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Research Article

On Extended Branciari b-Distance Spaces and Applications to Fractional Differential Equations

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In this work, we define new $\alpha - \lambda$-rational contractive conditions and establish fixed-points results based on aforesaid contractive conditions for a mapping in extended Branciari b-distance spaces. We furnish two examples to justify the work. Further, we discuss results on weak well-posed property, weak limit shadowing property, and generalized w-Ulam-Hyers stability in the underlying space. Finally, as an application of our main result, we obtain sufficient conditions for the existence of solutions of a nonlinear fractional differential equation with integral boundary conditions.

1. Introduction and Preliminaries

The distance notion in the metric fixed-point theory is introduced and generalized in different ways by many authors [1–5]. Bakhtin [6] defined the notion of b-metric space which is further used by Czerwik in [7, 8]. In [9], Branciari extended the metric space and introduced the notion of the Branciari distance by changing the property of triangle inequality with quadrilateral one.

Definition 1 [9]. Let $\Xi \neq \emptyset$ be a set and let $b : \Xi^2 \longrightarrow \mathbb{R}_+$ such that, for all $\vartheta, \nu \in \Xi$ and all $u, v \in \Xi \setminus \{\vartheta, \nu\}$

(bd1) $b(\vartheta, \nu) = 0$ if and only if $\vartheta = \nu$ (self-distance/indistancy)

(bd2) $b(\vartheta, \nu) = b(\nu, \vartheta)$ (symmetry)

(bd3) $b(\vartheta, \nu) \leq b(\vartheta, u) + b(u, v) + b(v, \nu)$ (quadrilateral inequality).

The symbol (Ξ, b) the denotes Branciari distance space and abbreviated as "BDS."

In [10], Kamran et al. introduced the notion of extended b-metric space as a generalization of b-metric space and proved the following result.

Definition 2 [10]. Let $\Xi \neq \emptyset$ be a set and $w : \Xi^2 \longrightarrow \mathbb{R}_+ \setminus (0, 1)$. We say that a function $\rho_e : \Xi^2 \longrightarrow \mathbb{R}_+$ is an extended b-metric (ρ_e-metric, in short) if it satisfies

(eb1) $\rho_e(\vartheta, \nu) = 0$ if and only if $\vartheta = \nu$

(eb2) $\rho_e(\vartheta, \nu) = \rho_e(\nu, \vartheta)$ (symmetry) (eb3) $\rho_e(\vartheta, \nu) \le w(\vartheta, \nu)[\rho_e(\vartheta, \upsilon) + \rho_e(\upsilon, \nu)]$,

for all $\vartheta, \nu, \upsilon \in \Xi$. The symbol (Ξ, ρ_e) denotes a ρ_e-metric space.

Theorem 3 [10]. *Let (Ξ, ρ_e) be a complete extended b-metric space such that ρ_e is a continuous functional. Let $\mathfrak{T} : \Xi \longrightarrow \Xi$ satisfy $\rho_e(\mathfrak{T}\vartheta, \mathfrak{T}\nu) \le k\,\rho_e(\vartheta, \nu)$ for all $\vartheta, \nu \in \Xi$ where $k \in [0, 1)$ such that for each $\vartheta_0 \in \Xi$, $\lim_{n,m\longrightarrow\infty} w(\vartheta_n, \vartheta_m) < 1/k$, here $\vartheta_n = \mathfrak{T}^n\vartheta_0$, $n = 1, 2, \cdots$. Then $\mathfrak{T}$ has precisely one fixed-point ϑ. Moreover, for each $\nu \in \Xi$, $\mathfrak{T}^n\nu \longrightarrow \vartheta$.*

In [3], Mitrović et al. extended Theorem 3 and proved the following:

Theorem 4 [3]. *Let (Ξ, ρ_e) be a complete extended b-metric space such that ρ_e is a continuous functional. Let $\mathfrak{T} : \Xi \longrightarrow \Xi$ satisfy*

$$\rho_e(\mathfrak{T}\vartheta, \mathfrak{T}\nu) \le a\,\rho_e(\vartheta, \nu) + b\,\rho_e(\vartheta, \mathfrak{T}\vartheta) + c\,\rho_e(\nu, \mathfrak{T}\nu), \quad (1)$$

for all $\vartheta, \nu \in \Xi$ where a, b, c are nonnegative real numbers with $a + b + c < 1$. Then, $\mathfrak{T}$ has a unique fixed-point ϑ. Moreover, there exists a sequence $\{\vartheta_n\}_{n\in\mathbb{N}}$ in Ξ which converges to ϑ such that $\vartheta_{n+1} = \mathfrak{T}\vartheta_n$ for every $n \in \mathbb{N}$.

In [11], Abdeljawad et al. defined the notion of extended Branciari b-distance (EBbDS, in short) by combining the extended b-metric and Branciari distance.

Definition 5 [11]. Let $\Xi \ne \varnothing$ be a set and $w : \Xi^2 \longrightarrow \mathbb{R}_+ \setminus (0, 1)$. We say that a function $e_b : \Xi^2 \longrightarrow \mathbb{R}_+$ is an extended Branciari b-metric (e_b-metric, in short) if it satisfies

(ebb1) $e_b(\vartheta, \upsilon) = 0$ if and only if $\vartheta = \upsilon$

(ebb2) $e_b(\vartheta, \upsilon) = e_b(\upsilon, \vartheta)$

(ebb3) $e_b(\vartheta, \upsilon) \le w(\vartheta, \upsilon)[e_b(\vartheta, \nu) + e_b(\nu, \rho) + e_b(\rho, \upsilon)]$,

for all $\vartheta, \upsilon \in \Xi$, all distinct $\nu, \rho \in \Xi \setminus \{\vartheta, \upsilon\}$. The symbol (Ξ, e_b) denotes the extended Branciari b-distance space. For $w(\vartheta, \upsilon) = 1$, (Ξ, e_b) will be called a Branciari b-distance space (BbDS, in short).

Example 1. Let $\Xi = C([0, 1], \mathbb{R})$ and define $e_b : \Xi^2 \longrightarrow \mathbb{R}_+$ by $e_b(P, Q) = \int_0^1 (P(t) - Q(t))^2 dt$ with $w(P, Q) = |P(t)| + |Q(t)| + 2$. Note that $e_b(P, Q) \ge 0$ for all $P, Q \in \Xi$, and $e_b(P, Q) = 0$ if and only if $P = Q$. Also, $e_b(P, Q) = e_b(Q, P)$. Hence, it is clear that (Ξ, e_b) is an EBbDS, but it is neither an BDS nor metric space.

Definition 6 [11]. Let $\Xi \ne \varnothing$ be a set endowed with extended Branciari b-distance e_b.

(a) A sequence $\{\vartheta_n\}$ in Ξ converges to ϑ if for every $\varepsilon > 0$ there exists $N = N(\varepsilon) \in \mathbb{N}$ such that $e_b(\vartheta_n, \vartheta) < \varepsilon$ for all $n \ge N$. For this particular case, we write $\lim_{n\longrightarrow\infty} \vartheta_n = \vartheta$

(b) A sequence $\{\vartheta_n\}$ in Ξ is called Cauchy if for every $\varepsilon > 0$ there exists $N = N(\varepsilon) \in \mathbb{N}$ such that $e_b(\vartheta_m, \vartheta_n) < \varepsilon$ for all $m, n \ge N$

(c) An e_b-metric space (Ξ, e_b) is complete if every Cauchy sequence in Ξ is convergent.

On the other hand, in [12], Samet et al. define the notion of α-admissible mappings which is further extended by Sintunavarat [13] and named as weakly α-admissible mapping.

Definition 7. For a $\Xi \ne \varnothing$ set, let $\alpha : \Xi \times \Xi \longrightarrow [0,\infty)$ and $\mathfrak{T} : \Xi \longrightarrow \Xi$ be two mappings. Then $\mathfrak{T}$ is called

(1) [12] α-admissible if

$$\vartheta, \nu \in \Xi \text{ with } \alpha(x, \nu) \ge 1 \Rightarrow \alpha(\mathfrak{T}\vartheta, \mathfrak{T}\nu) \ge 1 \quad (2)$$

(2) [13] weakly α-admissible if

$$\vartheta \in \Xi \text{ with } \alpha(\vartheta, \mathfrak{T}\vartheta) \ge 1 \Rightarrow \alpha(\mathfrak{T}\vartheta, \mathfrak{T}\mathfrak{T}\vartheta) \ge 1. \quad (3)$$

For a $\Xi \ne \varnothing$ set and a mapping $\alpha : \Xi \times \Xi \longrightarrow [0,\infty)$, we use

$$\mathscr{A}(\Xi, \alpha) \coloneqq \text{The set of all } \alpha-\text{admissible mappings on } \Xi,$$
$$\mathscr{W}\mathscr{A}(\Xi, \alpha) \coloneqq \text{The set of all weakly } \alpha-\text{admissible mappings on } \Xi. \quad (4)$$

It is noted that

$$\mathscr{A}(\Xi, \alpha) \subset \mathscr{W}\mathscr{A}(\Xi, \alpha). \quad (5)$$

The notion of well-posedness of a fixed-point problem (fpp) has evoked much interest of several mathematicians, for example, Popa [14, 15] and others. In the paper [16], authors defined a weak well-posed (wwp) property in BbDS and in the papers [17, 18]; the authors have discussed limit shadowing property of fixed-point problems.

The aim of this work is to introduce $\alpha - \lambda$-rational contraction in an EBbDS and prove the existence of fixed points of such rational contraction in an EBbDS. We also discuss the weak well-posedness, limit shadowing property, and generalized weak-Ulam-Hyers stability of fixed-point problems in a EBbDS. As an application of our main result, we obtain sufficient conditions for the existence of solutions of a nonlinear fractional differential equation with integral boundary conditions. By doing these work, we generalize Theorems 3 and 4 in the sense that we use a more general contractive condition which depends on the variable (Lipschitz constants), function $w(x, y)$ on the left-side of contractive condition, and proved results on the weakly α-admissible mapping on more general space structures. It is justifies the usefulness of these terms through illustrations, and the results are real generalization as the considered distances are neither metric space not Branciari distance space.

2. Main Results

2.1. $\alpha-\lambda$-Rational Contractive Mapping and Fixed Points. We start with introducing the notion of $\alpha-\lambda$-rational contraction in a EBbDS as follows.

Definition 8. Let (Ξ, e_b) be an EBbDS and $\alpha : \Xi^2 \longrightarrow \mathbb{R}_+$ and $\lambda : \Xi \longrightarrow [0,1)$. A mapping $\mathfrak{T} : \Xi \longrightarrow \Xi$ is said to be an $\alpha-\lambda$ -rational contraction, if there exist

$$\vartheta, \nu \in \Xi, \tag{6}$$

with

$$\begin{gathered}\alpha(\vartheta, \nu) \geq 1, \\ e_b(\vartheta, \nu) > 0, e_b(\mathfrak{T}\vartheta, \mathfrak{T}\nu) > 0,\end{gathered} \tag{7}$$

which implies

$$\begin{aligned} &w(\vartheta, \nu) e_b(\mathfrak{T}\vartheta, \mathfrak{T}\nu) \\ &\leq \lambda(u) \max \left\{ \begin{array}{c} e_b(\vartheta, \nu), e_b(\vartheta, \mathfrak{T}\vartheta), e_b(\nu, \mathfrak{T}\nu), \\ \dfrac{e_b(\nu, \mathfrak{T}\nu)[1 + e_b(\vartheta, \mathfrak{T}\vartheta)]}{w(\vartheta, \nu)[1 + e_b(\vartheta, \nu)]}, \dfrac{e_b(\vartheta, \mathfrak{T}\vartheta).e_b(\nu, \mathfrak{T}\nu)}{w(\vartheta, \nu).e_b(\vartheta, \nu)} \end{array} \right\}. \end{aligned} \tag{8}$$

We denote by $\Lambda(\Xi, \alpha)$ the collection of all $\alpha-\lambda$-rational contractive mappings on (Ξ, e_b).

The set of all fixed points of a self-mapping $\mathfrak{T}$ on a set $\Xi \neq \emptyset$ will be denoted by $Fix(\mathfrak{T})$.

We are now in a position to state and prove the result.

Theorem 9. *Let (Ξ, e_b) be a complete EBbDS and $\alpha : \Xi \times \Xi \longrightarrow [0,\infty)$. Let $\mathfrak{T} : \Xi \longrightarrow \Xi$ be a mapping satisfying the following:*

(A1) $\mathfrak{T} \in \Lambda(\Xi, \alpha) \cap \mathscr{W}\mathscr{A}(\Xi, \alpha)$

(A2) There exists $u_0 \in \Xi$ such that $\alpha(u_0, \mathfrak{T}u_0) \geq 1$

(A3) $\mathfrak{T}$ is continuous.

Then, $Fix(\mathfrak{T}) \neq \emptyset$. Furthermore, for any $u_0 \in \Xi$, the sequence u_n satisfying $u_n = \mathfrak{T}u_{n-1}$ is convergent.

Proof. By virtue of condition (A2), there exists $u_0 \in \Xi$ such that $\alpha(u_0, \mathfrak{T}u_0) \geq 1$. Define the sequence $\{u_n\} \in \Xi$ by $u_{n+1} = \mathfrak{T}u_n$. If there exists $n_0 \in \mathbb{N}$ such that $u_{n_0} = u_{n_0+1}$, then $u_{n_0} \in Fix(\mathfrak{T})$, and we are complete. Therefore, we assume that $u_n \neq u_{n+1}$ for all $n \in \mathbb{N}$.

It follows that

$$e_b(u_n, u_{n+1}) > 0, \forall n \in \mathbb{N}. \tag{9}$$

It follows from $\mathfrak{T} \in \mathscr{W}\mathscr{A}(\Xi, \alpha)$ and $\alpha(u_0, \mathfrak{T}u_0) \geq 1$ that

$$\alpha(u_1, u_2) = \alpha(\mathfrak{T}u_0, \mathfrak{T}\mathfrak{T}u_0) \geq 1. \tag{10}$$

Continuing this process, we obtain

$$\alpha(u_n, u_{n+1}) \geq 1 \forall n \in \mathbb{N}. \tag{11}$$

Step 1. First, we prove that

$$\lim_{n\longrightarrow\infty} e_b(u_n, u_{n+1}) = 0. \tag{12}$$

It follows from $\mathfrak{T} \in \Lambda(\Xi, \alpha)$ that

$$\begin{aligned} &w(u_{n-1}, u_n) e_b(\mathfrak{T}u_{n-1}, \mathfrak{T}u_n)) \\ &\leq \lambda(u_{n-1}) \max \left\{ \begin{array}{c} e_b(u_{n-1}, u_n), e_b(u_{n-1}, \mathfrak{T}u_{n-1}), e_b(u_n, \mathfrak{T}u_n), \\ \dfrac{e_b(u_n, \mathfrak{T}u_n)[1 + e_b(u_{n-1}, \mathfrak{T}u_{n-1})]}{w(u_{n-1}, u_n)[1 + e_b(u_{n-1}, u_n)]}, \dfrac{e_b(u_{n-1}, \mathfrak{T}u_{n-1}).e_b(u_n, \mathfrak{T}u_n)}{w(u_{n-1}, u_n).e_b(u_{n-1}, u_n)} \end{array} \right\} \\ &= \lambda(u_{n-1}) \max \left\{ \begin{array}{c} e_b(u_{n-1}, u_n), e_b(u_{n-1}, u_n), e_b(u_n, u_{n+1}), \\ \dfrac{e_b(u_n, u_{n+1})[1 + e_b(u_{n-1}, u_n)]}{w(u_{n-1}, u_n)[1 + e_b(u_{n-1}, u_n)]}, \dfrac{e_b(u_{n-1}, u_n).e_b(u_n, u_{n+1})}{w(u_{n-1}, u_n).e_b(u_{n-1}, u_n)} \end{array} \right\}, \end{aligned} \tag{13}$$

$$\begin{aligned} &\leq \lambda(u_{n-1}) \max \left\{ \begin{array}{c} e_b(u_{n-1}, u_n), e_b(u_{n-1}, u_n), e_b(u_n, u_{n+1}), \\ e_b(u_n, u_{n+1}), e_b(u_n, u_{n+1}) \end{array} \right\} \\ &\leq \lambda(u_{n-1}) \max \{e_b(u_{n-1}, u_n), e_b(u_n, u_{n+1})\}. \end{aligned} \tag{14}$$

If $e_b(u_{n-1}, u_n) \leq e_b(u_n, u_{n+1})$ for some $n \in \mathbb{N}$, then from (13), we have $w(u_n, u_{n+1}) e_b(u_{n-1}, u_n) \leq \lambda(u_{n-1}) e_b(u_n, u_{n+1})$, which is a contradiction since $w \geq 1$ and $\lambda < 1$. Thus, $e_b(u_n, u_{n+1}) \leq e_b(u_{n-1}, u_n)$ for all $n \in \mathbb{N}$, and the sequence $\{e_b(u_n, u_{n+1})\}$ is a decreasing sequence of real numbers. Therefore, there exists ζ such that

$$\lim_{n\longrightarrow\infty} e_b(u_n, u_{n+1}) = \zeta. \tag{15}$$

Again applying the limit in (13), we get

$$\lim_{n\longrightarrow\infty} w(u_{n-1}, u_n)\zeta \leq \lim_{n\longrightarrow\infty} \lambda(u_{n-1})\zeta, \tag{16}$$

which leads to $\zeta = 0$ as $w \geq 1$. Thus, we get

$$\lim_{n\longrightarrow\infty} e_b(u_n, u_{n+1}) = 0. \tag{17}$$

Step 2. At this step, we will prove that $\{u_n\}$ is a Cauchy sequence, that is, for $m > n$, we prove

$$\lim_{n,m\longrightarrow\infty} e_b(u_n, u_m) = 0. \tag{18}$$

Using (ebb3), we have

$$\begin{aligned} e_b(u_n, u_m) &\leq w(u_n, u_m)[e_b(u_n, u_{n+1}) \\ &\quad + e_b(u_{n+1}, u_{n+2}) + e_b(u_{n+2}, u_{n+m})] \\ &\leq w(u_n, u_m)e_b(u_n, u_{n+1}) + w(u_n, u_m)e_b(u_n, u_{n+1}) \\ &\quad + w(u_n, u_m)e_b(u_{n+2}, u_m) \\ &\leq w(u_n, u_m)e_b(u_n, u_{n+1}) + w(u_n, u_m)e_b(u_n, u_{n+1}) \\ &\quad + w(u_n, u_m)w(u_{n+2}, u_m)[e_b(u_{n+2}, u_{n+3}) \\ &\quad + e_b(u_{n+3}, u_{n+4}) + e_b(u_{n+4}, u_m)] \\ &\leq w(u_n, u_m)e_b(u_n, u_{n+1}) + w(u_n, u_m)e_b(u_n, u_{n+1}) \\ &\quad + w(u_n, u_m)w(u_{n+2}, u_m)e_b(u_n, u_{n+1}) \\ &\quad + w(u_n, u_m)w(u_{n+2}, u_m)e_b(u_n, u_{n+1}) \\ &\quad + w(u_n, u_m)w(u_{n+2}, u_m)e_b(u_{n+4}, u_m) \vdots \\ &\leq w(u_n, u_m)e_b(u_n, u_{n+1}) + w(u_n, u_m)e_b(u_n, u_{n+1}) \\ &\quad + w(u_n, u_m)w(u_{n+2}, u_m)e_b(u_n, u_{n+1}) \\ &\quad + w(u_n, u_m)w(u_{n+2}, u_m)e_b(u_n, u_{n+1}) + \cdots \\ &\quad + w(u_n, u_m)w(u_{n+2}, u_m) \cdots w(u_{m-2}, u_m)e_b \\ &\quad \cdot (u_n, u_{n+1}) + w(u_n, u_m)w(u_{n+2}, u_m) \cdots w \\ &\quad \cdot (u_{m-2}, u_m)e_b(u_n, u_{n+1}) \\ &\leq w(u_n, u_m)e_b(u_n, u_{n+1}) + w(u_n, u_m)w \\ &\quad \cdot (u_{n+1}, u_m)e_b(u_n, u_{n+1}) + w(u_n, u_m)w \\ &\quad \cdot (u_{n+1}, u_m)w(u_{n+2}, u_m)e_b(u_n, u_{n+1}) \\ &\quad + w(u_n, u_m)w(u_{n+1}, u_m)w(u_{n+2}, u_m)w \\ &\quad \cdot (u_{n+3}, u_m)e_b(u_n, u_{n+1}) + \cdots + w(u_n, u_m)w \\ &\quad \cdot (u_{n+1}, u_m)w(u_{n+2}, u_m) \cdots w(u_{m-2}, u_m)e_b \\ &\quad \cdot (u_n, u_{n+1}) + w(u_n, u_m)w(u_{n+1}, u_m)w \\ &\quad \cdot (u_{n+2}, u_m) \cdots w(u_{m-2}, u_m)w(u_{m-1}, u_m)e_b(u_n, u_{n+1}). \end{aligned} \tag{19}$$

Applying $n, m \longrightarrow \infty$ and using (12), we get

$$\lim_{n,m \longrightarrow \infty} e_b(u_n, u_m) = 0. \tag{20}$$

Hence, $\{u_n\}$ is a Cauchy sequence. Since (Ξ, e_b) is a complete EBbDS, then there exists a point $u^* \in \Xi$ such that $u_n \longrightarrow u^*$ as $n \longrightarrow +\infty$, that is

$$\lim_{n \longrightarrow +\infty} e_b(u_n, u^*) = 0. \tag{21}$$

Next, we prove that $u^* \in Fix(\mathfrak{T})$. Indeed, we write

$$\begin{aligned} e_b(u^*, \mathfrak{T}u^*) &\leq w(u^*, \mathfrak{T}u^*)[e_b(u^*, u_n) \\ &\quad + e_b(u_n, u_{n+1}) + e_b(u_{n+1}, \mathfrak{T}u^*)]. \end{aligned} \tag{22}$$

Since $\mathfrak{T}$ is continuous, on letting $n \longrightarrow +\infty$, we obtain $e_b(u^*, \mathfrak{T}u^*) = 0$, that is, $\mathfrak{T}u^* = u^*$, and hence, u^* is a fixed point of $\mathfrak{T}$.

To prove the uniqueness of fixed-point u^*, we impose an additional requirement.

(A4) For every pair u^* and v^* of fixed points of $\mathfrak{T}$, $\alpha(u^*, v^*) \geq 1$.

Theorem 10. *In addition of condition (A4) in Theorem 9, Fi x($\mathfrak{T}$) is a singleton set.*

Proof. Following Theorem 9, $u^* \in Fix(\mathfrak{T})$. To prove $Fix(\mathfrak{T})$ is a singleton set, assume that there exist $u^*, v^* \in Fix(\mathfrak{T})$ with $u^* \neq v^*$, and by (A4), we have $\alpha(u^*, v^*) \geq 1$. It follows from $\mathfrak{T} \in \Lambda(\Xi, \alpha)$ that

$$\begin{aligned} &w(u^*, v^*)e_b(\mathfrak{T}u^*, \mathfrak{T}v^*) \\ &\leq \lambda(u^*) \max \left\{ \begin{array}{c} e_b(u^*, v^*), e_b(u^*, \mathfrak{T}u^*), e_b(v^*, \mathfrak{T}v^*), \\ \dfrac{e_b(v^*, \mathfrak{T}v^*)[1 + e_b(u^*, \mathfrak{T}v^*)}{w(u^*, v^*)[1 + e_b(u^*, v^*)]}, \dfrac{e_b(u^*, \mathfrak{T}u^*).e_b(v^*, \mathfrak{T}v^*)}{w(u^*, v^*)).e_b(u^*, v^*)} \end{array} \right\} \\ &\leq \lambda(u^*) \max \{ e_b(u^*, v^*), 0, 0, 0, 0 \}, \end{aligned} \tag{23}$$

which implies that

$$w(u^*, v^*)e_b(u^*, v^*) \leq \lambda(u^*)e_b(u^*, v^*), \tag{24}$$

a contradiction, and hence, $u^* = v^*$.

2.2. Illustrations

Example 2. Let $\Xi = \{0.2,0.25,0.3,0.5,1\}$. Define $e_b : \Xi^2 \longrightarrow \mathbb{R}_+$ so that $e_b(\zeta, \xi) = e_b(\xi, \zeta)$ for all $\zeta, \xi \in \Xi$, and

$$\begin{gathered} e_b(0.5,0.3) = 0.07, e_b(0.5,0.25) = 0.015, e_b(0.25,0.2) = 0.02, \\ e_b(0.3,0.25) = 0.02, e_b(0.3,0.2) = 0.02, \end{gathered} \tag{25}$$

$e_b(\zeta, \xi) = (\zeta - \xi)^2$, otherwise. Then (Ξ, e_b) is a EBbDS with $w(\zeta, \xi) = \zeta + \xi + 2$ but neither a BDS (Ξ, b) nor a metric space (Ξ, d). For instance

$$\begin{aligned} &e_b(0.5,0.3) = 0.07 \leq 0.035 = e_b(0.5,0.25) + e_b(0.25,0.3), \\ &e_b(0.5,0.3) = 0.07 \leq 0.055 = e_b(0.5,0.25) + e_b(0.25,0.2) + e_b(0.2,0.3), \end{aligned} \tag{26}$$

but

$$\begin{aligned} e_b(0.5,0.3) = 0.07 \leq 0.154 = w(\zeta, \nu)[&e_b(0.5,0.25) \\ &+ e_b(0.25,0.2) + e_b(0.2,0.3)]. \end{aligned} \tag{27}$$

Consider the self-mapping $\mathfrak{T}$ on Ξ, $\alpha : \Xi^2 \longrightarrow \mathbb{R}_+$ and $\lambda : \Xi \longrightarrow [0, 1)$

$$\mathfrak{T} : \begin{pmatrix} 0.2 & 0.25 & 0.3 & 0.5 & 1 \\ 0.3 & 0.5 & 0.2 & 0.5 & 0.25 \end{pmatrix},$$

$$\alpha(\zeta, \nu) = \begin{cases} 1, & (\zeta, \nu) \in (0.2,1) \cup (0.5,1) \cup (1,0.2) \cup (1,0.5) \\ 0, & \text{otherwise}, \end{cases} \tag{28}$$

and $\lambda(\zeta) = 2\zeta/3$ for all $\zeta \in \Xi$.

It is easy to see that $\mathfrak{T} \in \mathscr{W}\mathscr{A}(\Xi, \alpha)$. We will check that $\mathfrak{T}$ satisfies (8) for $\zeta \neq \xi$ with $\mathfrak{T}(\zeta) \neq \mathfrak{T}(\xi)$ and $\alpha(\zeta, \xi) > 1$. We demonstrate by three nontrivial possible cases. Here, $w(\zeta, \xi) \in [2.4,6]$.

Case 1. $\zeta = 0.5$, $\xi = 1$ (or vice versa if ζ, ξ change places). Then, $e_b(\mathfrak{T}\zeta, \mathfrak{T}\xi) = 0.015$, $w(\zeta, \xi) = 3.5$, $\lambda(\zeta) = 0.333$ and

$$\mathscr{M}_w(\zeta, \xi) = \max \left\{ 0.25,0, 0.5625 \frac{(0.5625)[1+0]}{(3.5)[1+0.25]}, \frac{(0)(0.5625)}{(3.5)(0.25)} \right\} = 0.5625. \tag{29}$$

Therefore, (8) implies that $0.0525 < 0.1873$, and (8) holds true.

Case 2. $\zeta = 0.2$, $\xi = 1$ (or vice versa if ζ, ξ change places). Then, $e_b(\mathfrak{T}\zeta, \mathfrak{T}\xi) = 0.02$, $w(\zeta, \xi) = 3.2$, $\lambda(\zeta) = 0.1333$ and

$$\mathscr{M}_w(\zeta, \xi) = \max \left\{ 0.64,0.02,0.5625, \frac{0.5625[1+0.02]}{3.2[1+0.64]}, \frac{(0.2)(0.5625)}{(3.2)(0.64)} \right\} = 0.64, \tag{30}$$

and it is easily seen that (8) is fulfilled.

Thus, all the conditions are fulfilled, and $\mathfrak{T}$ has a unique fixed point (which is $\zeta^* = 0.5$).

Note that in this example the use of weakly α-admissibility and $\lambda(\zeta)$ was crucial because, e.g., if we take $\zeta = 0.2$, $\xi = 0.5$, we get $e_b(\mathfrak{T}\zeta, \mathfrak{T}\xi) = 0.07$, $w(\zeta, \xi) = 2.7$ and

$$\mathscr{M}_w(\zeta, \xi) = \max \left\{ 0.09,0.02,0, \frac{(0)[1+0.02]}{2.7[1+0.09]}, \frac{(0.02)(0)}{(2.7)(0.09)} \right\} = 0.09, \tag{31}$$

and no contractive condition for any $\lambda(\zeta) < 1$ can be chosen which would holds for these points.

Example 3. Consider $\Xi = [0, 1]$ and define $e_b : \Xi^2 \longrightarrow \mathbb{R}_+$ by $e_b(\zeta, \xi) = |\zeta - \xi|^2$. Then, (Ξ, e_b) is a EBbDS with $w(\zeta, \xi) = \zeta + \xi + 2.5$ but neither a BDS (Ξ, b) not a metric space (Ξ, d). For instance

$$\begin{aligned} e_b(0,1) &= 1 \leq 0.5 = e_b(0,0.5) + e_b(0.5,1), \\ e_b(0,1) &= 1 \leq 0.4902 = e_b(0,0.5) + e_b(0.5,0.99) + e_b(0.99,1), \end{aligned} \tag{32}$$

but

$$\begin{aligned} e_b(\zeta, \xi) &= |\zeta - \xi|^2 = |\zeta - \mu + \mu - \upsilon + \upsilon - \xi|^2 \leq |\zeta - \mu|^2 \\ &\quad + |\mu - \upsilon|^2 + |\upsilon - \xi|^2 + 2|\zeta - \mu||\mu - \upsilon| \\ &\quad + 2|\mu - \upsilon||\upsilon - \xi| + 2|\upsilon - \xi||\zeta - \mu| \\ &\leq \left(\zeta + \xi + \frac{5}{2} \right) \left[|\zeta - \mu|^2 + |\mu - \upsilon|^2 + |\upsilon - \xi|^2 \right] \\ &= w(\zeta, \xi)[e_b(\zeta, \mu) + e_b(\mu, \upsilon) + e_b(\upsilon, \xi)], \end{aligned} \tag{33}$$

for all $\zeta, \xi, \mu, \upsilon \in \Xi$.

Consider the self-mapping $\mathfrak{T}$ on Ξ given by $\mathfrak{T}(\zeta) = \zeta^2/2$. Taking $\alpha : \Xi^2 \longrightarrow \mathbb{R}_+$ and $\lambda : \Xi \longrightarrow [0, 1)$ such that $\lambda(\zeta) = 8.95 + \zeta/10$ for all $\zeta \in \Xi$, and $\alpha(\zeta, \xi) = 1$ for $\zeta, \xi \in \Xi$, it is obvious to see $\mathfrak{T} \in \mathscr{W}\mathscr{A}(\Xi, \alpha)$. Here, $w(\zeta, \xi) \in (2, 4)$.

Then equation (8) for $\zeta \neq \xi$ would be of the form

$$\begin{aligned} &(\zeta + \xi + 2.5)\left|\frac{\zeta^2}{2} - \frac{\xi^2}{2}\right|^2 \\ &\leq \left(\frac{8.95 + \zeta}{10}\right) \max \left\{ \begin{array}{c} |\zeta - \xi|^2, \left|\zeta - \frac{\zeta^2}{2}\right|^2, \left|\xi - \frac{\xi^2}{2}\right|^2, \\ \frac{\left|\xi - \xi^2/2\right|^2 \left[1 + \left|\zeta - \zeta^2/2\right|^2\right]}{(\zeta + \xi + 2.5)\left[1 + |\zeta - \xi|^2\right]}, \frac{\left|\zeta - \zeta^2/2\right|^2 \cdot \left|\xi - \xi^2/2\right|^2}{(\zeta + \xi + 2.5) \cdot |\zeta - \xi|^2} \end{array} \right\} \end{aligned} \tag{34}$$

holds whenever $e_b(\mathfrak{T}\zeta, \mathfrak{T}\xi) > 0$ and $\alpha(\zeta, \xi) \geq 1$.

For example, we demonstrate (34) is true for two cases:

Case 1. $\zeta = 0$, $\xi = 1$ (or vice versa if ζ, ξ change places). Then, (34) will be

$$\begin{aligned} (3.5)(0.25) = 0.875 &\leq \left(\frac{8.95}{10}\right) \max \left\{ 1, 0, \frac{1}{4}, \frac{1/4(1+0)}{(3.5)(1+1)}, 0 \right\} \\ &= 0.895, \end{aligned} \tag{35}$$

which is true.

Case 2. $\zeta = 1$, $\xi = 0.9$ (or vice versa if ζ, ξ change places). Then, (34) will be

$$\begin{aligned} (4.4)(0.19)^2 = 0.15884 &\leq \left(\frac{9.95}{10}\right) \max \left\{ \begin{array}{c} 0.01,0.444,0.3969, \\ \frac{(0.3969)[1+0.444]}{(4.4)(1+0.01)}, \frac{(0.444)(0.3969)}{(4.4)(0.01)} \end{array} \right\} \\ &= 0.44178, \end{aligned} \tag{36}$$

which holds true.

Similarly, it can be verified for any $\zeta \neq \xi \in \Xi$ with $\alpha(\zeta, \xi) \geq 1$. Thus, all the conditions are fulfilled, and the $Fix(\mathfrak{T}) = \{0\}$ is a singleton set.

2.3. Weak Well-Posedness, Weak Limit Shadowing, and Generalized w-Ulam-Hyers Stability. The notion of well-posedness of an fpp has evoked much interest of several mathematicians, for example, Popa [14, 15] and others. In the paper [16], the authors defined a weak well-posed (wwp) property in BbDS. In what follows, we extend this notion to EBbDS.

Definition 11. Let (Ξ, e_b) be a complete EBbDS and $\mathfrak{T} : \Xi \longrightarrow \Xi$ be a mapping. The fpp of $\mathfrak{T}$ is said to be weak well-posed if it satisfies the following:

(1) $u^* \in Fix(\mathfrak{T})$ is a singleton set in Ξ

(2) For any sequence $\{u_p\}$ in Ξ with $\lim_{p\longrightarrow\infty} e_b(u_p, \mathfrak{T}(u_p)) = 0$ and

$$\lim_{p,q\longrightarrow\infty} e_b(\mathfrak{T}(u_p), \mathfrak{T}(u_q)) = 0, \text{ one has } \lim_{p\longrightarrow\infty} e_b(u_p, u^*) = 0. \tag{37}$$

Theorem 12. *Let (Ξ, e_b) be a complete EBbDS and $\mathfrak{T} : \Xi \longrightarrow \Xi$ be a mapping satisfying all the conditions of Theorem 9 and a sequence $\{u_n\}$ in Ξ such that $\lim_{n\longrightarrow\infty} e_b(u_n, \mathfrak{T}u_n) = 0$, $\lim_{n,m\longrightarrow\infty} e_b(\mathfrak{T}u_n, \mathfrak{T}u_m) = 0$, and $u^* \in Fix(\mathfrak{T})$. Then, the fpp of $\mathfrak{T}$ is wwp.*

Proof. Let $\{u_n\}$ be a sequence in Ξ such that $\lim_{n\longrightarrow\infty} e_b(u_n, \mathfrak{T}(u_n)) = 0$ and $\lim_{n,m\longrightarrow\infty} e_b(\mathfrak{T}u_n, \mathfrak{T}u_m) = 0$, for $m > n$; we obtain from (ebb3) that

$$e_b(u_n, u^*) \le w(u_n, u^*)\{e_b(u_n, \mathfrak{T}u_m) + e_b(\mathfrak{T}u_m, \mathfrak{T}u_n) + e_b(\mathfrak{T}u_n, u^*)\}. \tag{38}$$

Taking limit $n \longrightarrow \infty$

$$\lim_{n\longrightarrow\infty} e_b(u_n, u^*) \le \lim_{n\longrightarrow\infty} w(u_n, u^*)\{e_b(u_n, \mathfrak{T}u_m) + e_b(\mathfrak{T}u_n, u^*)\}. \tag{39}$$

WLOG, we can assume that there exists a distinct subsequence $\{\mathfrak{T}u_{n_k}\}$ of $\{\mathfrak{T}u_n\}$. Otherwise, there exists $u_0 \in \Xi$ and $n_1 \in \mathbb{N}$ such that $\mathfrak{T}u_n = u_0$ for $n \ge n_1$. Since $\lim_{n\longrightarrow\infty} e_b(u_n, \mathfrak{T}u_n) = 0$, we get $\lim_{n\longrightarrow\infty} e_b(u_n, u_0) = 0$. If $u_0 \ne u^*$, then $u_0 \ne \mathfrak{T}u_0$ due to uniqueness of the fixed point of $\mathfrak{T}$. For $n \ge n_1$, we obtain $u_0 = \mathfrak{T}u_n \ne \mathfrak{T}u_0$. So, we have

$$e_b(u_0, \mathfrak{T}u_0) = e_b(\mathfrak{T}u_n, \mathfrak{T}u_0) \le w(u_n, u_0) e_b(\mathfrak{T}u_n, \mathfrak{T}u_0). \tag{40}$$

For $\alpha(u_0, \mathfrak{T}u_0) \ge 1$ and $\mathfrak{T} \in \Lambda(\Xi, \alpha)$, we have

$$\begin{aligned} w(u_n, u_0) e_b(\mathfrak{T}u_n, \mathfrak{T}u_0) &\le \lambda(u_n) \max \left\{ \begin{array}{c} e_b(u_n, u_0), e_b(u_n, \mathfrak{T}u_n), e_b(u_0, \mathfrak{T}u_0), \\ \dfrac{e_b(u_0, \mathfrak{T}u_0)[1 + e_b(u_n, \mathfrak{T}u_n)]}{w(u_n, u_0))[1 + e_b(u_n, u_0)]}, \dfrac{e_b(u_n, \mathfrak{T}u_n).e_b(u_0, \mathfrak{T}u_0)}{w(u_n, u_0).e_b(u_n, u_0)} \end{array} \right\} \\ &\le \lambda(u_n) \max \{ e_b(u_n, u_0), e_b(u_n, u_0), e_b(u_0, \mathfrak{T}u_0), 0, e_b(u_0, \mathfrak{T}u_0) \} = \lambda(u_n) \max \{ e_b(u_n, u_0), e_b(u_0, \mathfrak{T}u_0) \}. \end{aligned} \tag{41}$$

Therefore, since $\lim_{n\longrightarrow\infty} e_b(u_n, u_0)) = 0$, we get

$$\lim_{n\longrightarrow\infty} w(u_n, u_0) e_b(u_0, \mathfrak{T}u_0) \le \lim_{n\longrightarrow\infty} \lambda(u_n) e_b(u_0, \mathfrak{T}u_0). \tag{42}$$

So $e_b(u_0, \mathfrak{T}u_0) = 0$, i.e., $u_0 = \mathfrak{T}u_0$, a contradiction. Hence, there exist $m, q, n > n_0 (m > q > n)$ such that $\mathfrak{T}u_m \ne \mathfrak{T}u_q \ne \mathfrak{T}u_n \ne u_n$. Then

$$e_b(u_n, \mathfrak{T}u_m) \le w(u_n, \mathfrak{T}u_m)\{e_b(u_n, \mathfrak{T}u_n) + e_b(\mathfrak{T}u_n, \mathfrak{T}u_q) + e_b(\mathfrak{T}u_q, \mathfrak{T}u_m)\}, \tag{43}$$

which $\longrightarrow 0$ as $n \longrightarrow \infty$. On replacing the value in (39), we get

$$\lim_{n\longrightarrow\infty} e_b(u_n, u^*) \le \lim_{n\longrightarrow\infty} w(u_n, u^*) e_b(\mathfrak{T}u_n, u^*). \tag{44}$$

Again, since $\alpha(u_n, u^*) \ge 1$ and $\mathfrak{T} \in \Lambda(\Xi, \alpha)$, we have

$$\begin{aligned} w(u_n, u^*) e_b(\mathfrak{T}u_n, \mathfrak{T}u^*) &\le \lambda(u_n) \max \left\{ \begin{array}{c} e_b(u_n, u^*), e_b(u_n, \mathfrak{T}u_n), e_b(u^*, \mathfrak{T}u^*), \\ \dfrac{e_b(u_n, \mathfrak{T}u_n)[1 + e_b(u^*, \mathfrak{T}u^*)]}{w(u_n, u^*)[1 + e_b(u_n, u^*)]}, \dfrac{e_b(u^*, \mathfrak{T}u^*).e_b(u_n, \mathfrak{T}u_n)}{w(u_n, u^*)).e_b(u_n, u^*)} \end{array} \right\} \\ &\le \lambda(u_n) \max \left\{ e_b(u_n, u^*), e_b(u_n, \mathfrak{T}u_n), 0, \frac{e_b(u_n, \mathfrak{T}u_n)}{w(u_n, u^*)[1 + e_b(u_n, u^*)]}, 0 \right\}, \end{aligned} \tag{45}$$

which implies

$$\lim_{n\longrightarrow\infty} w(u_n, u^*)e_b(\mathfrak{I}u_n, \mathfrak{I}u^*) \le \lim_{n\longrightarrow\infty} \lambda(u_n)e_b(u_n, u^*). \quad (46)$$

On placing in (39), we get

$$\begin{aligned}\lim_{n\longrightarrow\infty} e_b(u_n, u^*) &\le \lim_{n\longrightarrow\infty} w(u_n, u^*)e_b(\mathfrak{I}u_n, \mathfrak{I}u^*)\\ &\le \lim_{n\longrightarrow\infty} \lambda(u_n)e_b(u_n, u^*).\end{aligned} \quad (47)$$

Therefore, $\lim_{n\longrightarrow\infty} e_b(u_n, u^*) = 0$.

The limit shadowing property of fpps has been discussed in the papers [17, 18]. We define weak limit shadowing property (wlsp) in EBbDS.

Definition 13. Let (Ξ, e_b) be a complete EBbDS and $\mathfrak{I} : \Xi \longrightarrow \Xi$ be a mapping. The fpp of $\mathfrak{I}$ is said to have wlsp in Ξ if assuming that $\{u_n\}$ in Ξ satisfies $e_b(u_n, \mathfrak{I}u_n) \longrightarrow 0$ as $n \longrightarrow \infty$ and $e_b(\mathfrak{I}u_n, \mathfrak{I}u_m) \longrightarrow 0$, it follows that there exists $u \in \Xi$ such that $e_b(u_n, \mathfrak{I}^n u) \longrightarrow 0$ as $n \longrightarrow \infty$.

Theorem 14. *Let (Ξ, e_b) be a complete EBbDS and $\mathfrak{I} : \Xi \longrightarrow \Xi$ be an $\alpha-\lambda$-contractive mapping for $\alpha : \Xi^2 \longrightarrow \mathbb{R}_+$ and $\lambda : \Xi \longrightarrow [0, 1)$ with $\{u_n\}$ in Ξ such that $\lim_{n\longrightarrow\infty} e_b(u_n, \mathfrak{I}u_n) = 0$, $\lim_{n,m\longrightarrow\infty} e_b(\mathfrak{I}u_n, \mathfrak{I}u_m) = 0$ and $u^* \in Fix(\mathfrak{I})$. Then, $\mathfrak{I}$ has the wlsp.*

Proof. Since u^* is a fixed point of $\mathfrak{I}$, we have $e_b(u^*, \mathfrak{I}u^*) = 0$, and let $\{u_n\}$ in Ξ such that $\lim_{n\longrightarrow\infty} e_b(u_n, \mathfrak{I}u_n) = 0$, $\lim_{n,m\longrightarrow\infty} e_b(\mathfrak{I}u_n, \mathfrak{I}u_m) = 0$; then, by virtue of Theorem 12, we have $\lim_{n\longrightarrow\infty} e_b(u_n, u^*) = 0$, and therefore, we can write $\lim_{n\longrightarrow\infty} e_b(u_n, \mathfrak{I}^n u^*) = 0$.

In the following, we define the generalized w-Ulam-Hyers stability (Gw-UHS) of fixed-point problem (fpp) in EBbDS as an extension of b-metric space case discussed in [19, 20] (see also [21]).

Definition 15. Let (Ξ, e_b) be a complete EBbDS and $\mathfrak{I} : \Xi \longrightarrow \Xi$ be a mapping. The fixed-point equation (FPE)

$$u = \mathfrak{I}u, u \in \Xi \quad (48)$$

is called the generalized weak-Ulam-Hyers stable (Gw-UHS in short) in the setting of EBbDS if there exists an increasing function $\phi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, continuous at 0, with $\phi(0) = 0$, such that for each $\varepsilon > 0$ and an ε-solution $\upsilon \in \Xi$, that is

$$e_b(\upsilon, \mathfrak{I}\upsilon) \le \varepsilon, \quad (49)$$

there exists a solution $u^* \in \Xi$ of (48) such that

$$e_b(\upsilon, u^*) \le \phi(w(u^*, \upsilon)\varepsilon). \quad (50)$$

If $\phi(\xi) = \alpha\xi$ for all $\xi \in \mathbb{R}_+$, where $\alpha > 0$, then FPE (48) is said to be w-UHS in the setting of EBbDS.

Theorem 16. *Let (Ξ, e_b) be a complete EBbDS and $\mathfrak{I} : \Xi \longrightarrow \Xi$ be an $\alpha-\lambda$-contractive mapping for $\alpha : \Xi^2 \longrightarrow \mathbb{R}_+$ and $\lambda : \Xi \longrightarrow [0, 1)$ and also that the function $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is strictly increasing and onto. Then the FPE (48) is Gw-UHS.*

Proof. Following Theorem 14, we have $\mathfrak{I}u^* = u^*$, that is, $u^* \in \Xi$ is a solution of the FPE (48) with $e_b(u^*, u^*) = 0$. Let $\varepsilon > 0$ and $\upsilon^* \in \Xi$ be an ε-solution of FPE (48), that is

$$e_{4b}(\upsilon^*, \mathfrak{I}\upsilon^*) \le \varepsilon. \quad (51)$$

Since $e_b(u^*, \mathfrak{I}u^*) = e_b(u^*, u^*) = 0 \le \varepsilon$, u^* and υ^* are ε-solutions. Since we have $\alpha(u^*, \upsilon^*) \ge 1$, so

$$\begin{aligned}e_b(u^*, \upsilon^*) &\le w(u^*, \upsilon^*)[e_b(u^*, \mathfrak{I}u^*) + e_b(\mathfrak{I}u^*, \mathfrak{I}\upsilon^*) + e_b(\mathfrak{I}\upsilon^*, \upsilon^*)]\\ &\le w(u^*, \upsilon^*)e_b(\mathfrak{I}u^*, \mathfrak{I}\upsilon^*) + \varepsilon w(u^*, \upsilon^*)\\ &\le \lambda(u^*)\max\left\{\begin{array}{c} e_b(u^*, \upsilon^*), e_b(u^*, \mathfrak{I}u^*), e_b(\upsilon^*, \mathfrak{I}\upsilon^*),\\ \dfrac{e_b(\upsilon^*, \mathfrak{I}\upsilon^*)[1 + e_b(u^*, \mathfrak{I}u^*)]}{w(u^*, \upsilon^*)[1 + e_b(u^*, \upsilon^*)]}, \dfrac{e_b(u^*, \mathfrak{I}u^*).e_b(\upsilon^*, \mathfrak{I}\upsilon^*)}{w(u^*, \upsilon^*)).e_b(u^*, \upsilon^*)}\end{array}\right\} + \varepsilon w(u^*, \upsilon^*)\\ &\le \lambda(u^*)\ \max\{e_b(u^*, \upsilon^*), 0, \varepsilon, 0, 0\} + \varepsilon w(u^*, \upsilon^*).\end{aligned} \quad (52)$$

Let us discuss the two possible cases.

Case 1. If $e_b(u^*, \upsilon^*) > \varepsilon$, then we get

$$e_b(u^*, \upsilon^*) \le \lambda(u^*)e_b(u^*, \upsilon^*) + w(u^*, \upsilon^*)\varepsilon, \quad (53)$$

that is

$$e_b(u^*, \upsilon^*)[1 - \lambda(u^*)] \le w(u^*, \upsilon^*)\varepsilon, \quad (54)$$

which implies that

$$e_b(u^*, v^*) \le \frac{1}{1-\lambda(u^*)} w(u^*, v^*)\varepsilon = \phi(w(u^*, v^*)\varepsilon). \quad (55)$$

Case 2. If $e_b(u^*, v^*) < \varepsilon$, then (12) gives

$$\begin{aligned} e_b(u^*, v^*) &\le \lambda(u^*)\varepsilon + w(u^*, v^*)\varepsilon \\ &\le \lambda(u^*)w(u^*, v^*)\varepsilon + w(u^*, v^*)\varepsilon \\ &= (\lambda(u^*) + 1)w(u^*, v^*)\varepsilon = \phi(w(u^*, v^*)\varepsilon). \end{aligned} \quad (56)$$

It shows that the inequality (50) is true for all cases, and thus the FPE (48) is Gw-UHS.

3. Application

In this section, we discuss the existence of solutions of a nonlinear fractional differential equation (FDE) [22] as an application of Theorem 9. Some other FDE-related work can be seen in [23–25].

The Caputo fractional derivative of order β is defined as

$$\begin{aligned} {}^c\mathscr{D}^\beta(p(\rho)) &= \frac{1}{\Gamma(n-\beta)} \int_0^\rho (\rho-\sigma)^{n-\beta-1} p^{(n)}(\sigma) ds \\ &\cdot (n-1 < \beta < n, n = [\beta]+1), \end{aligned} \quad (57)$$

where $p : [0,\infty) \longrightarrow \mathbb{R}$ is a continuous function, $[\beta]$ denotes the integer part of the positive real number β, and Γ is the gamma function.

Consider the nonlinear FDE

$${}^c\mathscr{D}^\beta(\vartheta(\rho)) = \hbar(\rho, \vartheta(\rho))(0 < \rho < 1, 1 < \beta \le 2), \quad (58)$$

with the integral boundary conditions

$$\vartheta(0) = 0, \vartheta(1) = \int_0^\eta \vartheta(\sigma)\, ds\, (0 < \eta < 1), \quad (59)$$

where $J = [0, 1]$, $\vartheta \in C(J, \mathbb{R})$, and $\hbar : J \times \mathbb{R} \longrightarrow \mathbb{R}$ are a continuous function.

Let $\Xi = C(J, \mathbb{R})$ be endowed with the EBbDS function

$$e_b(\vartheta, \nu) = \max_{\rho\in J} |\vartheta(\rho) - \nu(\rho)|^2, \quad (60)$$

and $w(\vartheta, \nu) = |\vartheta(\rho)| + |\nu(\rho)| + 2$.

Theorem 17. *Let $\mathfrak{T} : \Xi \longrightarrow \Xi$ be the operator defined by*

$$\begin{aligned} \mathfrak{T}\vartheta(\rho) &= \frac{1}{\Gamma(\beta)} \int_0^\rho (\rho-\sigma)^{\beta-1} \hbar(\sigma, \vartheta(\sigma)) d\sigma \\ &\quad - \frac{2\rho}{(2-\eta^2)\Gamma(\beta)} \int_0^1 (1-\sigma)^{\beta-1} \hbar(\sigma, \vartheta(\sigma)) d\sigma \\ &\quad + \frac{2\rho}{(2-\eta^2)\Gamma(\beta)} \int_0^\eta \left(\int_0^\sigma (\sigma-\varsigma)^{\beta-1} \hbar(z, \vartheta(z))\, d\varsigma \right) d\sigma, \end{aligned} \quad (61)$$

for $\vartheta \in \Xi$, $\rho \in J$. Also, let $\zeta : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ be a given function. Assume the following:

(F1) $\hbar : J \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function, nondecreasing in the second variable

(F2) There exists $\vartheta_0 \in \Xi$ such that $\zeta(\vartheta_0(\rho), \mathfrak{T}\vartheta_0(\rho)) \ge 0$ for all $\rho \in J$

(F3) $(\vartheta, \nu) \in \Xi^2$ and $\zeta(\vartheta(\rho), \mathfrak{T}\vartheta(\rho)) \ge 0$ for all $\rho \in J$ imply that $\zeta(\mathfrak{T}\vartheta(\rho), \mathfrak{T}\mathfrak{T}\vartheta(\rho)) \ge 0$ for all $\rho \in J$

(F4) There exists $\lambda : \Xi \longrightarrow [0, 1)$ such that for $\vartheta, \nu \in \Xi$ with $\zeta(\vartheta, \nu) \ge 0$, and $\rho \in J$, we have

$$|\hbar(\rho, \vartheta(\rho)) - \hbar(\rho, \nu(\rho))|^2 \le \frac{\lambda(\vartheta(\rho))\Theta(\vartheta, \nu)(\rho)}{\theta \times \max_{\rho\in J}(|\vartheta(\rho)| + |\nu(\rho)| + 2)}, \quad (62)$$

where

$$\Theta(\vartheta, \nu)(\rho) = \max \left\{ \begin{array}{c} |\vartheta(\rho) - \nu(\rho)|^2, |\vartheta(\rho) - \mathfrak{T}\vartheta(\rho)|^2, |\nu(\rho) - \mathfrak{T}\nu(\rho)|^2, \\ \dfrac{\left(|\nu(\rho) - \mathfrak{T}\nu(\rho)|^2\right)\left(1 + |\vartheta(\rho) - \mathfrak{T}\vartheta(\rho)|^2\right)}{\max_{\rho\in J}(|\vartheta(\rho)| + |\nu(\rho)| + 2)\left(1 + \max_{\rho\in J}|\vartheta(\rho) - \nu(\rho)|^2\right)}, \\ \dfrac{\left(|\vartheta(\rho) - \mathfrak{T}\vartheta(\rho)|^2\right)\left(|\nu(\rho) - \mathfrak{T}\nu(\rho)|^2\right)}{\max_{\rho\in J}(|\vartheta(\rho)| + |\nu(\rho)| + 2)\left(\max_{\rho\in J}|\vartheta(\rho) - \nu(\rho)|^2\right)}, \end{array} \right\} \quad (63)$$

and $\theta = (2\beta - 1)\Gamma(\beta)\Gamma(\beta + 1)/2(5\beta + 2)$. Then, the problems (58) and (59) have at least one solution $\vartheta^* \in \Xi$.

Proof. Define a function $\alpha : \Xi^2 \longrightarrow [0,\infty)$ by

$$\alpha(\vartheta, \nu) = \begin{cases} 1, & \text{if for}\zeta(\vartheta(\rho), \nu(\rho)) \ge 0, \text{for all}\rho \in J \\ \gamma, & \text{otherwise,} \end{cases} \quad (64)$$

where $\gamma \in (0, 1)$. It is obvious to check that the assumption (F2) implies the condition (A2) of Theorem 9. Assumption (F3) clearly implies that $\mathfrak{T} \in \mathscr{W}\mathscr{A}(\Xi, \alpha)$.

Let $\vartheta, \nu \in \Xi$ be $\alpha(\vartheta, \nu) \ge 1$, i.e., $\zeta(\vartheta(\rho), \nu(\rho)) \ge 0$ for all $\rho \in J$. For each $\rho \in J$, by the definition (61) of operator $\mathfrak{T}$, we have (using Cauchy-Schwartz inequality)

$$
\begin{aligned}
&|\mathfrak{I}\vartheta(\rho)-\mathfrak{I}\nu(\rho)|^2\\
&=\left|\frac{1}{\Gamma(\beta)}\int_0^\rho(\rho-\sigma)^{\beta-1}\hbar(\sigma,\vartheta(\sigma))\,d\sigma\right.\\
&\quad-\frac{2\rho}{(2-\eta^2)\Gamma(\beta)}\int_0^1(1-\sigma)^{\beta-1}\hbar(\sigma,\vartheta(\sigma))d\sigma\\
&\quad+\frac{2\rho}{(2-\eta^2)\Gamma(\beta)}\int_0^\eta\left(\int_0^\sigma(\sigma-\varsigma)^{\beta-1}\hbar(\varsigma,\vartheta(z))\,d\varsigma\right)d\sigma\\
&\quad-\frac{1}{\Gamma(\beta)}\int_0^\rho(\rho-\sigma)^{\beta-1}\hbar(\sigma,\nu(\sigma))\,d\sigma\\
&\quad-\frac{2\rho}{(2-\eta^2)\Gamma(\beta)}\int_0^1(1-\sigma)^{\beta-1}\hbar(\sigma,\nu(\sigma))\,d\sigma\\
&\quad\left.+\frac{2\rho}{(2-\eta^2)\Gamma(\beta)}\int_0^\eta\left(\int_0^\sigma(\sigma-\varsigma)^{\beta-1}\hbar(\varsigma,\nu(\varsigma))\,d\varsigma\right)d\sigma\right|^2\\
&\le\frac{2}{\Gamma^2(\beta)}\left\{\int_0^\rho(\rho-\sigma)^{\beta-1}|\hbar(\sigma,\vartheta(\sigma))-\hbar(\sigma,\nu(\sigma))|^2\,d\sigma\right\}\\
&\quad+\frac{8\rho^2}{(2-\eta^2)^2\Gamma^2(\beta)}\left\{\int_0^1(1-\sigma)^{\beta-1}|\hbar(s,\vartheta(\sigma))\right.\\
&\quad\left.-\hbar(\sigma,\nu(\sigma))|^2\,d\sigma\right\}+\frac{8\rho^2}{(2-\eta^2)^2\Gamma^2(\beta)}\\
&\quad\cdot\left\{\int_0^\eta\left(\int_0^\sigma(\sigma-\varsigma)^{\beta-1}|\hbar(\varsigma,\vartheta(\varsigma))-\hbar(\varsigma,\nu(\varsigma))|^2\,d\varsigma\right)d\sigma\right\},
\end{aligned}
\tag{65}
$$

that is

$$
\begin{aligned}
|\mathfrak{I}\vartheta(\rho)-\mathfrak{I}\nu(\rho)|^2\le{}&\frac{2}{\Gamma^2(\beta)}\int_0^\rho(\rho-\sigma)^{2\beta-2}d\sigma\\
&\cdot\int_0^\rho|\hbar(\sigma,\vartheta(\sigma))-\hbar(\sigma,\nu(\sigma))|^2\,d\sigma\\
&+\frac{8\rho^2}{(2-\eta^2)^2\Gamma^2(\beta)}\int_0^1(1-\sigma)^{2\beta-2}d\sigma\\
&\cdot\int_0^1|\hbar(\sigma,\vartheta(\sigma))-\hbar(\sigma,\nu(\sigma))|^2\,d\sigma\\
&+\frac{8\rho^2}{(2-\eta^2)^2\Gamma^2(\beta)}\int_0^\eta\int_0^\sigma(\sigma-\varsigma)^{2\beta-2}d\varsigma\,d\sigma\\
&\times\int_0^\eta\int_0^\sigma|\hbar(\varsigma,\vartheta(\varsigma))-\hbar(\varsigma,\nu(\varsigma))|^2\,d\varsigma\,d\sigma.
\end{aligned}
\tag{66}
$$

Applying (F4) and small calculations, we get

$$
|\mathfrak{I}\vartheta(\rho)-\mathfrak{I}\nu(\rho)|^2\le\frac{\lambda(\vartheta(\rho))\Theta_{\mathfrak{I}}(\vartheta,\nu)(\rho)}{(|\vartheta(\rho)|+|\nu(\rho)|+2)}.\tag{67}
$$

This implies that

$$
\begin{aligned}
w(\vartheta,\nu)e_b(\mathfrak{I}\vartheta,\mathfrak{I}\nu)&=w(\vartheta,\nu)\max_{t\in I}\left(|(\mathfrak{I}\vartheta)(\rho)-(\mathfrak{I}\nu)(\rho)|^2\right)\\
&\le\lambda(\vartheta)\Theta_{\mathfrak{I}}(\vartheta,\nu),
\end{aligned}
\tag{68}
$$

for all $\vartheta,\nu\in\Xi$ with $e_b(\mathfrak{I}\vartheta,\mathfrak{I}\nu)>0$ where

$$
\Theta_{\mathfrak{I}}(\vartheta,\nu)=\max\left\{\begin{array}{c}e_b(\vartheta,\nu),e_b(\vartheta,\mathfrak{I}\vartheta),e_b(\nu,\mathfrak{I}\nu),\\ \dfrac{e_b(\nu,\mathfrak{I}\nu)[1+e_b(\vartheta,\mathfrak{I}\vartheta)]}{w(\vartheta,\nu)[1+e_b(\vartheta,\nu)]},\dfrac{e_b(\vartheta,\mathfrak{I}\vartheta).e_b(\nu,\mathfrak{I}\nu)}{w(\vartheta,\nu).e_b(\vartheta,\nu)}\end{array}\right\}.\tag{69}
$$

Thus, $\mathfrak{I}\in\Lambda(\Xi,\alpha)$. Therefore, all the requirements of Theorem 9 are fulfilled, and we conclude that there is a fixed-point $\vartheta^*\in\Xi$ of the operator $\mathfrak{I}$. It is well known (see, e.g., [22], Theorem 17) that in this case ϑ^* is also a solution of the integral equation (61) and the FDE (58) with the condition (59).

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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