J. Indones. Math. Soc. (MIHMI)
Vol. 13, No. 1 (2007), pp. 27–38.

ON CERTAIN CLASSES OF P-VALENT FUNCTIONS DEFINED BY MULTIPLIER TRANSFORMATION AND DIFFERENTIAL OPERATOR

A. Tehranchi , S. R. Kulkarni and G. Murugusundaramoorthy

Abstract. In this paper, we discuss the p-valent functions that satisfy the differential subordinations $\frac{z(I_p(r,\lambda)f(z))^{(j+1)}}{(p-j)(I_p(r,\lambda)f(z))^{(j)}} \prec \frac{a+(aB+(A-B)\beta)z}{a(1+Bz)}$. We also obtain coefficient inequalities, extreme points, integral representation and arithmetic mean. Further we investigate some interesting properties of operators defined on $A_p(r,j,\beta,a,A,B)$.

1. INTRODUCTION

Denote by A the class of functions $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$ analytic in $D = \{z \in C : |z| < 1\}$. Also denote A_p the class of all analytic functions of the form

$$f(z) = kz^p + \sum_{n=p-1}^{2p-1} t_{n-p+1} z^{n-p+1} - {}_2F_1(a,b;c;z), \quad |z| < 1 \tag{1}$$

where

$$\begin{aligned} {}_2F_1(a,b;c;z) &= \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(c,n)n!} z^n \\ (a,n) &= \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1,n-1), \quad c > b > 0, c > a+b \quad \text{and} \\ t_{n-p+1} &= \frac{(a,n-p+1)(b,n-p+1)}{(c,n-p+1)(n-p+1)!}, \quad k > 0. \end{aligned}$$

Received 14 August 2005, revised 28 July 2006, accepted 2 August 2006 .
2000 Mathematics Subject Classification: 30C45, 30C50.
Key words and Phrases: Multivalent function, Differential subordinations, multiplier transformation, differential operator.

These functions are analytic in the unit disk D (For details see [1], [5]).

Definition 1.1. *A function $f \in A_p$ is said to be in the class $S_p^*(\alpha)$, p-valently starlike functions of order α, if it satisfies Re $\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \quad (0 \le \alpha < p, \ \ z \in D)$. We note that $S_p^*(0) = S_p^*$, the class of p-valently starlike functions in D. A function $f \in A_p$ is said to be in the class $C_p(\alpha)$ of p-valently convex of order α, if it satisfies Re $\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, \ \ (0 \le \alpha < p, z \in D)$.*

Let $h(z)$ be analytic and $h(0) = 1$. A function $f \in A_p$ is in the class $S_p^*(h)$ if

$$\frac{zf'(z)}{f(z)} \prec h(z), \quad z \in \Delta. \tag{2}$$

The class $S_p^*(h)$ and a corresponding convex class $C_p(h)$ are defined by Ma and Minda [6]. But results about the convex class can be obtained easily from the corresponding result of functions in $S_p^*(h)$. If

$$h(z) = \frac{1+z}{1-z}, \tag{3}$$

then the classes reduce to the usual classes of starlike and convex functions. If $h(z) = \frac{1+(1-2\alpha)z}{1-z}$, $0 \le \alpha < p$, then the classes reduce to the usual classes of starlike and convex functions of order α. If $h(z) = \frac{1+Az}{1+Bz}, \quad -1 \le B < A \le 1$, then the classes reduce to the class of Janowski starlike function $S_p^*[A, B]$ defined by

$$S_p^*[A, B] = \left\{ f \in A_p : \frac{zf'}{f} \prec \frac{1+Az}{1+Bz}, \quad -1 \le B < A \le 1, z \in D \right\}. \tag{4}$$

If $h(z) = \left(\frac{1+z}{1-z}\right)^{\alpha}, \ \ 0 < \alpha \le 1$, then the classes reduce to the classes of strongly starlike and convex function of order α that consists of univalent functions $f \in A$ satisfying

$$\left| arg \left(\frac{zf'(z)}{f(z)} \right) \right| < \frac{\alpha\pi}{z}, \ \ 0 < \alpha \le 1, z \in \Delta$$

or equivalently we have

$$SS^*(\alpha) = \left\{ f \in A_p : \frac{zf'}{f} \prec \left(\frac{1+z}{1-z}\right)^{\alpha}, \ 0 < \alpha \le 1, z \in \Delta \right\}. \tag{5}$$

Obradovič and Owa [8], Silverman [11], Obradovič and Tuneski [9] and Tuneski [12] have studied the properties of classes of functions defined in terms of the ratio of $1 + \frac{zf''(z)}{f'(z)}$ and $\frac{zf'(z)}{f(z)}$.

Definition 1.2. *A function $f \in A_p$ is said to be p-valent Bazilevic of type η and order α if there exists a function $g \in S_p^*$ such that*

$$Re\left\{\frac{zf'(z)}{f^{1-\eta}(z)g^{\eta}(z)}\right\} > \alpha, \quad (z \in \Delta) \tag{6}$$

for some $\eta(\eta \geq 0)$ and $\alpha(0 \leq \alpha < p)$. We denote by $B_p(\eta,\alpha)$, the subclass of A_p consisting of all such functions. In particular, a function in $B_p(1,\alpha) = B_p(\alpha)$ is said to be p-valently close-to-convex of order α in Δ.

Definition 1.3. [7] *For two functions f and g, analytic in Δ we say f is subordinate to g denoted by $f \prec g$ if there exists a Schwarz function $w(z)$, analytic in Δ with $w(0) = 1$ and $|w(z)| < 1$, such that $f(z) = g(w(z)), z \in \Delta$. In particular, if the function g is univalent in Δ, the above subordination is equivalent to $f(0) = g(0), f(\Delta) \subset g(\Delta)$. Also, we say that g is superordinate to f.*

Using the techniques of Cho and Srivastava[4],Cho and Kim[3] and Uralegadi and Somanatha[12] we define the following transformation.

Definition 1.4. *We define the multiplier transformation operator $I_p(r,\lambda)$ on the infinite series $f(z) = z^p - \sum_{n=p+1}^{\infty} a_n z^n$ as*

$$I_p(r,\lambda)f(z) = z^p - \sum_{n=1+p}^{\infty}\left(\frac{n+\lambda}{p+\lambda}\right)^r a_n z^n, \quad (\lambda \geq 0). \tag{7}$$

We note that Sălăgean derivative operators [9] is closely related to the operators $I_p(r,\lambda)$ when the coefficient of $f(z)$ is positive. Also note that the class $I_1(r,1) = I_r$ [12] ,$I_1(r,\lambda) = I_r^{\lambda}$ the classes studied in [4] and [3].

Definition 1.5. For each $f(z) = z^p - \sum_{n=p+1}^{\infty} a_n z^n$ we have

$$f^{(j)}(z) = \frac{p!}{(p-i)!}z^{p-j} - \sum_{n=p+1}^{\infty}\frac{n!}{(n-j)!}a_n z^{n-j} \tag{8}$$

where $n, p \in N, p > j$, and $j \in N_0 = \{0\} \cup N$. For $j = 0$ we have $f^{(0)}(z) = f(z)$.

Definition 1.6. *A function $f \in A_p$ is said to be in the class $A_p(r,j;h)$ if it satisfies*

$$\frac{z(I_p(r,\lambda)f(z))^{(j+1)}}{(p-j)(I_p(r,\lambda)f(z))^{(j)}} \prec h(z) \tag{9}$$

where

$$h(z) = 1 + \frac{A-B}{a}\frac{\beta z}{1+Bz}, \quad z \in \Delta,$$

and $-1 \leq B < A \leq 1, 0 < \beta < p, a > 0$ *, we denote* $A_p(r,j;h) = A_p(r,j,\beta,a,A,B)$.

We say that $f(z)$ is superordinate to $h(z)$ if $f(z)$ satisfies the following

$$h(z) \prec \frac{z(I_p(r,\lambda)f(z))^{(j+1)}}{(p-j)(I_p(r,\lambda)f(z))^{(j)}}$$

where $h(z)$ is analytic in Δ and $h(0) = 1$.

We note that if

$$\frac{z(I_p(r,\lambda)f(z))^{(j+1)}}{(I_p(r,\lambda)f(z))^{(j)}} \prec \frac{(p-j)[a+(aB+(A-B)\beta)z]}{a(1+Bz)}$$

so $h(0) = 1$ by choosing $j = r = 0, p = 1$, then $f(z) \in S_p^*(h)$. Also $a = A = \beta = 1, B = -1, f(z) \in S_p^*(1)$. But if $a = \beta = 1$ and $-1 \leq B < A \leq 1$ then $f(z) \in S^*[A,B]$, class of Janowski starlike function. If we put $p = a = \beta = A = 1, B = -1$ then $f(z) \in SS^*(1)$ classes of strongly starlike.

By Definition 1.2., if $g(z) \in S^*$, univalent starlike and $j = r = 0$ and $p = a = A = \beta = 1, B = -1$ and if $Re\left\{\frac{zf'(z)}{f(z)^{-1}g^2(z)}\right\} > 1$, then $f(z) \in B(2,1)$ class Bazilevic function of type $\eta = 2$ and order $\alpha = 1$.

2. MAIN RESULTS

In this section we obtain sharp coefficient estimates for functions in $A_p(r,j,\beta,a,A,B)$.

Theorem 2.1. *Let* $f(z)$ *be of the form (1). Then* $f \in A_p(r,j,\beta,a,A,B)$ *if and only if*

$$\sum_{m=p+1}^{\infty} \gamma^r(m,p)\left[\frac{a(1+B)(\delta(m,j)(p-j)-\delta(m,j+1))}{\beta\kappa(A-B)\delta(p,j+1)} - \frac{\delta(m,j)(p-j)}{\kappa\delta(p,j+1)}\right]k_m < 1 \tag{10}$$

where

$$\gamma^r(m,p) = \left(\frac{m+\lambda}{p+\lambda}\right)^r, \quad \lambda \geq 0, \quad p,r \in N$$

and

$$\delta(m,j) = \frac{m!}{(m-j)!}, \quad -1 \leq B < A \leq 1, \quad 0 < \beta < p, \ j < p.$$

Proof. The function $f(z)$ of the form (1) can be expressed as $f(z) = \kappa z^p - \sum\limits_{n=2p}^{\infty} k_{n-p+1} z^{n-p+1}$, or

$$f(z) = \kappa z^p - \sum_{m=p+1}^{\infty} k_m z^m \tag{11}$$

where $m = n - p + 1$ and $k_m = \frac{(a,m)(b,m)}{(c,m)m!}$, and also we have for all $r, j \in N_0$

$$\begin{aligned}(I_p(r,\lambda)f(z))^{(j)} &= \frac{\kappa p!}{(p-j)!} z^{p-j} - \sum_{m=p+1}^{\infty} \left(\frac{m+\lambda}{p+\lambda}\right)^r \frac{m!}{(m-j)!} k_m z^{m-j} \\ &= \kappa\delta(p,j) z^{p-j} - \sum_{m=p+1}^{\infty} \gamma^r(m,p)\delta(m,j) k_m z^{m-j}. \end{aligned} \tag{12}$$

Let $f(z) \in A_p(r, j, a, \beta, A, B)$ then

$$\left| \frac{az(I_p(r,\lambda)f(z))^{(j+1)} - a(p-j)(I_p(r,\lambda)f(z))^{(j)}}{R(p-j)(I_p(r,\lambda)f(z))^{(j)} - Baz(I_p(r,\lambda)f(z))^{(j+1)}} \right| < 1 \tag{13}$$

where $R = aB + (A - B)\beta$. Now, we can write

$$Re\left\{ \frac{a \sum\limits_{m=p+1}^{\infty} \gamma^r(m,p)(\delta(m,j)(p-j) - \delta(m,j+1))k_m z^{m-j}}{\beta(A-B)\delta(p,j+1)\kappa z^{p-j} - \sum\limits_{m=p+1}^{\infty} \gamma^r(m,p)(\delta(m,j)(p-j)R - \delta(m,j+1)Ba)k_m z^{m-j}} \right\} < 1.$$

We choose the values of z on the real axis and letting $z \to 1^-$ then we have

$$\frac{a \sum\limits_{m=p+1}^{\infty} \gamma^r(m,p)(\delta(m,j)(p-j) - \delta(m,j+1))k_m}{\beta\kappa(A-B)\delta(p,j+1) - \sum\limits_{m=p+1}^{\infty} \gamma^r(m,p)(aB(\delta(m,j)(p-j) - \delta(m,j+1)) - \delta(m,j)(A-B)\beta(p-j))k_m} < 1,$$

and

$$\begin{aligned}&\sum_{m=p+1}^{\infty} \gamma^r(m,p)[a(1+B)(\delta(m,j)(p-j) \\ &-\delta(m,j+1)) - \delta(m,j)(A-B)\beta(p-j)]k_m < \beta\kappa(A-B)\delta(p,j+1)\end{aligned}$$

Conversely, we assume that the condition (10) holds true. Hence it is sufficient to show that $f \in A_p(r, j, \beta, a, A, B)$, that is to prove that

$$\left| \frac{az(I_p(r,\lambda)f(z))^{(j+1)} - a(p-j)(I_p(r,\lambda)f(z))^{(j)}}{(p-j)R(I_p(r,\lambda)f(z))^{(j)} - Baz(I_p(r,\lambda)f(z))^{(j+1)}} \right| < 1.$$

But we have

$$\begin{aligned}
&\left|\frac{az(I_p(r,\lambda)f(z))^{(j+1)}-a(p-j)(I_p(r,\lambda)f(z))^{(j)}}{(p-j)R(I_p(r,\lambda)f(z))^{(j)}-Baz(I_p(r,\lambda)f(z))^{(j+1)}}\right|\\
&=|[a\sum_{m=p+1}^{\infty}\gamma^r(m,p)(\delta(m,j)(p-j)-\delta(m,j+1))k_m z^{m-j}]/\\
&[\beta(A-B)\delta(p,j+1)\kappa z^{p-j}\\
&-\sum_{m=p+1}^{\infty}\gamma^r(m,p)(aB(\delta(m,j)(p-j)-\delta(m,j+1))-\\
&\delta(m,j)(A-B)\beta(p-j))k_m]|\\
&<\{[a\sum_{m=p+1}^{\infty}\gamma^r(m,p)(\delta(m,j)(p-j)-\delta(m,j+1))k_m]/\\
&[\beta\kappa(A-B)\delta(p,j+1)\\
&-\sum_{m=p+1}^{\infty}\gamma^r(m,p)(aB(\delta(m,j)(p-j)-\delta(m,j+1))-\\
&\delta(m,j)(A-B)\beta(p-j))k_m]\}<1
\end{aligned}$$

and so proof is complete.
The inequality (10) is sharp for the function

$$f(z)=z^p-\frac{\beta\kappa(A-B)\delta(p,j+1)}{\gamma^r(q,p)[a(1+B)(\delta(q,j)(p-j)-\delta(q,j+1))-\delta(q,j)(A-B)\beta(p-j)]}z^q,$$

with $q\geq 1+p$.

Corollary 2.2. *Let $f\in A_p(r,j,\beta,a,A,B)$ then we have*

$$k_m<\frac{\kappa\beta(A-B)\delta(p,j+1)}{\gamma^r(m,p)[a(1+B)(\delta(m,j)(p-j)-\delta(m,j+1))-\delta(m,j)(A-B)\beta(p-j)]},\tag{14}$$

with $m\geq p+1$

In the next theorem we prove that the class $A_p(r,j,\beta,\alpha,A,B)$ is closed under linear combination.

Theorem 2.3. *Let $f_q(z)=\kappa z^p-\sum_{m=p+1}^{\infty}k_{m,q}z^m (q=1,2,\cdots,t)$ be in $A_p(r,j,\beta,a,A,B)$. Then the function $F(z)=\sum_{q=1}^{t}d_qf_q(z)$ where $\sum_{q=1}^{t}d_q=1$, is also in $A_p(r,j,\beta,a,A,B)$.*

Proof. We have

$$\begin{aligned} F(z) &= \sum_{k=1}^{q} d_k \left(\kappa z^p - \sum_{m=1+p}^{\infty} k_{m,q} z^m\right) = \kappa z^p - \sum_{q=1}^{t} d_q \left(\sum_{m=1+p}^{\infty} k_{m,q} z^m\right) \\ &= \kappa z^p - \sum_{m=1+p}^{\infty} \left(\sum_{q=1}^{t} d_q k_{m,q}\right) z^m. \end{aligned}$$

Hence we obtain

$$\sum_{m=1+p}^{\infty} \gamma^r(m,p) \left[\frac{a(1+B)(\delta(m,j)(p-j) - \delta(m,j+1)}{\beta\kappa(A-B)\delta(p,j+1)} - \frac{\delta(m,j)(p-j)}{\kappa\delta(p,j+1)}\right] \left(\sum_{q=1}^{t} d_q d_{m,q}\right)$$
$$= \sum_{q=1}^{t} \left(\sum_{m=1+p}^{\infty} \gamma^r(m,p) \left[\frac{a(1+B)(\delta(m,j)(p-j) - \delta(m,j+1))}{\beta\kappa(A-B)\delta(p,j+1)} - \frac{\delta(m,j)(p-j)}{\kappa\delta(p,j+1)}\right] k_{m,q}\right) d_q$$
$$< \sum_{q=1}^{t} d_q = 1.$$

Now we prove that the class $A_p(r,j,\beta,a,A,B)$ is closed under arithmetic mean.

Theorem 2.4. *Let* $f_j(z) = \kappa z^p - \sum\limits_{m=1+p}^{\infty} k_{m,q} z^m$ $(q = 1, 2, \cdots, s)$ *are in* $A_p(r,j,\beta,a,A,B)$. *Then the function* $F(z) = \kappa z^p - \sum\limits_{m=1+p}^{\infty} b_m z^m$ *where* $b_m = \frac{1}{s}\sum\limits_{q=1}^{s} k_{m,q}$, *is also in* $A_p(r,j,\beta,a,A,B)$.

Proof. Since $f_j(z) \in A_p(r,j,\beta,a,A,B)$, then by (10) we have

$$\sum_{m=1+p}^{\infty} \gamma^r(m,p) \left[\frac{a(1+B)(\delta(m,j)(p-j) - \delta(m,j+1))}{\beta\kappa(A-B)\delta(p,j+1)} - \frac{\delta(m,j)(p-j)}{\kappa\delta(p,j+1)}\right]$$
$$k_{m,q} < 1, q = 1, 2, 3, \cdots s.$$

Therefore

$$\begin{aligned} &\textstyle\sum_{m=1+p}^{\infty} \gamma^r(m,p) \left[\frac{a(1+B)(\delta(m,j)(p-j)-\delta(m,j+1))}{\beta\kappa(A-B)\delta(p,j+1)} - \frac{\delta(m,j)(p-j)}{\kappa\delta(p,j+1)}\right] b_m \\ &= \textstyle\sum_{m=1+p}^{\infty} \gamma^r(m,p) \left[\frac{a(1+B)(\delta(m,j)(p-j)-\delta(m,j+1))}{\beta\kappa(A-B)\delta(p,j+1)} - \frac{\delta(m,j)(p-j)}{\kappa\delta(p,j+1)}\right] \left(\frac{1}{s}\sum_{q=1}^{s} k_{m,q}\right) \\ &\le \textstyle\frac{1}{s}\sum_{q=1}^{s} \left(\sum_{m=1+p}^{\infty} \gamma^r(m,q) \left[\frac{a(1+B)(\delta(m,j)(p-j)-\delta(m,j+1))}{\beta\kappa(A-B)\delta(p,j+1)} - \frac{\delta(m,j)(p-j)}{\kappa\delta(p,j+1)}\right] k_{m,q}\right) \\ &\le \textstyle\sum_{q=1}^{s} \frac{1}{s} = 1 \end{aligned}$$

and this completes the proof.

Theorem 2.5. (Extreme Points) *Let* $f_p(z) = z^p$ *and for* $m \ge 1+p$

$$f_m(z) = \kappa z^p - \frac{\beta\kappa(A-B)\delta(p,j+1)}{\gamma^r(m,p)[a(1+B)(\delta(m,j)(p-j) - \delta(m,j+1)) - \delta(m,j)(A-B)\beta(p-j)]} z^m,$$

Then the function $f(z) \in A_p(r,j,\beta,a,A,B)$ *if and only if it can be expressed in the form*

$$f(z) = \sum_{m=p}^{\infty} \mu_m f_m(z) \tag{15}$$

where $\mu_m \geq 0$ *and* $\sum\limits_{m=p}^{\infty} \mu_m = 1$.

Proof. Suppose that f can be expressed in the form (15) then we have

$$\begin{aligned} f(z) &= \sum_{m=p}^{\infty} \mu_m f_m(z) \\ &= \mu_p f_p(z) + \sum_{m=p+1}^{\infty} \mu_m f_m(z) \\ &= \mu_p \kappa z^p + \sum_{m=p+1}^{\infty} \mu_m (\kappa z^p - \\ &\quad \frac{\beta\kappa(A-B)\delta(p,j+1)}{\gamma^r(m,p)[a(1+B)(\delta(m,j)(p-j)-\delta(m,j+1))-\delta(m,j)(A-B)\beta(p-j)]} z^m) \\ &= \kappa z^p - \sum_{m=1+p}^{\infty} \frac{\beta\kappa(A-B)\delta(p,j+1)}{\gamma^r(m,p)[a(1+B)(\delta(m,j)(p-j)-\delta(m,j+1))-\delta(m,j)(A-B)\beta(p-j)]} z^m \mu_m \end{aligned}$$

Consequently

$$\begin{aligned} &\sum_{m=p+1}^{\infty} \gamma^r(m,p) \left[\frac{a(1+B)(\delta(m,j)(p-j)-\delta(m,j+1))}{\beta\kappa(A-B)\delta(p,j+1)} - \frac{\delta(m,j)(p-j)}{\kappa\delta(p,j+1)} \right] \\ &\frac{\beta\kappa(A-B)\delta(p,j+1)}{\gamma^r(m,p)[a(1+B)(\delta(m,j)(p-j)-\delta(m,j+1))-\delta(m,j)(A-B)\beta(p-j)]} \mu_n \\ &= \sum_{m=1+p}^{\infty} \mu_m = 1 - \mu_p < 1. \end{aligned}$$

Therefore we conclude the result.

Conversely, let $f \in A_p(r,j,\beta,a,A,B)$ since by (10) we may set

$$\mu_m = \gamma^r(m,p) k_m \frac{a(1+B)(\delta(m,j)(p-j)-\delta(m,j+1))-\delta(m,j)(A-B)\beta(p-j)}{\beta\kappa(A-B)\delta(p,j+1)},$$

with $m \geq 1+p$. Therefore $\mu_m \geq 0$ and if we set $\mu_p = 1 - \sum\limits_{m=1+p}^{\infty} \mu_n$ then we can write

$$\begin{aligned} f(z) &= \kappa z^p - \sum_{m=1+p}^{\infty} k_m z^m \\ &= \kappa z^p - \sum_{m=1+p}^{\infty} \frac{\beta\kappa(A-B)\delta(p,j+1)}{\gamma^r(m,p)[a(1+B)(\delta(m,j)(p-j)-\delta(m,j+1))-\delta(m,j)(A-B)\beta(p-j)]} \mu_m z^m \end{aligned}$$

$$
\begin{aligned}
&= \kappa z^p - \sum_{m=1+p}^{\infty} \mu_m(\kappa z^p - f_m(z)) \\
&= \kappa z^p(1 - \sum_{m=1+p}^{\infty} \mu_m) - \sum_{m=1+p}^{\infty} \mu_m f_m(z) \\
&= \sum_{m=p}^{\infty} \mu_m f_m(z).
\end{aligned}
$$

Remark 2.6. The extreme points of the class $A_p(r, j, \beta, a, A, B)$ are the function $f_p(z), f_{m+p}(z), m \geq 1+p$ as in Theorem 2.1.

In the following theorem, we obtain the integral representation for $A_p(r, j, \beta, a, A, B)$.

Theorem 2.7. *Let $f(z) \in A_p(r, j, \beta, a, A, B)$ then*

$$f(z) = \exp\left[\int_0^z \frac{p(\psi(t)R + a)}{t(1 + B\psi(t))} dt\right]$$

where $|\psi(z)| < 1, z \in U$ and $R = aB + (A - B)\beta$. Also

$$f(z) = z^p \exp\left[\int_X \log(1 - Bxz)^{\frac{p(A-B)\beta}{aB}} d\mu_{(x)}\right]$$

where $\mu_{(x)}$ is the probability measure on $X = \{x : |x| = 1\}$.

Proof. Set

$$\frac{z(I_p(r, \lambda)f(z))^{(j+1)}}{(p - j)(I_p(r, \lambda)f(z))^{(j)}} = Q(z)$$

Since $f(z) \in A_p(r, j, \beta, a, A, B)$ so $\left|\frac{a(Q(z)-1)}{R - aBQ(z)}\right| < 1$ where $R = aB + (A - B)\beta$.

Consequently we put $\frac{a(Q(z)-1)}{R-aBQ(z)} = \psi(z), |\psi(z)| < 1$.

Finally we can write $Q(z) = \frac{\psi(z)R + a}{a + aB\psi(z)}$ or

$$\frac{(I_p(r, \lambda)f(z))^{(j+1)}}{(I_p(r, \lambda)f(z))^{(j)}} = \frac{(p - j)(\psi(z)R + a)}{az(1 + B\psi(z))}$$

Then we have

$$\log(I_p(r, \lambda)f(z))^{(j)} = \int_0^z \frac{(p - j)(\psi(t)R + a)}{t(1 + B\psi(t))} dt$$

$$(I_p(r, \lambda)f(z))^{(j)} = \exp\left[\int_0^z \frac{(p - j)(\psi(t)R + a)}{t(1 + B\psi(t))} dt\right]$$

for $r = j = 0$ we have

$$f(z) = \exp\left[\int_0^z \frac{p(\psi(t)R + a)}{t(1 + B\psi(t))}dt\right].$$

For obtaining the second representation let $X = \{x : |x| = 1\}$ then we have $\frac{a(Q(z)-1)}{R-aBQ(z)} = xz, z \in \Delta$ and then we conclude that

$$\begin{aligned}\frac{(I_p(r,\lambda)(f(z))^{(j+1)}}{(I_p(r,\lambda)f(z))^{(j)}} &= \frac{(p-j)(Rxz + a)}{az(1 + Bxz)} = \frac{(p-j)}{z} + \frac{x(p-j)(A-B)\beta}{a(1 + Bxz)} \\ &= (p-j)\left(\frac{1}{z} + \frac{x(A-B)\beta}{a(1 + Bxz)}\right)\end{aligned}$$

$$\log(I_p(r,\lambda)f(z))^{(j)} = (p-j)\left(\log z + \frac{(A-B)\beta}{aB}\log(1 + Bxz)\right)$$

$$\log\frac{(I_p(r,\lambda)f(z))^{(j)}}{z^{p-j}} = \frac{(p-j)(A-B)\beta}{aB}\log(1 + Bxz)$$

$$(I_p(r,\lambda)f(z))^{(j)} = z^{p-j}\exp\left[\int_X \log(1 - Bxz)^{\frac{(p-j)(A-B)\beta}{aB}} d\mu_{(x)}\right]$$

where $\mu_{(x)}$ is probability measure on X. For $j = r = 0$ we have

$$f(z) = z^p \exp\left[\int_X \log(1 - Bxz)^{\frac{p(A-B)\beta}{aB}} d\mu_{(x)}\right].$$

Now, we introduce an integral operator due to Bernardi [2]

$$L_c(f(z)) = \frac{p + c}{z^c}\int_0^z f(t)t^{c-1}dt, \quad (c > -p)$$

and we study the effect of this operator on class $A_p(r, j, \beta, a, A, B)$.

Theorem 2.8. *If $f \in A_p(r, j, \beta, a, A, B)$ then $L_c(f(z))$ is also in $A_p(r, j, \beta, a, A, B)$.*

Proof. If $f(z) = \kappa z^p - \sum_{m=1+p}^{\infty} k_m z^m$ then

$$\begin{aligned}L_c(f(z)) &= \frac{p + c}{z^c}\int_0^z \left(\kappa t^p - \sum_{m=1+p}^{\infty} k_m t^m\right) t^{c-1}dt \\ &= \kappa z^p - \sum_{m=1+p}^{\infty} \frac{p + c}{m + c} k_m z^m.\end{aligned}$$

Since $m > p$ then $\frac{p+c}{m+c} \leq 1$ so we have

$$\sum_{m=1+p}^{\infty} \gamma^r(m,p)\left[\frac{a(1+B)(\delta(m,j)(p-j)-\delta(m,j+1))}{\beta\kappa(A-B)\delta(p,j+1)} - \frac{\delta(m,j)(p-j)}{\kappa\delta(p,j+1)}\right]\left(\frac{p+c}{m+c}\right)k_m$$
$$\leq \sum_{m=1+p}^{\infty} \gamma^r(m,p)\left[\frac{a(1+B)(\delta(m,j)(p-j)-\delta(m,j+1))}{\beta\kappa(A-B)\delta(p,j+1)} - \frac{\delta(m,j)(p-j)}{\kappa\delta(p,j+1)}\right]k_m < 1.$$

Thus $L_c(f(z)) \in A_p(r,j,\beta,a,A,B)$.

REFERENCES

1. C. E. Andrews, R. Askey and R. Roy, *Special functions*, Cambridge University Press, Cambridge, 1991.
2. S. D. Bernardi, "Convex and starlike univalent functions", *Trans. Amer. Math. Soc.* **135** (1969), 429-446.
3. N. E. Cho and T. H. Kim,, "Multiplier transformations and strongly close-to-convex functions ", *Bull. Korean Math. Soc.* **40**(3) (2003), 399-410.
4. N. E. Cho and H. M. Srivastava,, "Argument estimates of certain analytic functions defined by a class of multiplier transformations ", *Math. Comput. Modelling.* **37** (1-2) (2003), 39-49.
5. Y. C. Kim and F. Rønning,, "Integral transforms of certain subclasses of analytic functions ", *J. Math. Anal. Appl.* **258** (2001), 466-489.
6. W. C. Ma and D. Minda., *A unified treatment of some special classes of univalent functions.*, Proceedings of the Conference on Complex Analysis (Tianjin, 1992), Conf. Proc. Lecture Notes Anal. I. pages 157-169, Cambridge, MA, 1994, Internat. Press.
7. S. S. Miller and P. T. Mocanu, "Subordinants of differential superordinations", *Complex Var. Theory Appl.* **48**(10) (2003), 815-826.
8. M. Obradovič and S. Owa, "A criterion for starlikeness", *Math. Nachr.* **140** (1989), 97-102.
9. M. Obradovič, N. Tuneski, "On the starlike criteria defined Silverman", *Zesz, Nauk. Politech. Rzeszowskiej Mat.* **181**(24) (2000), 59-64.
10. G. S. Sălăgean, "Subclasses of univalent functions in Complex analysis" *Fifth Romanian - Finnish Seminar, Part 1 (Bucharest, 1981), 362-372,* Lecture Notes in Math., 1013, Springer, Berlin.
11. H. Silverman, "Convex and starlike criteria", *Int. J. Math. Math. Sci.* **22** (1999), 75-79.
12. N. Tuneski, "On the quotient of the representations of convexity and starlikeness", *Math. Nachr.*, **248-249** (2003), 200-203.
13. B.A. Uralegaddi and C. Somanatha, "Certain classes of univalent functions", *Current topics in analytic function theory,* 371-374, World Sci. Publishing, River Edge, NJ, 1992.

A. Tehranchi and S. R. Kulkarni : Department of Mathematics, Fergusson College, Pune University, Pune - 411004, India.
E-mail: Tehranchiab@yahoo.co.uk and kulkarni_ferg@yahoo.com

G.Murugusundaramoorthy: School of Science and Humanities, Vellore Institute of Technology, Deemed University, Vellore-632 014 ,India.
E-mail: gmsmoorthy@yahoo.com