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On certain subclasses of analytic functions associated with hypergeometric functions

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ABSTRACT

In this paper, we find the necessary and sufficient conditions for functions $zF(a, b; c; z)$ in the generalized class of β uniformly starlike and β uniformly convex functions of order α and also consequences of the results are pointed out.

1. Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic and univalent in the open disc $U = \{z : z \in \mathcal{C}\ |z| < 1\}$. A function $f \in A$ is called *starlike of order* α $(0 \leq \alpha < 1)$ if and only if $\Re(\frac{zf'(z)}{f(z)}) > \alpha \quad (z \in U)$. A function $f \in A$ is called *convex of order* α $(0 \leq \alpha < 1)$ if and only if $\Re(1 + \frac{zf''(z)}{f'(z)}) > \alpha \quad (z \in U)$. We denote the class of all starlike functions of order α by $S^*(\alpha)$ and the class convex functions of order α by $K(\alpha)$. Denote by T the subclass of A consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \quad z \in U, \tag{1.2}$$

$T^*(\alpha)$ and $C(\alpha)$ are the class of starlike and convex functions of order α $(0 \leq \alpha < 1)$, introduced and studied by Silverman [1].

The class β-*UCV* was introduced by Kanas and Wisniowska [2], where its geometric definition and connections with the conic domains were considered. The class β-*UCV* was defined pure geometrically as a subclass of univalent functions, that map each circular arc contained in the unit disk U with a center ξ, $|\xi| \leq \beta$ $(0 \leq \beta < 1)$, onto a convex arc. The notion of

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β-uniformly convex function is a natural extension of the classical convexity. Observe that, if $\beta = 0$ then the center ξ is the origin and the class β-UCV reduces to the class of convex univalent functions K. Moreover for $\beta = 1$ corresponds to the class of uniformly convex functions UCV introduced by Goodman [3,4], and studied extensively by Rønning [5,6]. The class β-S_P is related to the class β-UCV by means of the well-known Alexander equivalence between the usual classes of convex K and starlike S^* functions. Further the analytic criterion for functions in these classes is given as below.

For $-1 < \alpha \leq 1$ and $\beta \geq 0$ a function $f \in A$ is said to be in the class

(i) β-uniformly starlike functions of order α is denoted by $S_P(\alpha, \beta)$ if it satisfies the condition

$$\Re\left(\frac{zf'(z)}{f(z)} - \alpha\right) > \beta\left|\frac{zf'(z)}{f(z)} - 1\right|, \quad z \in U \tag{1.3}$$

and

(ii) β-uniformly convex functions of order α is denoted by $UCV(\alpha, \beta)$, if it satisfies the condition

$$\Re\left(1 + \frac{zf''(z)}{f'(z)} - \alpha\right) > \beta\left|\frac{zf''(z)}{f'(z)}\right|, \quad z \in U. \tag{1.4}$$

Indeed it follows from (1.3) and (1.4) that

$$f \in UCV(\alpha, \beta) \Leftrightarrow zf' \in S_P(\alpha, \beta). \tag{1.5}$$

Remark 1.1. It is of interest to state that $UCV(\alpha, 0) = K(\alpha)$ and $S_P(\alpha, 0) = S^*(\alpha)$

Motivated by above definitions we define the following subclasses of A.

For $0 \leq \lambda < 1$, $0 \leq \alpha < 1$ and $\beta \geq 0$, we let $S_P(\lambda, \alpha, \beta)$ be the subclass of $\mathcal{A}$ consisting of functions of the form (1.1) and satisfying the analytic criterion

$$\mathrm{Re}\left\{\frac{zf'(z)}{(1-\lambda)f(z) + \lambda zf'(z)} - \alpha\right\} > \beta\left|\frac{zf'(z)}{(1-\lambda)f(z) + \lambda zf'(z)} - 1\right|, \quad z \in U, \tag{1.6}$$

and also, let $UCV(\lambda, \alpha, \beta)$ be the subclass of $\mathcal{A}$ consisting of functions of the form (1.1) and satisfying the analytic criterion

$$\mathrm{Re}\left\{\frac{f'(z) + zf''(z)}{f'(z) + \lambda zf''(z)} - \alpha\right\} > \beta\left|\frac{f'(z) + zf''(z)}{f'(z) + \lambda zf''(z)} - 1\right|, \quad z \in U. \tag{1.7}$$

We further let $TS_P(\lambda, \alpha, \beta) = S_P(\lambda, \alpha, \beta) \cap T$ and $UCT(\lambda, \alpha, \beta) = UCV(\lambda, \alpha, \beta) \cap T$. Suitably specializing the parameters we note that

(1) $TS_P(0, \alpha, \beta) = TS_P(\alpha, \beta)$ [7]
(2) $TS_P(0, 0, \beta) = TS_P(\beta)$ [8]
(3) $TS_P(0, \alpha, 1) = TS_P(\alpha)$ [7]
(4) $TS_P(\lambda, \alpha, 0) = T^*(\lambda, \alpha)$ [9]
(5) $TS_P(0, \alpha, 0) = T^*(\alpha)$ [1]
(6) $UCT(0, \alpha, \beta) = UCT(\alpha, \beta)$ [7]
(7) $UCT(0, 0, \beta) = UCT(\beta)$ [10]
(8) $UCT(0, \alpha, 1) = UCT(\alpha)$ [7]
(9) $UCT(\lambda, \alpha, 0) = C(\lambda, \alpha)$ [9]
(10) $UCT(0, \alpha, 0) = C(\alpha)$ [1].

We recall the Gaussian hypergeometric function $F(a, b; c; z)$ defined by

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{(1)_n}, \tag{1.8}$$

where $a, b, c \in \mathbb{C}$ with $c \neq 0, -1, -2, \ldots$ and $(a)_n$ is the Pochhammer symbol defined in terms of the Gamma functions, by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{Bmatrix} 1 & n = 0 \\ a(a+1)(a+2)\ldots(a+n-1), & n \in N \end{Bmatrix}. \tag{1.9}$$

It is known that

$$F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad Re(c-a-b) > 0 \tag{1.10}$$

and the function $F(a, b; c; 1)$ converges if $Re(c - a - b) > 0$.

Carlson and Schaffer [11] studied the class of starlike functions and pre starlike functions involving hypergeometric functions. In 1993, Silverman [12] gave necessary and sufficient conditions for $zF(a, b; c; z)$ to be in $T^*(\alpha)$ and $C(\alpha)$. Motivated by Silverman [12], Swaminathan [13] and Mostafa [14] in this paper, we find the necessary and sufficient conditions for $zF(a, b; c; z)$ to be in $TS_P(\lambda, \alpha, \beta)$ and $UCT(\lambda, \alpha, \beta)$ when $f \in TS_P(\lambda, \alpha, \beta)$ and $f \in UCT(\lambda, \alpha, \beta)$ respectively for a given a, b, c such that $Re(c - a - b) > 0$.

2. Preliminary results

In this section we obtain the characterization properties for the classes $S_P(\lambda, \alpha, \beta)$, $TS_P(\lambda, \alpha, \beta)$, $UCV(\lambda, \alpha, \beta)$ and $UCT(\lambda, \alpha, \beta)$.

Theorem 2.1. *A function $f(z)$ of the form (1.1) is in $S_P(\lambda, \alpha, \beta)$ if*

$$\sum_{n=2}^{\infty}[n(1+\beta)-(\alpha+\beta)(1+n\lambda-\lambda)]|a_n| \leq 1-\alpha. \tag{2.1}$$

Proof. It suffices to show that

$$\beta\left|\frac{zf'(z)}{(1-\lambda)f(z)+\lambda zf'(z)}-1\right| - \mathrm{Re}\left\{\frac{zf'(z)}{(1-\lambda)f(z)+\lambda zf'(z)}-\alpha\right\} \leq 1-\alpha.$$

Now we consider,

$$\begin{aligned}&\beta\left|\frac{zf'(z)}{(1-\lambda)f(z)+\lambda zf'(z)}-1\right| - \mathrm{Re}\left\{\frac{zf'(z)}{(1-\lambda)f(z)+\lambda zf'(z)}-\alpha\right\}\\ &\quad\leq (1+\beta)\left|\frac{zf'(z)}{(1-\lambda)f(z)+\lambda zf'(z)}-1\right|\\ &\quad\leq \frac{(1+\beta)\sum_{n=2}^{\infty}(n-1-n\lambda+\lambda)|a_n||z|^{n-1}}{1-\sum_{n=2}^{\infty}(1+n\lambda-\lambda)|a_n||z|^{n-1}}.\end{aligned}$$

The last expression is bounded above by $(1-\alpha)$ if

$$\sum_{n=2}^{\infty}[n(1+\beta)-(\alpha+\beta)(1+n\lambda-\lambda)]|a_n| \leq 1-\alpha$$

and the proof is complete. □

Theorem 2.2. *A function $f(z)$ of the form (1.2) is in $TS_P(\lambda, \alpha, \beta)$ if and only if*

$$\sum_{n=2}^{\infty}[n(1+\beta)-(\alpha+\beta)(1+n\lambda-\lambda)]|a_n| \leq 1-\alpha. \tag{2.2}$$

Proof. In view of Theorem 2.1, we need only to prove the necessity. If $f \in TS_P(\lambda, \alpha, \beta)$ and z is real then

$$\frac{1-\sum_{n=2}^{\infty}na_nz^{n-1}}{1-\sum_{n=2}^{\infty}[1+n\lambda-\lambda]a_nz^{n-1}} - \alpha > \beta\left|\frac{\sum_{n=2}^{\infty}(n-1-n\lambda+\lambda)a_nz^{n-1}}{1-\sum_{n=2}^{\infty}[1+n\lambda-\lambda]a_nz^{n-1}}\right|.$$

Letting $z \to 1$ along the real axis, we obtain the desired inequality

$$\sum_{n=2}^{\infty}[n(1+\beta)-(\alpha+\beta)(1+n\lambda-\lambda)]|a_n| \leq 1-\alpha,$$

which gives the required result and the proof is complete. □

Now, we state the following theorems without proof.

Theorem 2.3. *A function $f(z)$ of the form (1.1) is in $UCV(\lambda, \alpha, \beta)$ if*

$$\sum_{n=2}^{\infty}n[n(1+\beta)-(\alpha+\beta)(1+n\lambda-\lambda)]|a_n| \leq 1-\alpha. \tag{2.3}$$

Theorem 2.4. *A function $f(z)$ of the form (1.2) is in $UCT(\lambda, \alpha, \beta)$ if and only if*

$$\sum_{n=2}^{\infty}n[n(1+\beta)-(\alpha+\beta)(1+n\lambda-\lambda)]|a_n| \leq 1-\alpha. \tag{2.4}$$

Remark 2.2. It is of interest to note that by suitably specializing the parameters $\phi_1, \phi_2, \varphi_1, \varphi_2$ and A, B as mentioned below in the function class $\mathcal{W}_0(\phi_1; \varphi_1; A; B; k)$ with varying arguments of coefficients introduced and studied extensively by Dziok and Srivastava [15]

(1) $S_P(\lambda, \alpha, \beta) = \mathcal{W}_0(\phi_1; \varphi_1; 2\alpha - 1; 1; \beta)$ and $TS_P(\lambda, \alpha, \beta) = \mathcal{T}\mathcal{W}_0(\phi_1; \varphi_1; 2\alpha - 1; 1; \beta)$ where $\phi_1 = \frac{z}{(1-z)^2}$ and $\varphi_1 = z + \sum_{n=2}^{\infty}(1 + n\lambda - \lambda)z^n$, $(z \in U)$

(2) $UCV(\lambda, \alpha, \beta) = \mathcal{W}_0(\phi_2; \varphi_2; 2\alpha - 1; 1; \beta)$ and $UCT(\lambda, \alpha, \beta) = \mathcal{W}_0(\phi_2; \varphi_2; 2\alpha - 1; 1; \beta)$ where $\phi_2 = z + \sum_{n=2}^{\infty} n^2 z^n$ and $\varphi_2 = z + \sum_{n=2}^{\infty} n(1 + n\lambda - \lambda)z^n$, $(z \in U)$

one can deduce the above results proved in Theorems 2.1–2.4.

3. Main results and their consequences

Theorem 3.5. *Let $a, b, c \in \mathbb{C}$.*

(i) *If $a, b > -1$, $c > 0$ and $ab < 0$, then $zF(a, b; c; z) \in TS_P(\lambda, \alpha, \beta)$ if and only if*

$$c > a + b + 1 - \frac{[1 + \beta - \lambda(\alpha + \beta)]ab}{1 - \alpha}. \tag{3.1}$$

(ii) *If $a, b > 0$, $c > a + b + 1$, then $F_1(a, b; c; z) = z[2 - F(a, b; c; z)] \in TS_P(\lambda, \alpha, \beta)$ if and only if*

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left[1 + \frac{(1-\lambda\alpha)ab}{(1-\alpha)(c-a-b-1)}\right] \le 2. \tag{3.2}$$

Proof. (i) Since

$$\begin{aligned} zF(a, b; c; z) &= z + \frac{ab}{c}\sum_{n=2}^{\infty}\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}z^n, \\ &= z - \left|\frac{ab}{c}\right|\sum_{n=2}^{\infty}\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}z^n, \end{aligned} \tag{3.3}$$

according to Theorem 2.2, we must show that

$$\sum_{n=2}^{\infty}[n(1+\beta) - (\alpha+\beta)(1+n\lambda-\lambda)]\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}} \le \left|\frac{c}{ab}\right|(1-\alpha). \tag{3.4}$$

Note that the left side of (3.4) diverges if $c < a + b + 1$. Now

$$\begin{aligned} &\sum_{n=2}^{\infty}[n(1+\beta) - (\alpha+\beta)(1+n\lambda-\lambda)]\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}} \\ &= [1+\beta-\lambda(\alpha+\beta)]\sum_{n=0}^{\infty}(n+1)\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}} + (1-\alpha)\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}} \\ &= [1+\beta-\lambda(\alpha+\beta)]\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n} + \frac{(1-\alpha)c}{ab}\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n} \\ &= [1+\beta-\lambda(\alpha+\beta)]\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)} + \frac{(1-\alpha)c}{ab}\left[\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} - 1\right]. \end{aligned}$$

Hence, (3.4) is equivalent to

$$\begin{aligned} &\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)}\left[[1+\beta-\lambda(\alpha+\beta)] + \frac{(1-\alpha)(c-a-b-1)}{ab}\right] \\ &\le (1-\alpha)\left[\left|\frac{c}{ab}\right| + \frac{c}{ab}\right] = 0. \end{aligned} \tag{3.5}$$

Thus, (3.5) is valid if and only if

$$[1+\beta-\lambda(\alpha+\beta)] + \frac{(1-\alpha)(c-a-b-1)}{ab} \le 0,$$

or, equivalently,

$$c \ge a + b + 1 - \frac{[1+\beta-\lambda(\alpha+\beta)]}{(1-\alpha)}.$$

(ii) Since

$$F_1(a,b;c;z) = z - \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} z^n,$$

according to Theorem 2.2, we must show that

$$\sum_{n=2}^{\infty}[n(1+\beta)-(\alpha+\beta)(1+n\lambda-\lambda)]\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \le (1-\alpha).$$

Now,

$$\begin{aligned}
&\sum_{n=2}^{\infty}[n(1+\beta)-(\alpha+\beta)(1+n\lambda-\lambda)]\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \\
&= \sum_{n=2}^{\infty}(n-1)[1+\beta-\lambda(\alpha+\beta)+(1-\alpha)]\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \\
&= [1+\beta-\lambda(\alpha+\beta)]\sum_{n=1}^{\infty} n\frac{(a)_n(b)_n}{(c)_n(1)_n} + (1-\alpha)\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n} \\
&= [1+\beta-\lambda(\alpha+\beta)]\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_{n-1}} + (1-\alpha)\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}.
\end{aligned} \tag{3.6}$$

Note that $(a)_n = a(a+1)_{n-1}$ then, (3.6) expressed as

$$\begin{aligned}
&[1+\beta-\lambda(\alpha+\beta)]\frac{ab}{c}\sum_{n=1}^{\infty}\frac{(a+1)_{n-1}(b+1)_{n-1}}{(c+1)_{n-1}(1)_{n-1}} + (1-\alpha)\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n} \\
&= [1+\beta-\lambda(\alpha+\beta)]\frac{ab}{c}\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n} + (1-\alpha)\left[\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n} - 1\right] \\
&= [1+\beta-\lambda(\alpha+\beta)]\frac{ab}{c}\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)} + (1-\alpha)\left[\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} - 1\right] \\
&= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left[(1-\alpha) + \frac{ab[1+\beta-\lambda(\alpha+\beta)]}{c-a-b-1}\right] - (1-\alpha).
\end{aligned}$$

But the last expression is bounded above by $1-\alpha$ if and only if (3.2) holds. □

Remark 3.3. (i) For $\beta = 0$, Theorem 3.5, yields, the result as stated in [14].

Theorem 3.6. *Let $a, b, c \in \mathbb{C}$.*

(i) *If $a, b > -1$, $ab < 0$ and $c > a+b+2$, then $zF(a,b;c;z) \in UCT(\lambda,\alpha,\beta)$ if and only if*

$$\begin{aligned}
&[1+\beta-\lambda(\alpha+\beta)](a)_2(b)_2 + (3+2\beta-\alpha-2(\alpha+\beta)\lambda)ab(c-a-b-2) \\
&\quad + (1-\alpha)(c-a-b-2)_2 \ge 0.
\end{aligned} \tag{3.7}$$

(ii) *If $a, b > 0$, $c > a+b+2$, then $F_1(a,b;c;z) = z[2 - F(a,b;c;z)] \in UCT(\lambda,\alpha,\beta)$ if and only if*

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left[1 + \frac{[1+\beta-\lambda(\alpha+\beta)](a)_2(b)_2}{(1-\alpha)(c-a-b-2)_2} + \frac{[3+2\beta-\alpha-2(\alpha+\beta)\lambda]ab}{(1-\alpha)(c-a-b-1)}\right] \le 2. \tag{3.8}$$

Proof. (i) Since $zF(a,b;c;z)$ has the form (3.3), we see from Theorem 2.4, that our conclusion is equivalent to

$$\sum_{n=2}^{\infty} n[n(1+\beta)-(\alpha+\beta)(1+n\lambda-\lambda)]\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}} \le \left|\frac{c}{ab}\right|(1-\alpha). \tag{3.9}$$

Note that the left side of (3.9) converges if $c > a+b+2$. For brevity,

$$\begin{aligned}
&(n+2)[(n+2)(1+\beta)-(1+\lambda(n+2)-\lambda)(\alpha+\beta)] \\
&\quad = (n+1)^2[1+\beta-\lambda(\alpha+\beta)] + (n+1)[2+\beta-\alpha-(\alpha+\beta)\lambda] + (1-\alpha),
\end{aligned}$$

we see that

$$\sum_{n=0}^{\infty}(n+2)[(n+2)(1+\beta)-(1+\lambda(n+2)-\lambda)(\alpha+\beta)]\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}}$$
$$=[1+\beta-\lambda(\alpha+\beta)]\sum_{n=0}^{\infty}(n+1)^2\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}}$$
$$+[2+\beta-\alpha-(\alpha+\beta)\lambda]\sum_{n=0}^{\infty}(n+1)\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}}+(1-\alpha)\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}}$$
$$=[1+\beta-\lambda(\alpha+\beta)]\sum_{n=0}^{\infty}(n+1)\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n}+[2+\beta-\alpha-(\alpha+\beta)\lambda]\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n}$$
$$+(1-\alpha)\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}}$$
$$=[1+\beta-\lambda(\alpha+\beta)]\sum_{n=0}^{\infty}n\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n}+[3+2\beta-\alpha-2(\alpha+\beta)\lambda]\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n}$$
$$+(1-\alpha)\sum_{n=1}^{\infty}\frac{(a+1)_{n-1}(b+1)_{n-1}}{(c+1)_{n-1}(1)_n}$$
$$=\frac{[1+\beta-\lambda(\alpha+\beta)](a+1)(b+1)}{(c+1)}\sum_{n=0}^{\infty}\frac{(a+2)_n(b+2)_n}{(c+2)_n(1)_n}$$
$$+[3+2\beta-\alpha-2(\alpha+\beta)\lambda]\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n}+\frac{(1-\alpha)c}{ab}\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}$$
$$=\frac{\Gamma(c+1)\Gamma(c-a-b-2)}{\Gamma(c-a)\Gamma(c-b)}\Bigg[[1+\beta-\lambda(\alpha+\beta)](a+1)(b+1)+[3+2\beta-\alpha$$
$$-2(\alpha+\beta)\lambda](c-a-b-2)+\frac{(1-\alpha)(c-a-b-2)_2}{ab}\Bigg]-\frac{(1-\alpha)c}{ab}.$$

The last expression is bounded above by $\left|\frac{c}{ab}\right|(1-\alpha)$ if and only if

$$[1+\beta-\lambda(\alpha+\beta)](a+1)(b+1)+[3+2\beta-\alpha-2(\alpha+\beta)\lambda](c-a-b-2)$$
$$+\frac{(1-\alpha)(c-a-b-2)_2}{ab}\le 0$$

which is equivalent to (3.7).

(ii) In view of Theorem 2.4, we need to show that

$$\sum_{n=2}^{\infty}n[n(1+\beta)-(\alpha+\beta)(1+n\lambda-\lambda)]\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}\le(1-\alpha).$$

Now

$$\sum_{n=2}^{\infty}n[n(1+\beta)-(\alpha+\beta)(1+n\lambda-\lambda)]\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}$$
$$=\sum_{n=0}^{\infty}(n+2)[(n+2)(1+\beta)-(1+\lambda(n+2)-\lambda)(\alpha+\beta)]\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}$$
$$=[1+\beta-\lambda(\alpha+\beta)]\sum_{n=0}^{\infty}(n+2)^2\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}-(\alpha+\beta)(1-\lambda)\sum_{n=0}^{\infty}(n+2)\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}$$
$$=[1+\beta-\lambda(\alpha+\beta)]\left\{\sum_{n=0}^{\infty}(n+1)^2\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}+2\sum_{n=0}^{\infty}(n+1)\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}+\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}\right\}$$
$$-(\alpha+\beta)(1-\lambda)\left\{\sum_{n=0}^{\infty}(n+1)\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}+\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}\right\}$$

$$= [1+\beta-\lambda(\alpha+\beta)]\left\{\sum_{n=0}^{\infty}(n+1)\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_n}+2\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_n}+\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}\right\}$$
$$-(\alpha+\beta)(1-\lambda)\left\{\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_n}+\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}\right\}$$
$$= [1+\beta-\lambda(\alpha+\beta)]\left\{\sum_{n=1}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n-1}}+3\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_n}+\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}\right\}$$
$$-(\alpha+\beta)(1-\lambda)\left\{\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_n}+\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}\right\}$$
$$= [1+\beta-\lambda(\alpha+\beta)]\sum_{n=0}^{\infty}\frac{(a)_{n+2}(b)_{n+2}}{(c)_{n+2}(1)_n}+[3+2\beta-\alpha-2\lambda(\alpha+\beta)]\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_n}$$
$$+(1-\alpha)\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}$$
$$= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left[\frac{[1+\beta-\lambda(\alpha+\beta)](a)_2(b)_2}{(c-a-b-2)_2}+\frac{[3+2\beta-\alpha-2\lambda(\alpha+\beta)]ab}{(c-a-b-1)}+1-\alpha\right]$$
$$-(1-\alpha). \tag{3.10}$$

By a simplification, we see that the last expression is bounded above by $(1-\alpha)$ if and only if (3.8) holds. □

Remark 3.4. (i) For $\beta = 0$, Theorem 3.6, yields, the result as stated in [14].
(ii) For $\beta = 0$ and $\lambda = 0$, Theorem 3.6 yields, the result obtained by Silverman [12].

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