one-to-one, continuous and (3) holds for all $x, y, u, v \in X$ and $\lambda > \frac{1}{4}$. Further, all the conditions of Theorem 2.2 are satisfied. Therefore, S has a unique coupled fixed point $(0, 0)$.

Theorem 2.4. *Let (X, d) be a complete cone metric space, P be a solid cone with $P \cup (-P) = E$ and $T : X \to X$ be a continuous, one-to-one mapping and $S : X \times X \to X$ be a mapping such that*

$$d(TS(x, y), TS(\tilde{x}, \tilde{y})) \preceq \lambda \max\{d(TS(x, y), Tx), d(TS(\tilde{x}, \tilde{y}), T\tilde{x})\} \tag{10}$$

for all $x, y, \tilde{x}, \tilde{y} \in X$ where $0 \leq \lambda < 1$. Then the conclusions of Theorem 2.2 hold.

Proof. The proof is similar to that of Theorem 2.2. □

Theorem 2.5. *Let (X, d) be a complete cone metric space, P be a solid cone with $P \cup (-P) = E$ and $T : X \to X$ be a continuous, one-to-one mapping and $S : X \times X \to X$ be a mapping such that*

$$d(TS(x, y), TS(\tilde{x}, \tilde{y})) \preceq \lambda \max\{d(TS(x, y), T\tilde{x}), d(TS(\tilde{x}, \tilde{y}), Tx)\} \tag{11}$$

for all $x, y, \tilde{x}, \tilde{y} \in X$ where $0 \leq \lambda < 1$. Then the conclusions of Theorem 2.2 hold.

Proof. We omit the proof as it is immediate from Theorem 2.2. □

Remarks 2.6. Theorems 2.2, 2.4 and 2.5 generalize the following results:

(i) Theorems 3, 4 and 5 of Rahimi et al. [28].
(ii) Theorems 2.2, 2.5 and 2.6 of Sabetghadam et al. [37].

Remarks 2.7. Note that the main results of Shatanawi [39] can be proved in a total ordering cone P under the weaker contractive condition of the type (3).

Now we obtain the following corollaries as consequences of Theorems 2.2, 2.4 and 2.5.

Corollary 2.8. *Let (X, d) be a complete cone metric space, P be a solid cone and $T : X \to X$ be a continuous, one-to-one mapping and $S : X \times X \to X$ be a mapping such that*

$$d(TS(x, y), TS(\tilde{x}, \tilde{y})) \preceq \lambda \max\left\{d(Tx, T\tilde{x}), d(Ty, T\tilde{y}), \frac{d(Tx, T\tilde{x}) + d(Ty, T\tilde{y})}{2}\right\}$$

for all $x, y, \tilde{x}, \tilde{y} \in X$ where $0 \leq \lambda < 1$. Then the conclusions of Theorem 2.2 hold.

Corollary 2.9. *Let (X, d) be a complete cone metric space, P be a solid cone and $T : X \to X$ be a continuous, one-to-one mapping and $S : X \times X \to X$ be a mapping such that*

$$d(TS(x, y), TS(\tilde{x}, \tilde{y})) \preceq \lambda \max\left\{d(TS(x, y), Tx), d(TS(\tilde{x}, \tilde{y}), T\tilde{x}), \frac{d(TS(x, y), Tx) + d(TS(\tilde{x}, \tilde{y}), T\tilde{x})}{2}\right\}$$

for all $x, y, \tilde{x}, \tilde{y} \in X$ where $0 \leq \lambda < 1$. Then the conclusions of Theorem 2.2 hold.

Corollary 2.10. *Let (X, d) be a complete cone metric space, P be a solid cone and $T : X \to X$ be a continuous, one-to-one mapping and $S : X \times X \to X$ be a mapping such that*

$$d(TS(x, y), TS(\tilde{x}, \tilde{y})) \preceq \lambda \max\left\{d(TS(x, y), T\tilde{x}), d(TS(\tilde{x}, \tilde{y}), Tx), \frac{d(TS(x, y), T\tilde{x}) + d(TS(\tilde{x}, \tilde{y}), Tx)}{2}\right\}$$

for all $x, y, \tilde{x}, \tilde{y} \in X$ where $0 \leq \lambda < 1$. Then the conclusions of Theorem 2.2 hold.

Next, we explain a general approach to our previous results.

Lemma 2.11. *Let (X, d) be a cone metric space. Then we have the following:*

(i) *$(X \times X, d_1)$ is a cone metric space with*

$$d_1((x, y), (u, v)) = \max\{d(x, u), d(y, v)\}.$$

In addition, (X, d) is complete if and only if $(X \times X, d_1)$ is complete.

(ii) *The mapping $S : X \times X \to X$ has a coupled fixed point if and only if the mapping $F_S : X \times X \to X \times X$ defined by $F_S(x, y) = (S(x, y), S(y, x))$ has a fixed point in $X \times X$.*

Proof.

(i) Notice that (i) and (ii) of Definition 1.3 are satisfied. Now it suffices to prove the triangle inequality. Since (X, d) is a cone metric space, we obtain

$$\begin{aligned} d_1((x, y), (u, v)) &= \max\{d(x, u), d(y, v)\} \\ &\preceq \max\{d(x, s) + d(s, u), d(y, t) + d(t, v)\} \\ &\preceq \max\{d(x, s), d(y, t)\} + \max\{d(s, u), d(t, v)\} \\ &= d_1((x, y), (s, t)) + d_1((s, t), (u, v)) \end{aligned}$$

for all $(x, y), (u, v), (s, t) \in X \times X$. Hence, $(X \times X, d_1)$ is a cone metric space. The completeness part can be easily proved.

(ii) Suppose that (x, y) is a coupled fixed point of S, that is, $S(x, y) = x$ and $S(y, x) = y$. Then

$$F_S(x, y) = (S(x, y), S(y, x)) = (x, y)$$

which shows that $(x, y) \in X \times X$ is a fixed point of F_S. Conversely, assume that $(x, y) \in X \times X$ is a fixed point of F_S, then $F_S(x, y) = (x, y)$ which implies that $S(x, y) = x$ and $S(y, x) = y$. $\square$

Theorem 2.12. *Let (X, d) be a complete cone metric space, P be a total ordering solid cone and $T : X \to X$ be a continuous and one-to-one mapping. Moreover, let $S : X \times X \to X$ be a mapping satisfying*

$$\max\{d(TS(x, y), TS(\tilde{x}, \tilde{y})), d(TS(y, x), TS(\tilde{y}, \tilde{x}))\} \preceq \lambda \max\big\{d(Tx, T\tilde{x}), d(Ty, T\tilde{y})\big\} \tag{12}$$

for all $x, y, \tilde{x}, \tilde{y} \in X$, where $0 \le \lambda < 1$. Then the conclusions of Theorem 2.2 hold.

Proof. Let us define $T_1 : X \times X \to X \times X$ by $T_1(x, y) = (Tx, Ty)$. Note that T_1 is continuous and one-to-one. Now, applying $Y = (x, y)$, $V = (u, v) \in X \times \times X$ and (ii) of Lemma 2.11, (12) becomes

$$d_1(T_1 F_S(Y), T_1 F_S(V)) \preceq \lambda d_1(T_1 Y, T_1 V).$$

It can be viewed that the conclusions follow by setting $a_1 = \lambda$ and $a_2 = a_3 = a_4 = a_5 = 0$ in Theorem 2.1 of [19] as $\lambda < 1$. $\square$

Remarks 2.13. The cone metric defined in Theorem 2.2 is the generalized form of the cone metric defined in [28]. Further, Theorem 2.12 generalizes Theorem 6 of Rahimi et al. [28].

Example 2.14. Let $X = [0, 1]$ and $E = C^1_{\mathbb{R}}[0, 1]$ with $d(x, y) = |x - y|e^t$ where $e^t \in E$ on $P = \{\varphi \in E : \varphi \ge 0\}$. Then (X, d) is a cone metric space. Suppose that $T : X \to X$ defined by $Tx = \frac{x}{2}$ and $S : X \times X \to X$ defined by $S(x, y) = \frac{x-y}{10}$. Note that

$$d(TS(x, y), TS(u, v)) = \frac{e^t}{20}|(x - u) - (y - v)| \tag{13}$$

and

$$d(Tx, Tu) = \frac{e^t}{2}|x - u| \ \text{ and } \ d(Ty, Tv) = \frac{e^t}{2}|y - v|. \tag{14}$$

From (13) and (14), it can be easily seen that the condition (12) holds for all $x, y, u, v \in X$ and $\lambda > \frac{1}{10}$. Further, all the conditions of Theorem 2.12 are satisfied. Hence, $(0, 0)$ is a unique coupled fixed point of S.

3. An application to integral equations

The purpose of this section is to study the existence of solution of a system of nonlinear integral equations using the results we obtained.

Let $X = C\big([0, T], \mathbb{R}\big)$ (the set of continuous functions defined on $[0, T]$ and taking values in $\mathbb{R}$) be together with the metric given by

$$d(x, y) = \sup_{t \in [0,T]} |x(t) - y(t)|, \ \forall x, y \in X.$$

Consider the following system of integral equations, for $t \in [0, T]$, $T > 0$,

$$F(x, y)(t) = \int_0^T G(t, s) f(t, x(s), y(s))ds + g(t), \tag{15}$$

$$F(y, x)(t) = \int_0^T G(t, s) f(t, y(s), x(s))ds + g(t). \tag{16}$$

Theorem 3.1. *Suppose that the following hold:*

(i) $G : [0, T] \times [0, T] \to \mathbb{R}$ *is a continuous function.*
(ii) $g \in C([0, T], \mathbb{R})$.
(iii) $f : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is a continuous function.*

(iv) *For all* $x, y, u, v \in X$ *and* $t \in [0, T]$, *we have*

$$|f(t, x(t), y(t)) - f(t, u(t), v(t))| \leq \lambda \max\{|x(t) - u(t)|, |y(t) - v(t)|\},$$

where $0 \leq \lambda < 1$.

(v) $\int_0^T |G(t, s)| \leq 1$.

Then the system (15)–(16) *has at least one solution in* $C\big([0, T], \mathbb{R}\big)$.

Proof. It is easy to see that (x, y) is a solution to (15)–(16) if and only if (x, y) is a coupled fixed point of F. Existence of such a point follows from Theorem 2.2, by taking T as identity mapping. So we have to check that all the conditions of Theorem 2.2 hold. For all $x, y, u, v \in X$ and $t \in [0, T]$, we have

$$\begin{aligned}|F(x, y)(t) - F(u, v)(t)| &\leq \int_0^T \big|G(t, s)\big|\big|f(t, x(s), y(s)) - f(t, u(s), v(s))\big|ds,\\ &\leq \int_0^T \big|G(t, s)\big|\lambda \max\big\{|x(t) - u(t)|, |y(t) - v(t)|\big\}ds,\\ &\leq \left(\int_0^T \big|G(t, s)\big|ds\right)\lambda \max\big\{d(x, u), d(y, v)\big\},\end{aligned}$$

which yields that

$$d(F(x, y), F(u, v)) \leq \lambda \max\big\{d(x, u), d(y, v)\big\}, \quad \forall x, y, u, v \in X.$$

This shows that the contractive condition of Theorem 2.2 holds. Therefore, F has a unique coupled fixed point $(\tilde{x}, \tilde{y}) \in C\big([0, T], \mathbb{R}\big) \times C\big([0, T], \mathbb{R}\big)$ which is the unique solution of (15)–(16). □

4. An application to Markov process

Let $\mathbb{R}^n_+ = \{x = (x_1, x_2, \ldots, x_n) : x_i > 0, i = 1, 2, \ldots, n\}$ and $\Delta^2_{n-1} = \{z = (x, y) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ : \sum_{i=1}^n z_i = \sum_{i=1}^n (x_i + y_i) = 1\}$ denote the $2(n-1)$ dimensional unit simplex. Note that any $z \in \Delta^2_{n-1}$ may be regarded as a probability over the $2n$ possible states. A random process in which one of the $2n$ states is realized in each period $t = 1, 2, \ldots$ with the probability conditioned on the current realized state is called Markov Process. Let a_{ij} denote the conditional probability that state i is reached in succeeding period starting in state j. Then, given the prior probability vector z^t in period t, the posterior probability in period $t+1$ is given by $z_i^{t+1} = \sum_j a_{ij} z_j^t$ for each $i = 1, 2, \ldots, n$. To express this in matrix notation, we let z^t denote a column vector. Then, $z^{t+1} = Az^t$. Observe that the properties of conditional probability require each $a_{ij} \geq 0$ and $\sum_{i=1}^n a_{ij} = 1$ for each j. If for any period t, $z^{t+1} = z^t$ then z^t is a stationary distribution of the Markov process. Thus, the problem of finding a stationary distribution is equivalent to the fixed point problem $Az^t = z^t$.

For each i, let $\varepsilon_i = \min_j a_{ij}$ and define $\varepsilon = \sum_{i=1}^n \varepsilon_i$.

Theorem 4.1. *Under the assumption* $a_{ij} > 0$, *a unique stationary distribution exists for the Markov process.*

Proof. Let $d : \Delta^2_{n-1} \times \Delta^2_{n-1} \to \mathbb{R}^2$ be given by

$$d(s, t) = d((x, y), (u, v)) = \left(\sum_{i=1}^n \big(|x_i - u_i| + |y_i - v_i|\big), \alpha \sum_{i=1}^n \big(|x_i - y_i| + |y_i - v_i|\big)\right)$$

for all $s, t \in \Delta^2_{n-1}$ and some $\alpha \geq 0$.

Note that $d(s, t) \geq (0, 0)$ for all $s, t \in \Delta^2_{n-1}$ and $d(s, t) = (0, 0) \Rightarrow \left(\sum_{i=1}^n \big(|x_i - u_i| + |y_i - v_i|\big), \alpha \sum_{i=1}^n \big(|x_i - u_i| + |y_i - v_i|\big)\right) = (0, 0) \Rightarrow \big(|x_i - u_i| + |y_i - v_i|\big) = 0$ for all i, which implies that $s = t$. Assume $s = t$ then $x_i = u_i$

and $y_i = v_i$ for all i which implies that $|x_i - y_i| = |y_i - v_i| = 0 \Rightarrow \sum_{i=1}^{n}\big(|x_i - u_i| + |y_i - v_i|\big) \Rightarrow d(s,t) = (0,0)$.

$$
\begin{aligned}
d(s,t) &= \left(\sum_{i=1}^{n}\big(|x_i - u_i| + |y_i - v_i|\big), \alpha\sum_{i=1}^{n}\big(|x_i - u_i| + |y_i - v_i|\big)\right)\\
&= \left(\sum_{i=1}^{n}\big(|u_i - x_i| + |v_i - y_i|\big), \alpha\sum_{i=1}^{n}\big(|u_i - x_i| + |v_i - y_i|\big)\right) = d(t,s).
\end{aligned}
$$

Now

$$
\begin{aligned}
d(s,t) &= \left(\sum_{i=1}^{n}\big(|x_i - u_i| + |y_i - v_i|\big), \alpha\sum_{i=1}^{n}\big(|x_i - u_i| + |y_i - v_i|\big)\right)\\
&= \Bigg(\sum_{i=1}^{n}\big(|(x_i - p_i) + (p_i - u_i)| + |(y_i - q_i) + (q_i - v_i)|\big),\\
&\quad \alpha\sum_{i=1}^{n}\big(|(x_i - p_i) + (p_i - u_i)| + |(y_i - q_i) + (q_i - v_i)|\big)\Bigg)\\
&\preceq \Bigg(\sum_{i=1}^{n}\big(|(x_i - p_i)| + |(p_i - u_i)| + |(y_i - q_i)| + |(q_i - v_i)|\big),\\
&\quad \alpha\sum_{i=1}^{n}\big(|(x_i - p_i)| + |(p_i - u_i)| + |(y_i - q_i)| + |(q_i - v_i)|\big)\Bigg)\\
&= \left(\sum_{i=1}^{n}\big(|x_i - p_i| + |y_i - q_i|\big), \alpha\sum_{i=1}^{n}\big(|x_i - p_i| + |y_i - q_i|\big)\right)\\
&\quad + \left(\sum_{i=1}^{n}\big(|p_i - u_i| + |q_i - v_i|\big), \alpha\sum_{i=1}^{n}\big(|p_i - u_i| + |q_i - v_i|\big)\right)\\
&= d(s,r) + d(r,t) \ \text{ for } \ s = (x,y), r = (p,q), t = (u,v) \in \Delta_{n-1}^2.
\end{aligned}
$$

Thus (Δ_{n-1}^2, d) is a cone metric space with $P = \{(x_1, x_2, \ldots, x_n) : x_i \geq 0, \forall i = 1, 2, \ldots, n\}$. For $z \in \Delta_{n-1}$, let $t = Az$. Then each $\beta_i = \sum_{j=1}^{n} a_{ij} z_j \geq 0$. Further more, since each $\sum_{j=1}^{n} a_{ij} = 1$, we have

$$
\sum_{i=1}^{n}\beta_i = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} z_j = \beta_i = \sum_{j=1}^{n} a_{ij}\sum_{j=1}^{n}(x_j + y_j) = \sum_{j=1}^{n}(x_j + y_j) = 1
$$

which shows that $t \in \Delta_{n-1}^2$. Thus, we see that $A : \Delta_{n-1}^2 \to \Delta_{n-1}^2$. We shall show that A is a contraction. Let A_i denote the ith row of A. Then for any $(x,y), (u,v) \in \Delta_{n-1}$, we have

$$
\begin{aligned}
d(A(x,y), A(u,v)) &= \Bigg(\sum_{i=1}^{n}\Big|\sum_{j=1}^{n}(a_{ij}(x_j + y_j) - a_{ij}(u_j + v_j))\Big|,\\
&\quad \alpha\sum_{i=1}^{n}\Big|\sum_{j=1}^{n}(a_{ij}(x_j + y_j) - a_{ij}(u_j + v_j))\Big|\Bigg)\\
&= \Bigg(\sum_{i=1}^{n}\Big|\sum_{j=1}^{n}(a_{ij} - \epsilon_i)((x_j + y_j) - (u_j + v_j)) + \epsilon_i((x_j + y_j) - (u_j + v_j))\Big|,\\
&\quad \alpha\sum_{i=1}^{n}\Big|\sum_{j=1}^{n}(a_{ij} - \epsilon_i)((x_j + y_j) - (u_j + v_j)) + \epsilon_i((x_j + y_j) - (u_j + v_j))\Big|\Bigg)
\end{aligned}
$$

$$\leq \Bigg(\sum_{i=1}^{n} \Big| \sum_{j=1}^{n} (a_{ij} - \epsilon_i)((x_j + y_j) - (u_j + v_j)) \Big| + \Big| \sum_{j=1}^{n} \epsilon_i ((x_j + y_j) - (u_j + v_j)) \Big|,$$

$$\alpha \sum_{i=1}^{n} \Big| \sum_{j=1}^{n} (a_{ij} - \epsilon_i)((x_j + y_j) - (u_j + v_j)) \Big| + \Big| \sum_{j=1}^{n} \epsilon_i ((x_j + y_j) - (u_j + v_j)) \Big| \Bigg)$$

$$\leq \Bigg(\sum_{i=1}^{n} \sum_{j=1}^{n} (a_{ij} - \epsilon_i)(|x_j - u_j| + |y_j - v_j|),$$

$$\alpha \sum_{i=1}^{n} \sum_{j=1}^{n} (a_{ij} - \epsilon_i)(|x_j - u_j| + |y_j - v_j|) \Bigg)$$

$$= \Bigg(\sum_{j=1}^{n} (|x_j - u_j| + |y_j - v_j|) \sum_{i=1}^{n} (a_{ij} - \epsilon_i),$$

$$\alpha \sum_{j=1}^{n} (|x_j - u_j| + |y_j - v_j|) \sum_{i=1}^{n} (a_{ij} - \epsilon_i) \Bigg)$$

$$= \Bigg(\sum_{j=1}^{n} (|x_j - u_j| + |y_j - v_j|)(1 - \epsilon), \alpha \sum_{j=1}^{n} (|x_j - u_j| + |y_j - v_j|)(1 - \epsilon) \Bigg)$$

$$= (1 - \epsilon) d((x, y), (u, v))$$

which establishes that A is a contraction mapping. Thus, Theorem 2.2 with T as identity mapping ensures a unique stationary distribution for the Markov Process. Moreover, for any $z^* \in \Delta_{n-1}$, the sequence $\langle A^n z^* \rangle$ converges to the unique stationary distribution. □

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