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Numerical treatment for the solution of singularly perturbed pseudo-parabolic problem on an equidistributed grid

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Abstract: The initial-boundary value problem for a pseudo-parabolic equation exhibiting initial layer is considered. For solving this problem numerically independence of the perturbation parameter, we propose a difference scheme which consists of the implicit-Euler method for the time derivative and a central difference method for the spatial derivative on uniform mesh. The time domain is discretized with a nonuniform grid generated by equidistributing a positive monitor function. The performance of the numerical scheme is tested which confirms the expected behavior of the method. The existing method is compared with other methods available in the recent literature.

Keywords: Pseudo-parabolic problem, singular perturbation, adaptive grid, boundary layer, uniform convergence

1 Introduction

This paper concerns with the following singularly perturbed pseudo-parabolic initial boundary value problem (IBVP) in the domain $\overline{D} = \overline{\Omega} \bigcup [0, T]$, $\Omega = (0, l)$, $D = \Omega \bigcup (0, T]$,

$$\begin{cases} Lu := \varepsilon L_1\left[\frac{\partial u}{\partial t}\right] - \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial u}{\partial x}\right) + b(x,t)\frac{\partial u}{\partial x} \\ \qquad + c(x,t)u = f(x,t), \quad (x,t) \in D, \\ u(x,0) = s(x), \quad x \in \Omega, \\ u(0,t) = u(l,t) = 0, \quad t \in (0,t], \end{cases} \tag{1.1}$$

where $L_1\left[\frac{\partial u}{\partial t}\right] = -\frac{\partial^3 u}{\partial t \partial x^2} + \frac{\partial u}{\partial t}$. Here ε is a small positive parameter.

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Singularly perturbed pseudo-parabolic problems arise in the various field of Physics, Mechanics applied Mathematics. Typical models include problems such as fluid flow in fissured porous media, the emission of radiation from a gas, flow of second-order fluids. *etc,* [24]. The study of pseudo-parabolic problems is important because one can relate the solution of the parabolic IBVP with the limit of some sequence of solutions of the corresponding pseudo-parabolic problems [19, 23]. The existence and uniqueness results have been extensively studied for pseudo-parabolic equation [11, 15].

The equation (1.1) is an example of Sobolev equation characterized by having mixed time and space derivatives appearing in the highest order terms. These type of problems were first studied by Sobolev. Ewing [11] gave a difference scheme to solve the pseudo-parabolic equation and proved its convergence in L^2– norm with an $(\triangle x^2 + \triangle t^2)$ discretization error. Grubb and Solonnikov [14] developed a method to overcome the degeneracy of parabolic problem by transferring the model to pseudo-parabolic nondegenerate problems. Various types of numerical methods for a parameter-free version of IBVP(1.1) has been studied by many researchers. Very limited literature exists [1, 3] for the problem of type (1.1) with the presence of the parameter ε which makes it singularly perturbed in nature. Singularly perturbed problems are more intrinsic and challenging due to the layer behavior of the solution. Duru [10] analyzed Sobolev type equations involving a single space variable through a finite-difference method to tackle the boundary layers. Amiraliyev et. al. [2] developed parameter uniform method for IBVP(1.1) on the standard Shishkin mesh [12, 17]. But to generate the Shishkin mesh, one needs to have aprior information about the location and the width of the layers. It is always desirable to provide an efficient numerical method in the simplest way. In order to serve this purpose, we have developed a finite difference scheme on a nonuniform mesh which is generated adaptively using equidistribution principle. One may refer [4, 13] and the references therein for more details on this argument. The nonuniform mesh is so generated that we hardly need any *aprior* information of the solution which is the advantage over the method discussed in [2]. The proposed scheme consists of the central difference scheme for the spatial derivative on an uniform mesh and implicit Euler scheme for the time derivative on an adaptively generated nonuniform mesh. The fully discrete scheme for IBVP (1.1) is analyzed on the adaptive grid. It is shown numerically to be uniformly convergent with respect to ε, however the detailed converengence analysis theoretically is yet to be done. Comparison results are shown which ensures the efficiency of the proposed scheme.

Notations: 'C' has been used as a generic positive constant which is independent of ε, the mesh points and the mesh size throughout this paper.

2 Analytic behaviour of solution

Here, in this section we deal with the analytical properties of the exact solution which are needed later in the the study for the numerical aspects.

We assume that the functions $a(x,t)$, $b(x,t)$, $c(x,t)$, $f(x,t)$ and $s(x)$ are sufficiently smooth functions satisfying certain regularity conditions with $a(x,t) \geq \alpha > 0$. At the two corner points $(0,0)$ and $(l,0)$, the functions $s(x)$ and $f(x,t)$ also satisfy the compatibility conditions given by:

$$s(0) = 0, \; s(l) = 0 \tag{2.1}$$

and

$$\begin{cases} -\dfrac{\partial}{\partial x}\left(a(0,0)\dfrac{ds(0)}{dx}\right) + b(0,0)\dfrac{ds(0)}{dx} + cs(0) = f(0,0), \\ -\dfrac{\partial}{\partial x}\left(a(l,0)\dfrac{ds(l)}{dx}\right) + b(l,0)\dfrac{ds(l)}{dx} + cs(l) = f(l,0). \end{cases} \tag{2.2}$$

Assuming (2.1) and (2.2) hold true, the IBVP (1.1) admits a solution which possesses exponential layer for small values of ε along the line $t = 0$ (refer [2]).

Lemma 2.1. *Assuming* $\frac{\partial a}{\partial x}, \frac{\partial b}{\partial x}, \frac{\partial a}{\partial t}, \frac{\partial^2 a}{\partial t \partial x}, \frac{\partial b}{\partial t}, c, \frac{\partial c}{\partial x}, f, \frac{\partial f}{\partial t} \in C(\overline{D})$, $s \in C^2(\overline{D})$,

$$\alpha + \pi^{-2} l^2 \min_{\overline{D}}\left(c - \frac{1}{2}\frac{\partial b}{\partial x}\right) > 0.$$

Under these assumptions, the solution $u(x,t)$ of (1.1) satisfies the following inequalities:

$$\left|\frac{\partial^{l+m} u}{\partial t^l \partial x^m}\right| \leq C\varepsilon^{-l}, \;\; (x,t) \in \overline{D}, \; l = 0,1,2; \; m = 0,1,2.$$

Moreover,

$$\left\|\frac{\partial^{l+m} u}{\partial t^l \partial x^m}\right\|_0 \leq C\left\{1 + \varepsilon^{-l}\exp(-\alpha_0 t/\varepsilon)\right\},$$
$$t \in [0,T], \; l = 1,2; \; m = 0,1,2,$$

where $\alpha_0 = \frac{1}{2}\lambda_0$ *and*

$$\lambda_0 = \begin{cases} \alpha, & \alpha \leq \min_{\overline{D}}\left(c - \dfrac{1}{2}\dfrac{\partial b}{\partial x}\right); \\ \dfrac{\alpha l^{-2}\pi^2 + \min_{\overline{D}}\left(c - \frac{1}{2}\frac{\partial b}{\partial x}\right)}{1 + l^{-2}\pi^2}, & \alpha > \min_{\overline{D}}\left(c - \dfrac{1}{2}\dfrac{\partial b}{\partial x}\right). \end{cases}$$

Proof: The proof follows from the idea given in Lemma 2.1 and Lemma 2.2 of [2]. □

The above bounds are are needed for the convergence analysis. It is important to observe that the second order derivative with respect to time is used for generating the nonuniform mesh. The difference scheme and the idea of generating the nonuniform mesh is discussed in the following section.

3 Discrete problem and the adaptive grid

We discretize the spatial mesh with uniform spatial step h such that

$$\Omega^N = \{x_i = ih, \; i = 0,1,2,\ldots N, \; h = l/N\}$$

where N is the number of subdivisions in space. Consider the finite difference approximation of time domain $[0,T]$ on a nonuniform mesh

$$\overline{\wedge}_t^M = \{0 = t_0 \leq t_1 \leq \ldots \leq t_M = 1\},$$

and the step size is

$$\tau_j = t_j - t_{j-1}, \quad j = 1,\ldots,M.$$

Furthermore, denote $v(x_i,t_j) = v_i^j$, define the backward, forward and central operators in space by

$$D^- v_i^j = \frac{v_i^j - v_{i-1}^j}{h} \quad D^+ v_i^j = \frac{v_{i+1}^j - v_i^j}{h} \quad D^0 v_i^j = \frac{v_{i+1}^j - v_{i-1}^j}{2h},$$

and second order approximation is given as

$$D^- D^+ v_i^j = \frac{v_{i+1}^j - 2v_i^j + v_{i-1}^j}{h^2}.$$

The difference operator in time direction is defined as

$$\delta^- v_i^j = \frac{v_i^j - v_i^{j-1}}{\tau_j}, \quad \delta^+ v_i^j = \frac{v_i^{j+1} - v_i^j}{\tau_{j+1}}.$$

The finite difference scheme for approximating (1.1) takes the following form: For $j = 1,2,\ldots,M$:

$$\begin{cases} L_N^M U := \varepsilon(\delta^- D^- D^+ U_i^j - \delta^- U_i^j) + D^+(a_{i-1/2}^j D^- U_i^j) \\ \qquad + b_i^j D^0 U_i^j + c_i^j U_i^j = f_i^j, \quad i = 1,2,\ldots N-1, \\ U_i^0 = s(x_i), \quad x_i \in \overline{\Omega}^N, \\ U_0^j = U_N^j = 0, \quad t_j \in \overline{\wedge}_t^M, \end{cases} \tag{3.1}$$

where $a^j_{i-1/2} = a(x_i - \frac{h}{2}, t_j)$.

In order to obtain ε-uniform convergent difference scheme, we need to use the appropriate nonuniform mesh. We need to have *a priori* idea to generate S-mesh and B-S-mesh [7, 18, 22]. Here, we have used another kind of nonuniform mesh *i.e.* adaptive grid for which we do not need to have the *a priori* knowledge of the location and the width of the layer. Adaptive grid is such a nonuniform mesh which is obtained using the idea of equidistribution of a positive monitor function. A grid $\wedge_t^M$ is said to be equidistributing, if

$$\int_{t_{j-1}}^{t_j} \Theta\big(u(x_m,s),s\big)\,ds = \int_{t_j}^{t_{j+1}} \Theta\big(u(x_m,s),s\big)\,ds, \quad j = 1,\dots,M-1, \tag{3.2}$$

where $M\big(u(x,t),x\big) > 0$ is a strictly positive, L_1-integrable function. Equivalently,

$$\int_{t_{j-1}}^{t_j} \Theta\big(u(x_m,s),s\big)\,ds = \frac{1}{M}\int_0^1 \Theta\big(u(x_m,s),s\big)\,ds, \quad j = 1,\dots,M-1. \tag{3.3}$$

Here, we consider the following monitor function

$$\Theta\big(u(x_m,s),x\big) = 1 + |u_{tt}(x_m,s)|^{1/p}, \quad p \ge 2. \tag{3.4}$$

Let us fixed the spatial nodal point x_m, since the boundary layer exhibits near $t = 0$ and has no role in the spatial component. One can refer [4] to see the effect of increasing 'p' to smoothen the monitor function.

In order to compute the approximation of $\Theta\big(u(x_m,t),t\big)$ at some fixed space level $x_m = mh$, $0 \le m \le N$, we assume

$$\Theta_j = 1 + |\delta^-\delta^+ U^j_m|^{1/p}, \quad \text{for } j = 1,\dots,M-1.$$

The discrete problem (3.1) and the equation (3.2) with the help of above equation are to be solved simultaneously to obtain the desired solution and the nonuniform grid. This approach is also used by many researchers for different class of problems (refer [9, 13, 16]). Recently Shakti and Mohapatra used this approach to solve nonlinear singularly perturbed problems [20, 21].

Many adaptive mesh generation based method have been developed for the several parabolic problem [5, 6, 8]. We use the following adaptive algorithm to generate the nonuniform mesh.

Adaptive Algorithm

Step 1: Let us consider the starting mesh $\{t_j^{(0)} : 0, 1/M, 2/M, \dots, 1\}$ as the uniform mesh.

Step 2: For $k = 0, 1, \dots$ assuming $\{t_j^{(k)}\}$ is given, the discrete solution is calculated $U_m^{j,(k)}$ at $x_m = mh$ from the discrete problem.

Step 3: Compute the discretized monitor function $\Theta_m^{j,(k)}$. Now

$$l_m^{j,(k)} = \tau_j^{(k)}\left(\frac{\Theta_m^{j-1,(k)} + \Theta_m^{j,(k)}}{2}\right) \quad \text{for } j = 1,\dots,M,$$

where $\Theta_m^{j,(k)} = 1 + [\delta^-\delta^+ U_m^{j,(k)}]^{1/p}$ and set $\Theta_m^{0,(k)} = \Theta_m^{1,(k)}$, $\Theta_m^{M,(k)} = \Theta_m^{M-1,(k)}$, $L_0 = 0$ and $L_m^{j,(k)} := \sum_{j=1}^{M} l_m^{j,(k)}$.

Step 4: Let C_0 be a constant (usually chosen by the user with $C_0 > 1$). If,

$$\frac{\max l_m^{j,(k)}}{L_m^M} \le \frac{C_0}{M},$$

then go to Step 6, else continue to Step 5.

Step 5: The new mesh can be generated through the equidistribution of the proposed monitor function using the current computed solution from Step 2 and $\Theta_m^{j,(k)}$ from Step 3. Set $Y_m^{j,(k)} = jl_m^{M,(k)}/M$ for $j = 0,\cdots,M$. Now interpolate $(t_j^{(k+1)}, Y_m^{(k)}(t_j^{(k)}))$ to $(t_j^{(k)}, l_m^{j,(k)})$ using the piecewise linear interpolation. Generate a new mesh $t_j^{(k+1)} =: \{0 = t_0^{(k+1)} < t_1^{(k+1)} < \dots < t_M^{(k+1)} = 1\}$ and return to Step 2.

Step 6: Set $t^\star = \{t_0, t_1, \dots, t_M\} = t_j^{(k+1)}$ and $U = U_i^{j,(k)}$, where U is the desired solution and $t^\star$ is the desired nonuniform mesh capturing the layer. STOP.

From the idea of [2] and [8], we can state the following result.

Proposition 3.1. *Let $u(x_i,t_j)$ and $U^{N,M}(x_i,t_j)$ be the exact solution and the numerical solution obtained by the proposed scheme on the adaptive grid (defined in Section 3) respectively. Then, there exists a constant C such that the following bounds holds:*

$$\max_{(x_i,t_j)\in\overline{G}^{N,M}} |u(x_i,t_j) - U^{N,M}(x_i,t_j)| \le C(N^{-2} + M^{-1}),$$

where $\overline{G}^{N,M} = \overline{\Omega}^N \times \overline{\wedge}_t^M$.

4 Numerical results and discussion

This section provides some numerical results to show the applicability and efficiency of the proposed scheme.

Example 4.1. *Consider the following test problem:*

$$\begin{cases} -\varepsilon\dfrac{\partial^3 u}{\partial t\partial x^2}+\varepsilon\dfrac{\partial u}{\partial t}-\dfrac{\partial^2 u}{\partial x^2}+\pi^4 u=\dfrac{\pi^4}{2}x^2(1-x)+1-3x, \\ \quad (x,t)\in(0,1)\times(0,1], \\ u(x,0)=\sin(\pi x)-0.5x^2(1-x), \quad x\in(0,1), \\ u(0,t)=u(1,t)=0, \quad t\in[0,1]. \end{cases}$$

The exact solution is given by $u(x,t) = \frac{1}{2}x^2(1-x) + \exp(\frac{t\pi^2}{\varepsilon})\sin(\pi x)$. For each value of ε, N and M, we calculate the maximum point-wise error by

$$E_\varepsilon^{N,M} = \max_{(x_i,t_j)\in\overline{G}^{N,M}} |u(x_i,t_j) - U^{N,M}(x_i,t_j)|,$$

and the corresponding rate of convergence is defined by,

$$r_\varepsilon^{N,M} = \log_2\left(\frac{E_\varepsilon^{N,M}}{E_\varepsilon^{N,2M}}\right).$$

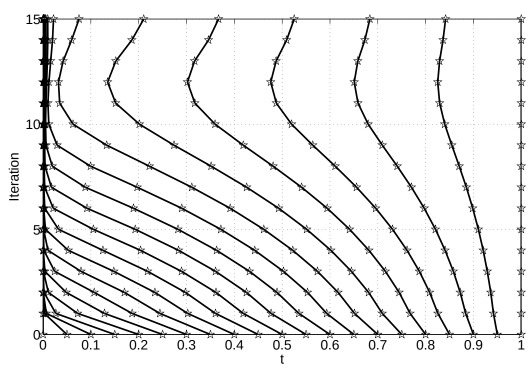

Figure 1: *Mesh movement with $\varepsilon = 10^{-2}$, $x_m = 0.5$, $N = 10$ and $M = 20$*

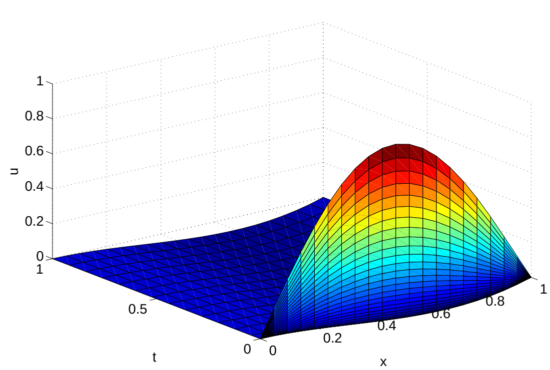

Figure 2: *Numerical solution with $\varepsilon = 10^{-2}$ for Example 4.1*

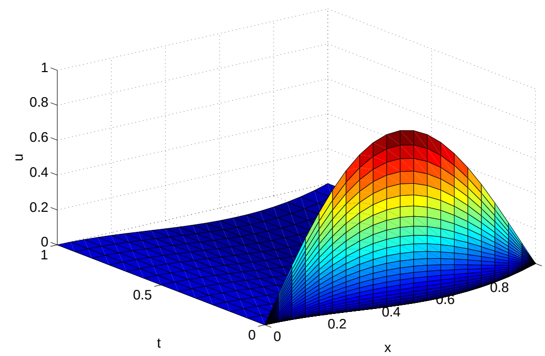

Figure 3: *Numerical solution with $\varepsilon = 10^{-4}$ for Example 4.1*

Figure 1 shows the movement of mesh points towards the initial layer. It can be observed that the mesh points start moving towards the initial layer and the final mesh is dense near the layer. In Figures 2 and 3, the numerical solution is plotted with $N = 20$ and $M = 40$ for $\varepsilon = 10^{-2}$ and $\varepsilon = 10^{-4}$ respectively, which clearly shows the existence of the initial layer near $t = 0$. The computed maximum point-wise errors and the corresponding rate of convergence for Example 4.1 on adaptive grid are precisely presented in Table 1. Clearly, from the results given in Table 1, we observe that the computed maximum errors decrease monotonically as M increases and is independent for sufficiently small values of ε. This ensures that the proposed scheme is parameter uniform. In order to show the efficiency of the proposed scheme, the results obtained by the proposed scheme is compared with the results given in [2] in Table 2. From this comparison, it is evident that the error obtained in the proposed scheme is less whereas the corresponding rate is more as compared to the result of [2]. The numerical outcome indicates that the method yields highly accurate results and is computationally more efficient.

In this article, we propose a numerical method to solve one-dimensional singularly perturbed pseudo-parabolic problem with initial layer of the form (1.1). We develop the method for solving this problem, which generate parameter uniform convergent approximation to the solution. To discretize the domain, we use nonuniform adaptive grid obtained by equidistribution of a positive monitor function in the temporal direction which is fitted to the initial layer and uniform mesh in the spatial direction. First, we use the backward Euler method for the discretization of time derivative and a central difference scheme for the spatial derivatives. The detailed theoretical analysis and the error bound is yet to be obtained. The proposed scheme having an advantage over the scheme proposed in [2] where we need to know about the prior information about the location of the layer to construct the nonuniform mesh. Numerical experiments are carried out to show the efficiency and accuracy of the proposed method.

Table 1: $E_\varepsilon^{N,M}$ and the corresponding $r_\varepsilon^{N,M}$ for Example 4.1 with $N = 10$

ε	32		64		128	
	$E_\varepsilon^{N,M}$	$r_\varepsilon^{N,M}$	$E_\varepsilon^{N,M}$	$r_\varepsilon^{N,M}$	$E_\varepsilon^{N,M}$	$r_\varepsilon^{N,M}$
2^{-2}	2.0571e-02	1.2359	8.7338e-03	1.1329	3.9826e-03	2.0198
2^{-4}	2.1546e-02	1.1940	9.4173e-03	1.4188	3.5222e-03	1.1445
2^{-6}	2.2339e-02	1.2113	9.6476e-03	1.3951	3.6681e-03	0.7723
2^{-8}	2.2612e-02	1.2417	9.5623e-03	1.3447	3.7651e-03	1.7639
2^{-10}	2.2987e-02	1.1819	1.0132e-02	1.3046	4.1018e-03	1.6480
2^{-12}	2.3023e-02	1.2285	9.8256e-03	1.3188	3.9387e-03	1.6379
2^{-14}	2.3238e-02	1.1862	1.0212e-02	1.3321	4.0562e-03	1.8430
2^{-16}	2.3682e-02	1.2795	9.7557e-03	1.3001	3.9619e-03	1.8091
2^{-18}	2.3992e-02	1.2290	1.0235e-02	1.4269	3.8066e-03	1.7734
2^{-20}	2.3900e-02	1.1825	1.0530e-02	1.4387	3.8845e-03	1.5739

Table 2: Comparison of the numerical results

ε	64		128	
	Results given in [2]	Our method	Results given in [2]	Our method
2^{-12}	1.0817e-02	1.1529e-02	6.5030e-03	5.5822e-03
	0.7331	1.0464	0.77843	1.0365
2^{-14}	1.0812e-02	1.1596e-02	6.4931e-03	5.3805e-03
	0.7333	1.1078	0.7775	1.0935
2^{-16}	1.0811e-02	1.2073e-02	6.4835e-03	5.3485e-03
	0.7333	1.1746	0.7809	1.1092
2^{-18}	1.0805e-02	1.1551e-02	6.4806e-03	5.3140e-03
	0.7325	1.1201	0.7834	1.1076
2^{-20}	1.0812e-02	1.1835e-02	6.4815e-03	5.1269e-03
	0.7382	1.2069	0.7792	1.1068

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