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On multivalued weakly Picard operators in partial Hausdorff metric spaces

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Abstract
We discuss multivalued weakly Picard operators on partial Hausdorff metric spaces. First, we obtain Kikkawa-Suzuki type fixed point theorems for a new type of generalized contractive conditions. Then, we prove data dependence of a fixed points set theorem. Finally, we present sufficient conditions for well-posedness of a fixed point problem. Our results generalize, complement and extend classical theorems in metric and partial metric spaces.

MSC: 47H10; 54H25

Keywords: data dependence; fixed point; multivalued operator; partial metric space

1 Introduction and preliminaries

In 1937, von Neumann [1] initiated the fixed point theory for multivalued mappings in the study of game theory. Indeed, the fixed point theorems for multivalued mappings are quite useful in control theory and have been frequently used in solving many problems of economics. In 1969, Nadler [2] initiated the development of the metric fixed point theory for multivalued mappings. Nadler used the concept of Hausdorff metric to establish the multivalued contraction principle containing the Banach contraction principle as a special case. Also, for the basic problems of fixed point theory for multivalued mappings, we refer to [3].

Let (X, d) be a metric space and let $\mathrm{CB}(X)$ be the family of all nonempty, closed and bounded subsets of X. For $A, B \in \mathrm{CB}(X)$, $x \in X$, let

$$H(A,B) = \max\Big\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\Big\},$$

where $d(x,B) = \inf\{d(x,y) : y \in B\}$ and $H : \mathrm{CB}(X) \times \mathrm{CB}(X) \to \mathbb{R}^+$ is the Hausdorff metric induced by d.

Now on, the letters $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{N}$ will denote the set of all real numbers, the set of all non-negative real numbers and the set of all positive integers, respectively. Also, $\mathrm{CL}(X)$ is the collection of nonempty closed subsets of X.

Definition 1.1 Let X be a nonempty set. If $T : X \to \mathrm{CB}(X)$ is a multivalued operator, then an element $x \in X$ is called

(i) fixed point of T if $x \in Tx$;
(ii) strict fixed point of T if $\{x\} = Tx$.

In the sequel, we denote by $\mathrm{Fix}(T) := \{x \in X : x \in Tx\}$ the set of all fixed points of T and by $S\,\mathrm{Fix}(T) := \{x \in X : \{x\} = Tx\}$ the set of all strict fixed points of T.

Definition 1.2 ([4]) Let (X,d) be a metric space and $T : X \to \mathrm{CL}(X)$ be a multivalued operator. T is called a multivalued weakly Picard operator (briefly MWP operator) if for all $x \in X$ and all $y \in Tx$, there exists a sequence $\{x_n\}$ such that:

(i) $x_0 = x$, $x_1 = y$;
(ii) $x_{n+1} \in Tx_n$ for all $n \in \mathbb{N} \cup \{0\}$;
(iii) the sequence $\{x_n\}$ is convergent and its limit is a fixed point of T.

A sequence $\{x_n\}$ satisfying (i) and (ii) is also called a sequence of successive approximations (briefly s.s.a.) of T starting from x_0.

For interested readers, the theory of MWP operators was presented in [4–7].

In 2008 Suzuki [8] introduced a new type of mappings in order to generalize the well-known Banach contraction principle. This result has led to some important contributions in metric fixed point theory (see, for instance, [9] and the references therein).

As we mentioned above, Nadler proved the following multivalued version of the Banach contraction principle.

Theorem 1.3 ([2]) *Let (X,d) be a complete metric space and $T : X \to \mathrm{CB}(X)$ be a multivalued mapping satisfying $H(Tx,Ty) \le kd(x,y)$ for all $x,y \in X$ and $k \in (0,1)$. Then T has a fixed point.*

In the last decades, a number of fixed point results (see [10–17]) have been obtained in attempts to generalize Theorem 1.3.

One of the most significant fixed point theorems for multivalued mappings appeared in [9], Theorem 2.1. This theorem merges the ideas of Suzuki [8] and Nadler [2] into a consistent framework.

Theorem 1.4 ([9]) *Let (X,d) be a complete metric space and $T : X \to \mathrm{CB}(X)$. Consider the non-increasing function $\psi : [0,1) \to (0,1]$ defined by*

$$\psi(r) = \begin{cases} 1 & \text{if } 0 \le r < \frac{1}{2}, \\ 1 - r & \text{if } \frac{1}{2} \le r < 1. \end{cases}$$

Assume that there exists $r \in [0,1)$ such that

$$\psi(r)d(x,Tx) \le d(x,y) \quad \Longrightarrow \quad H(Tx,Ty) \le rM_d(x,y)$$

for all $x,y \in X$, where

$$M_d(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty) + d(y,Tx)}{2}\right\}.$$

Then T has a fixed point.

We remark that the right-hand side of the above implication is known as Ćirić type contractive condition, see [10, 11, 18]. Also, for our further use, we recall the following refinement of Nadler's theorem, see [14].

Theorem 1.5 *Let $\eta : [0,1) \to (\frac{1}{2},1]$ be a function defined by $\eta(r) = \frac{1}{1+r}$. Let (X,d) be a complete metric space and $T : X \to \mathrm{CB}(X)$ be an r-KS multivalued operator, that is, there exists $r \in [0,1)$ such that*

$$\eta(r)d(x,Tx) \le d(x,y) \implies H(Tx,Ty) \le rd(x,y)$$

for all $x,y \in X$. Then T is an MWP operator.

The other basic notion for the development of our work is the concept of partial metric space, which was introduced by Matthews [19] as a part of the study of denotational semantics of dataflow networks. Matthews presented a modified version of the Banach contraction principle, more suitable in this context, see also [20, 21]. For more reading on interesting approaches to partial metric spaces and related contexts, we refer to [22–24]. Now, the (complete) partial metric spaces constitute a suitable framework to model several distinguished examples of the theory of computation and also to model metric spaces via domain theory, see [18, 19, 25–37]. In this direction, Aydi *et al.* [38] introduced the concept of partial Hausdorff metric and extended Nadler's fixed point theorem in the setting of partial metric spaces.

Consistent with [19, 38–40], the following definitions and results will be needed in the sequel.

Definition 1.6 ([19]) Let X be any nonempty set. A function $p : X \times X \to \mathbb{R}^+$ is said to be a partial metric if and only if for all $x,y,z \in X$ the following conditions are satisfied:

(P1) $p(x,x) = p(y,y) = p(x,y)$ if and only if $x = y$;
(P2) $p(x,x) \le p(x,y)$;
(P3) $p(x,y) = p(y,x)$;
(P4) $p(x,z) \le p(x,y) + p(y,z) - p(y,y)$.

The pair (X,p) is called a partial metric space. If $p(x,y) = 0$, then (P1) and (P2) imply that $x = y$, but the converse does not hold in general. A trivial example of a partial metric space is the pair $(\mathbb{R}^+,p)$, where $p : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is given by $p(x,y) = \max\{x,y\}$, see also [41].

Example 1.7 ([19]) Let $X = \{[a,b] : a,b \in \mathbb{R}, a \le b\}$. It is easy to show that the function $p : X \times X \to \mathbb{R}^+$ given by $p([a,b],[c,d]) = \max\{b,d\} - \min\{a,c\}$ defines a partial metric on X.

For further examples, we refer to [34, 35, 39, 42, 43]. Note that each partial metric p on X generates a T_0 topology τ_p on X which has as a base the family of the open balls (p-balls) $\{B_p(x,\epsilon) : x \in X, \epsilon > 0\}$, where for all $x \in X$ and $\epsilon > 0$,

$$B_p(x,\epsilon) = \{y \in X : p(x,y) < p(x,x) + \epsilon\}.$$

A sequence $\{x_n\}$ in a partial metric space (X,p) is called convergent to a point $x \in X$, with respect to τ_p, if and only if $p(x,x) = \lim_{n\to+\infty} p(x,x_n)$, see [19] for details. If p is a partial metric on X, then the function

$$p^S(x,y) = 2p(x,y) - p(x,x) - p(y,y)$$

defines a metric on X. Further a sequence $\{x_n\}$ converges in the metric space (X,p^S) to a point $x \in X$ if and only if

$$p(x,x) = \lim_{n\to+\infty} p(x,x_n) = \lim_{n,m\to+\infty} p(x_n,x_m).$$

Definition 1.8 ([19]) Let (X,p) be a partial metric space. Then:

(i) A sequence $\{x_n\}$ in X is called Cauchy if and only if $\lim_{n,m\to+\infty} p(x_n,x_m)$ exists and is finite.

(ii) A partial metric space (X,p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x,x) = \lim_{n,m\to+\infty} p(x_n,x_m)$.

Lemma 1.9 ([19, 39]) *Let (X,p) be a partial metric space. Then*:

(i) *A sequence $\{x_n\}$ in X is Cauchy in (X,p) if and only if it is Cauchy in (X,p^S).*

(ii) *A partial metric space (X,p) is complete if and only if the metric space (X,p^S) is complete.*

Consistent with [38], let (X,p) be a partial metric space and let $\mathrm{CB}^p(X)$ be the family of all nonempty, closed and bounded subsets of the partial metric space (X,p), induced by the partial metric p. Note that the closedness is taken from (X,τ_p) (τ_p is the topology induced by p) and the boundedness is given as follows: A is a bounded subset in (X,p) if there exist $x_0 \in X$ and $M \geq 0$ such that for all $a \in A$, we have $a \in B_p(x_0,M)$, that is, $p(x_0,a) < p(x_0,x_0) + M$. For $A,B \in \mathrm{CB}^p(X)$, $x \in X$, $\delta_p : \mathrm{CB}^p(X) \times \mathrm{CB}^p(X) \to \mathbb{R}^+$ define

$$\begin{aligned}
&p(x,A) = \inf\bigl\{p(x,a) : a \in A\bigr\},\\
&p(A,B) = \inf\bigl\{p(x,y) : x \in A, y \in B\bigr\},\\
&\delta_p(A,B) = \sup\bigl\{p(a,B) : a \in A\bigr\},\\
&H_p(A,B) = \max\bigl\{\delta_p(A,B), \delta_p(B,A)\bigr\},
\end{aligned}$$

where $H_p : \mathrm{CB}^p(X) \times \mathrm{CB}^p(X) \to \mathbb{R}^+$ is called the partial Hausdorff metric induced by p. Also, it is easy to show that $p(x,A) = 0$ implies that $p^S(x,A) = 0$, where

$$p^S(x,A) = \inf\bigl\{p^S(x,a) : a \in A\bigr\}.$$

Lemma 1.10 ([39]) *Let (X,p) be a partial metric space and A be any nonempty subset of X, then $a \in \overline{A}$ if and only if $p(a,A) = p(a,a)$.*

Lemma 1.11 *Let (X,p) be a partial metric space and A be any nonempty subset of X. If A is closed in (X,p), then A is closed in (X,p^S).*

Proof Let $\{x_n\}$ be a sequence converging to some $x \in X$ in (X,p^S). Then we have

$$p(x,x) = \lim_{n\to+\infty} p(x,x_n),$$

which implies, by definition, that $\{x_n\}$ converges to $x \in X$ also in (X,p). Now, since A is closed in (X,p), then $x \in A$ and so we deduce that A is closed also in (X,p^S). □

Proposition 1.12 ([38]) *Let (X,p) be a partial metric space. For any $A,B,C \in \mathrm{CB}^p(X)$, we have*:

(i) $\delta_p(A,A) = \sup\{p(a,a) : a \in A\}$;

(ii) $\delta_p(A,A) \leq \delta_p(A,B)$;

(iii) $\delta_p(A,B) = 0 \Longrightarrow A \subseteq B$;

(iv) $\delta_p(A,B) \leq \delta_p(A,C) + \delta_p(C,B) - \inf_{c\in C} p(c,c)$.

Proposition 1.13 ([38]) *Let (X,p) be a partial metric space. For any $A,B,C \in \mathrm{CB}^p(X)$, we have*:

(h1) $H_p(A,A) \leq H_p(A,B)$;

(h2) $H_p(A,B) = H_p(B,A)$;

(h3) $H_p(A,B) \leq H_p(A,C) + H_p(C,B) - \inf_{c\in C} p(c,c)$;

(h4) $H_p(A,B) = 0 \Longrightarrow A = B$.

Notice that each Hausdorff metric is a partial Hausdorff metric but the converse is not true, see Example 2.6 in [38].

Lemma 1.14 ([38]) *Let (X,p) be a partial metric space, $A,B \in \mathrm{CB}^p(X)$ and $q > 1$, then for any $a \in A$, there exists $b(a) \in B$ such that $p(a,b(a)) \leq qH_p(A,B)$.*

Theorem 1.15 ([38]) *Let (X,p) be a partial metric space. If $T : X \to \mathrm{CB}^p(X)$ is a multivalued mapping such that for all $x,y \in X$, we have $H_p(Tx,Ty) \leq kp(x,y)$, where $k \in (0,1)$, then T has a fixed point.*

In view of the above considerations and following the ideas in [44], the aim of this paper is to discuss multivalued weakly Picard operators on partial Hausdorff metric spaces, see also [45] for other interesting results. First, we obtain Kikkawa-Suzuki type fixed point theorems for a new type of generalized contractive conditions. Then, we prove data dependence of a fixed points set theorem. Finally, we present sufficient conditions for well-posedness of a fixed point problem. The presented results extend and unify some recently obtained comparable results for multivalued mappings (see [9] and the references therein).

2 Fixed point theorems in partial Hausdorff metric spaces

In this section we present several theorems which characterize MWP operators, defined in the previous section, in terms of different contractive conditions. Results of this section are generalizations of Theorem 1.4, Theorem 1.15 (and so Nadler's Theorem 1.3), Ćirić's theorem in [10] and others.

2.1 Result - I

To provide the first theorem we introduce the notion of (s,r)-contractive multivalued operator in partial Hausdorff metric spaces as follows.

Definition 2.1 Let $p : X \times X \to \mathbb{R}^+$ be a partial metric and $T : X \to \mathrm{CB}^p(X)$ be a multivalued mapping. T is called an (s,r)-contractive multivalued operator if there exist $r \in [0,1)$ and $s \geq r$ such that

$$p(y,Tx) \leq sp(y,x) \quad \Longrightarrow \quad H_p(Tx,Ty) \leq rM_p(x,y) \tag{1}$$

for all $x, y \in X$, where

$$M_p(x,y) = \max\left\{p(x,y), p(x,Tx), p(y,Ty), \frac{p(x,Ty) + p(y,Tx)}{2}\right\}.$$

Now we state and prove our theorem.

Theorem 2.2 *Let (X,p) be a complete partial metric space and $T : X \to \mathrm{CB}^p(X)$ be an (s,r)-contractive multivalued operator with $s \geq 1$. Then T is an MWP operator.*

Proof Let r_1 be a real number such that $0 \leq r < r_1 < 1$. Let $u_1 \in X$. As Tu_1 is nonempty, we can choose $u_2 \in Tu_1$. Clearly, if $u_2 = u_1$ the proof is finished and so we assume $u_2 \neq u_1$. Then we get

$$p(u_2, Tu_1) = p(u_2, u_2) \leq p(u_2, u_1) \leq sp(u_2, u_1). \tag{2}$$

Next, we choose $u_3 \in Tu_2$ such that

$$p(u_3, u_2) \leq \frac{r_1}{r} H_p(Tu_2, Tu_1).$$

Now, from (2) and by using condition (1) we write

$$\begin{aligned} p(u_3, u_2) &\leq \frac{r_1}{r} H_p(Tu_2, Tu_1) \leq r_1 M_p(u_1, u_2) \\ &= r_1 \max\left\{p(u_1,u_2), p(u_1,Tu_1), p(u_2,Tu_2), \frac{p(u_1,Tu_2) + p(u_2,Tu_1)}{2}\right\} \\ &\leq r_1 \max\left\{p(u_1,u_2), p(u_1,u_2), p(u_2,u_3), \frac{p(u_1,u_3) + p(u_2,u_2)}{2}\right\} \\ &\leq r_1 \max\left\{p(u_1,u_2), p(u_2,u_3), \frac{p(u_1,u_2) + p(u_2,u_3)}{2}\right\} \\ &\leq r_1 \max\left\{p(u_1,u_2), p(u_2,u_3)\right\}. \end{aligned}$$

Thus, we obtain

$$p(u_2, u_3) \leq r_1 \max\left\{p(u_1,u_2), p(u_2,u_3)\right\}.$$

If $\max\{p(u_1,u_2), p(u_2,u_3)\} = p(u_2,u_3)$, then $p(u_2,u_3) \leq r_1 p(u_2,u_3)$ implies that $p(u_2,u_3) = 0$, and we obtain $p^S(u_2,u_3) \leq 2p(u_2,u_3) = 0$, which further implies that $p^S(u_2,u_3) = 0$. Hence $u_2 = u_3 \in Tu_2$ and the proof is finished. On the contrary, if $\max\{p(u_1,u_2), p(u_2,u_3)\} = p(u_1,u_2)$, then we have

$$p(u_2, u_3) \leq r_1 p(u_1, u_2).$$

Continuing this process, we can construct a sequence $\{u_n\}$ in X such that $u_{n+1} \in Tu_n$, $u_{n+1} \neq u_n$ and

$$p(u_n, u_{n+1}) \leq r_1^{n-1} p(u_1, u_2)$$

for every $n > 1$. This shows that $\lim_{n\to+\infty} p(u_n, u_{n+1}) = 0$.

Now, since

$$p(u_n, u_n) + p(u_{n+1}, u_{n+1}) \leq 2p(u_n, u_{n+1}),$$

then we get

$$\lim_{n\to+\infty} p(u_n, u_n) = 0 \quad \text{and} \quad \lim_{n\to+\infty} p(u_{n+1}, u_{n+1}) = 0. \tag{3}$$

Let $\epsilon > 0$ and pick $N \in \mathbb{N}$ large enough so that for $n \geq N$ we have

$$2r_1^{n-1} \frac{1}{1 - r_1} p(u_1, u_2) < \epsilon.$$

Then, for every positive integer $k > n \geq N$, there is some $m \in \mathbb{N}$ such that $k = n + m$, and we have

$$\begin{aligned} p^S(u_n, u_{n+m}) &\leq 2p(u_n, u_{n+m}) \\ &\leq 2\bigl[p(u_n, u_{n+1}) + \cdots + p(u_{n+m-1}, u_{n+m})\bigr] \\ &\leq 2\bigl[r_1^{n-1} p(u_1, u_2) + \cdots + r_1^{n+m-2} p(u_1, u_2)\bigr] \\ &\leq 2r_1^{n-1} \frac{1}{1 - r_1} p(u_1, u_2) < \epsilon. \end{aligned}$$

It is immediate to deduce that $\{u_n\}$ is a Cauchy sequence in (X, p^S), but by Lemma 1.9 $\{u_n\}$ is Cauchy also in (X, p). Moreover, since (X, p) is complete, again by Lemma 1.9 we deduce the completeness of (X, p^S). It follows that there exists $z \in X$ such that $\lim_{n\to+\infty} u_n = z$ in (X, p^S). Therefore $\lim_{n\to+\infty} p^S(u_n, z) = 0$ implies

$$p(z, z) = \lim_{n\to+\infty} p(u_n, z) = \lim_{m,n\to+\infty} p(u_n, u_m).$$

Now, since $\{u_n\}$ is a Cauchy sequence in (X, p^S), then we have

$$\lim_{m,n\to+\infty} p^S(u_n, u_m) = 0$$

and so

$$\lim_{m,n\to+\infty} 2p(u_n, u_m) - \lim_{m\to+\infty} p(u_m, u_m) - \lim_{n\to+\infty} p(u_n, u_n) = 0.$$

It follows from (3) that

$$\lim_{m\to+\infty} p(u_m, u_m) = \lim_{n\to+\infty} p(u_n, u_n) = 0,$$

which further implies that

$$\lim_{m,n\to+\infty} p(u_n, u_m) = 0$$

and hence

$$p(z,z) = \lim_{n\to+\infty} p(u_n,z) = \lim_{n\to+\infty} p(u_n,u_n) = 0.$$

Next, we will show that there exists a subsequence $\{u_{n(k)}\}$ of $\{u_n\}$ such that

$$p(z, Tu_{n(k)}) \leq sp(z, u_{n(k)})$$

for all $k \in \mathbb{N}$. Reasoning by contradiction, we assume that there exists a positive integer N such that $p(z, Tu_n) > sp(z, u_n)$ for all $n \geq N$. This implies $p(z, u_{n+1}) > sp(z, u_n)$ for all $n \geq N$. By induction, for all $n \geq N$ and $m' \geq 1$, we get that

$$p(z, u_{n+m'}) > s^{m'} p(z, u_n). \tag{4}$$

Since

$$\begin{aligned} p(u_{n+m'}, u_n) &\leq p(u_n, u_{n+1}) + p(u_{n+1}, u_{n+2}) + \cdots + p(u_{n+m'-1}, u_{n+m'}) - \sum_{i=n+1}^{m'-1} p(u_i, u_i) \\ &\leq p(u_n, u_{n+1}) + p(u_{n+1}, u_{n+2}) + \cdots + p(u_{n+m'-1}, u_{n+m'}) \\ &\leq p(u_n, u_{n+1})\bigl(1 + r_1 + r_1^2 + r_1^3 + \cdots + r_1^{m'-1}\bigr) \\ &\leq p(u_n, u_{n+1})\left[\frac{1 - r_1^{m'}}{1 - r_1}\right] \end{aligned}$$

for all $n \geq N$ and $m' \geq 1$, then we get

$$\begin{aligned} p(z, u_n) &\leq p(z, u_{n+m'}) + p(u_{n+m'}, u_n) - p(z,z) \\ &\leq p(z, u_{n+m'}) + p(u_n, u_{n+1})\left[\frac{1 - r_1^{m'}}{1 - r_1}\right]. \end{aligned}$$

Passing to the limit as $m' \to +\infty$, we have

$$p(z, u_n) \leq \frac{1}{1 - r_1} p(u_n, u_{n+1}) \quad \text{for all } n \geq N.$$

Then we obtain

$$p(z, u_{n+m'}) \leq \frac{1}{1 - r_1} p(u_{n+m'}, u_{n+m'+1}) \leq \frac{r_1^{m'}}{1 - r_1} p(u_n, u_{n+1}) \tag{5}$$

for all $n \geq N$ and $m' \geq 1$. By (4) and (5) we get

$$p(z, u_n) < \frac{(\frac{r_1}{s})^{m'}}{1 - r_1} p(u_n, u_{n+1})$$

for all $n \geq N$ and $m' \geq 1$. Next, passing to the limit as $m' \to +\infty$, we obtain that $p(z, u_n) = 0$ for all $n \geq N$. This contradicts (4) and therefore there exists a subsequence $\{u_{n(k)}\}$ of $\{u_n\}$ such that $p(z, Tu_{n(k)}) \leq sp(z, u_{n(k)})$ for all $k \in \mathbb{N}$.

Also, by hypothesis we have

$$\begin{aligned} &H_p(Tu_{n(k)}, Tz) \\ &\quad \le r\max\left\{p(u_{n(k)},z), p(u_{n(k)}, Tu_{n(k)}), p(z,Tz), \frac{p(u_{n(k)}, Tz)+p(z,Tu_{n(k)})}{2}\right\} \\ &\quad \le r\max\left\{p(u_{n(k)},z), p(u_{n(k)}, u_{n(k)+1}), p(z,Tz), \frac{p(u_{n(k)}, z)+p(z,Tz)+p(z,u_{n(k)+1})}{2}\right\}. \end{aligned}$$

Therefore, we write

$$\begin{aligned} &p(z,Tz) \\ &\quad \le p(z,u_{n(k)+1}) + H_p(Tu_{n(k)}, Tz) - p(u_{n(k)+1}, u_{n(k)+1}) \\ &\quad \le p(z,u_{n(k)+1}) \\ &\qquad + r\max\left\{p(u_{n(k)},z), p(u_{n(k)}, u_{n(k)+1}), p(z,Tz), \frac{p(u_{n(k)}, z)+p(z,Tz)+p(z,u_{n(k)+1})}{2}\right\}. \end{aligned}$$

On passing to the limit as $k \to +\infty$, we get

$$p(z,Tz) \le r\max\left\{p(z,Tz), \frac{p(z,Tz)}{2}\right\} = rp(z,Tz).$$

Since $r < 1$, it follows that

$$p(z,Tz) = 0.$$

Therefore $p(z,Tz) = 0 = p(z,z)$ and hence by Lemma 1.10 we deduce that $z \in Tz$, that is, z is a fixed point of T. □

The following example illustrates the use of Theorem 2.2.

Example 2.3 Let $X = \{0,1,2\}$ and $p : X \times X \to \mathbb{R}^+$ be defined by $p(0,0) = 0$, $p(1,1) = p(2,2) = \frac{1}{4}$, $p(1,0) = \frac{1}{3}$, $p(2,0) = \frac{3}{5}$, $p(2,1) = \frac{2}{5}$ and $p(y,x) = p(x,y)$ for all $x,y \in X$. Then (X,p) is a complete partial metric space. Also define $T : X \to \mathrm{CB}^p(X)$ by

$$Tx = \begin{cases} \{0\} & \text{if } x \neq 2, \\ \{0,1\} & \text{if } x = 2. \end{cases}$$

Therefore, we get

$$\begin{aligned} &H_p(T0,T0) = H_p(T0,T1) = H_p(T1,T1) = p(0,T0) = p(0,T1) = p(0,0), \\ &H_p(T0,T2) = H_p(T1,T2) = p(1,T0) = p(1,T1) = p(1,0), \\ &p(2,T2) = \min\{p(2,0), p(2,1)\} = p(2,1), \\ &H_p(T2,T2) = p(2,2). \end{aligned}$$

It follows easily that the inequality

$$p(y, Tx) \leq sp(x, y)$$

holds for all $x, y \in X$ with $s \geq \frac{8}{5}$. Also, for all $x, y \in X$ with $r \in [\frac{5}{6}, 1)$, we get

$$H_p(Tx, Ty) \leq rM_p(x, y).$$

Thus all the conditions of Theorem 2.2 are satisfied and 0 is a fixed point of T.

From Theorem 2.2 we deduce some corollaries.

Corollary 2.4 *Let (X, p) be a complete partial metric space and $T : X \to \mathrm{CB}^p(X)$ be a multivalued mapping. Assume that there exist $r \in [0, 1)$ and $s \geq 1$ such that*

$$p(y, Tx) \leq sp(x, y) \quad \Longrightarrow \quad H_p(Tx, Ty) \leq r \max\{p(x, y), p(x, Tx), p(y, Ty)\}$$

for all $x, y \in X$. Then T is an MWP operator.

Corollary 2.5 *Let (X, p) be a complete partial metric space and $T : X \to \mathrm{CB}^p(X)$ be a multivalued mapping. Assume that there exist $r \in [0, 1)$ and $s \geq 1$ such that*

$$p(y, Tx) \leq sp(x, y) \quad \Longrightarrow \quad H_p(Tx, Ty) \leq \frac{r}{3}\big[p(x, y) + p(x, Tx) + p(y, Ty)\big]$$

for all $x, y \in X$. Then T is an MWP operator.

In the case of single-valued mappings, Theorem 2.2 reduces to the following significant corollary.

Corollary 2.6 *Let (X, p) be a complete partial metric space and $T : X \to X$ be a single-valued mapping. Assume that there exist $r \in [0, 1)$ and $s \geq 1$ such that*

$$p(y, Tx) \leq sp(x, y) \quad \Longrightarrow \quad p(Tx, Ty) \leq rM_p(x, y), \tag{6}$$

where

$$M_p(x, y) = \max\left\{p(x, y), p(x, Tx), p(y, Ty), \frac{p(x, Ty) + p(y, Tx)}{2}\right\}.$$

Then T has a unique fixed point.

Proof The existence part of the proof follows easily by Theorem 2.2. Thus, we need to prove uniqueness of the fixed point. Suppose to the contrary that, for $s \geq 1$, there exist $x, y \in \mathrm{Fix}(T)$ with $x \neq y$. It follows immediately that

$$p(y, Tx) = p(y, x) \leq sp(y, x).$$

Thus, by hypothesis on T, we would have

$$\begin{aligned} p(Tx,Ty) &\leq r\max\left\{p(x,y),p(x,Tx),p(y,Ty),\frac{p(x,Ty)+p(y,Tx)}{2}\right\} \\ &= r\max\left\{p(x,y),p(x,x),p(y,y),\frac{p(x,y)+p(y,x)}{2}\right\} \\ &= rp(x,y). \end{aligned}$$

We deduce that $p(x,y)=0$, which further implies that $p^S(x,y)\leq 2p(x,y)=0$ and hence $x=y$, a contradiction. This completes the proof. □

The following two examples, adapted from [46], show the validity of Corollary 2.6.

Example 2.7 Let $X=[0,1]$ be endowed with the partial metric $p(x,y)=\max\{x,y\}$ for all $x,y\in X$. Then (X,p) is a complete partial metric space. Also define $T:X\to X$ by

$$Tx=\frac{x^2}{1+x}.$$

Take arbitrary elements $x,y\in X$ with $y\leq x$. Then we have

$$p(y,Tx)=\max\left\{y,\frac{x^2}{1+x}\right\}\leq sx=sp(x,y)$$

for all $x,y\in X$ and $s\geq 1$. On the other hand, we get

$$p(Tx,Ty)=\max\left\{\frac{x^2}{1+x},\frac{y^2}{1+y}\right\}=\frac{x^2}{1+x}$$

and

$$\begin{aligned} M_p(x,y) &= \max\left\{p(x,y),p(x,Tx),p(y,Ty),\frac{p(x,Ty)+p(y,Tx)}{2}\right\} \\ &= \max\left\{p(x,y),p\left(x,\frac{x^2}{1+x}\right),p\left(y,\frac{y^2}{1+y}\right),\frac{1}{2}\left(p\left(x,\frac{y^2}{1+y}\right)+p\left(y,\frac{x^2}{1+x}\right)\right)\right\} \\ &= \max\left\{x,x,y,\frac{1}{2}\left(x+\max\left\{y,\frac{x^2}{1+x}\right\}\right)\right\}=x. \end{aligned}$$

Thus the inequality $p(Tx,Ty)\leq rM_p(x,y)$ holds for all $x,y\in X$ with $y\leq x$ and for any $r\in[\frac{1}{2},1)$. Note that we obtain the same conclusion if we assume that $x\leq y$. Hence, all the conditions of Corollary 2.6 are satisfied and 0 is a fixed point of T.

The following example underlines the crucial role of the right-hand side of (6) in establishing existence of the fixed point.

Example 2.8 Let $X=\{1,2,3,4\}$ and $p:X\times X\to\mathbb{R}^+$ be defined by $p(x,x)=\frac{1}{2}$ for each $x\in X$, $p(1,2)=p(3,4)=2$, $p(1,3)=p(2,4)=1$, $p(1,4)=p(2,3)=\frac{3}{2}$ and $p(y,x)=p(x,y)$ for all $x, y \in X$. Then (X, p) is a complete partial metric space. Also define $T : X \to X$ by

$$T = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 1 & 2 \end{pmatrix}.$$

Trivially T has no fixed points, but we try to apply Corollary 2.6. It is easy to check that the inequality $p(y, Tx) \leq sp(y, x)$ certainly holds for all $x, y \in X$ with $s \geq 2$. Now we note that, for $x = 3$ and $y = 1$, we get

$$p(T3, T1) = p(1, 2) = 2$$

and

$$\begin{aligned} M_p(3,1) &= \max\left\{p(3,1), p(3,T3), p(1,T1), \frac{p(3,T1) + p(1,T3)}{2}\right\} \\ &= \max\left\{1, 1, 2, \frac{1}{2}\left(\frac{3}{2} + \frac{1}{2}\right)\right\} = 2. \end{aligned}$$

It follows that

$$p(T3, T1) = 2 \nleq 2r = rM_p(3,1),$$

whatever $r \in [0,1)$ is chosen. We conclude that Corollary 2.6 is not applicable in this case.

2.2 Result - II

Another interesting characterization of MWP operators is provided by the following theorem.

Theorem 2.9 *Let (X, p) be a complete partial metric space and $T : X \to \mathrm{CB}^p(X)$ be a multivalued operator. Assume that there exist $r, s \in [0,1)$, with $r < s$, such that*

$$\frac{1}{1+r} p(x, Tx) \leq p(x, y) \leq \frac{1}{1-s} p(x, Tx) \quad \Longrightarrow \quad H_p(Tx, Ty) \leq rM_p(x, y) \tag{7}$$

for all $x, y \in X$, where

$$M_p(x,y) = \max\left\{p(x,y), p(x,Tx), p(y,Ty), \frac{p(x,Ty) + p(y,Tx)}{2}\right\}.$$

Then T is an MWP operator.

Proof Let r_1 be a real number such that $0 \leq r < r_1 < s$. Also, let $u_1 \in X$ and $u_2 \in Tu_1$ be such that

$$p(u_1, u_2) \leq \frac{1 - r_1}{1 - s} p(u_1, Tu_1).$$

Then we have

$$\frac{1}{1+r} p(u_1, Tu_1) \leq p(u_1, Tu_1) \leq p(u_1, u_2) \leq \frac{1}{1-s} p(u_1, Tu_1)$$

and so, by using condition (7), we obtain

$$\begin{aligned} p(u_2, Tu_2) &\leq H_p(Tu_1, Tu_2) \\ &\leq r \max\left\{p(u_1, u_2), p(u_1, Tu_1), p(u_2, Tu_2), \frac{p(u_1, Tu_2) + p(u_2, Tu_1)}{2}\right\} \\ &\leq r \max\left\{p(u_1, u_2), p(u_2, Tu_2), \frac{p(u_1, Tu_2) + p(u_2, u_2)}{2}\right\} \\ &\leq r \max\left\{p(u_1, u_2), p(u_2, Tu_2), \frac{p(u_1, u_2) + p(u_2, Tu_2)}{2}\right\}. \end{aligned}$$

Thus

$$p(u_2, Tu_2) \leq r \max\{p(u_1, u_2), p(u_2, Tu_2)\}.$$

Now, if $\max\{p(u_1, u_2), p(u_2, Tu_2)\} = p(u_2, Tu_2)$, then $p(u_2, Tu_2) \leq rp(u_2, Tu_2)$ implies that $p(u_2, Tu_2) = 0$ and so we obtain

$$p^S(u_2, Tu_2) \leq 2p(u_2, Tu_2) = 0,$$

which further implies that $p^S(u_2, Tu_2) = 0$. In view of Lemma 1.11, $u_2 \in Tu_2$, and the proof is finished. On the contrary, if $\max\{p(u_1, u_2), p(u_2, u_3)\} = p(u_1, u_2)$, then we have

$$p(u_2, Tu_2) \leq rp(u_1, u_2).$$

It follows that there exists $u_3 \in Tu_2$ such that

$$p(u_2, u_3) \leq r_1 p(u_1, u_2) \quad \text{and} \quad p(u_2, u_3) \leq \frac{1 - r_1}{1 - s} p(u_2, Tu_2).$$

This implies

$$\frac{1}{1 + r} p(u_2, Tu_2) \leq p(u_2, u_3) \leq \frac{1}{1 - s} p(u_2, Tu_2).$$

Now, by using condition (7), we get $p(u_3, Tu_3) \leq rp(u_2, u_3)$. Continuing this process, we can construct a sequence $\{u_n\}$ in X with the following properties:

$$\begin{cases} u_{n+1} \in Tu_n, \\ p(u_{n+1}, Tu_{n+1}) \leq rp(u_n, u_{n+1}), \\ p(u_{n+1}, u_{n+2}) \leq r_1 p(u_n, u_{n+1}), \\ p(u_{n+1}, u_{n+2}) \leq \frac{1-r_1}{1-s} p(u_{n+1}, Tu_{n+1}) \end{cases}$$

for every $n \in \mathbb{N}$. Next, from $p(u_{n+1}, u_{n+2}) \leq r_1 p(u_n, u_{n+1})$ we deduce that

$$\lim_{n \to +\infty} p(u_n, u_{n+1}) = 0.$$

Since

$$p(u_n, u_n) + p(u_{n+1}, u_{n+1}) \leq 2p(u_n, u_{n+1}),$$

then we get

$$\lim_{n\to+\infty} p(u_n,u_n) = 0 \quad \text{and} \quad \lim_{n\to+\infty} p(u_{n+1},u_{n+1}) = 0.$$

Proceeding as in the proof of Theorem 2.2, one can show that the sequence $\{u_n\}$ is a Cauchy sequence in (X,p) converging to some $z \in X$ with $p(z,z) = 0$.

Now, since

$$p(u_n,u_{n+1}) \le \frac{1-r_1}{1-s} p(u_n,Tu_n),$$

then we have

$$p(z,u_n) \le \frac{1}{1-s} p(u_n,Tu_n)$$

for all $n \ge 1$. Then we assume that there exists a positive integer N such that

$$p(z,u_n) < \frac{1}{1+r} p(u_n,Tu_n)$$

holds for every $n \ge N$. Consequently, we have

$$\begin{aligned} p(u_n,u_{n+1}) &\le p(z,u_n) + p(z,u_{n+1}) - p(z,z) \\ &< \frac{1}{1+r}\bigl[p(u_n,Tu_n) + p(u_{n+1},Tu_{n+1})\bigr] \\ &< \frac{1}{1+r}\bigl[p(u_n,Tu_n) + rp(u_n,u_{n+1})\bigr] \\ &\le \frac{1}{1+r}\bigl[p(u_n,u_{n+1}) + rp(u_n,u_{n+1})\bigr] = p(u_n,u_{n+1}), \end{aligned}$$

which is a contradiction. Hence, there exists a subsequence $\{u_{n(k)}\}$ of $\{u_n\}$ such that

$$p(z,u_{n(k)}) \ge \frac{1}{1+r} p(u_{n(k)},Tu_{n(k)})$$

holds for every $k \ge N$. Since $p(z,u_n) \le \frac{1}{1-s}p(u_n,Tu_n)$ for all $n \ge 1$, by condition (7), we have $H_p(Tz,Tu_{n(k)}) \le rM_p(z,u_{n(k)})$. This implies

$$\begin{aligned} &p(z,Tz) \\ &\quad \le p(u_{n(k)+1},z) + H_p(Tu_{n(k)},Tz) \\ &\quad \le p(u_{n(k)+1},z) \\ &\qquad + r\max\biggl\{p(u_{n(k)},z), p(u_{n(k)},Tu_{n(k)}), p(z,Tz), \frac{p(u_{n(k)},Tz)+p(z,Tu_{n(k)})}{2}\biggr\} \\ &\quad \le r\max\biggl\{p(u_{n(k)},z), p(u_{n(k)},u_{n(k)+1}), p(z,Tz), \frac{p(u_{n(k)},Tz)+p(z,u_{n(k)+1})}{2}\biggr\}. \end{aligned}$$

On passing to the limit as $k \to +\infty$, we get

$$p(z,Tz) \le r\max\biggl\{p(z,Tz), \frac{p(z,Tz)}{2}\biggr\}.$$

Since $r < 1$, it follows that

$$p(z, Tz) = 0.$$

Therefore $p(z, Tz) = 0 = p(z, z)$ and hence by Lemma 1.10 we have $z \in Tz$, that is, z is a fixed point of T. □

In the case of single-valued mappings, Theorem 2.9 reduces to the following corollary.

Corollary 2.10 *Let (X, p) be a complete partial metric space and $T : X \to X$ be a single-valued mapping. Assume that there exists $r \in [0, 1)$ such that*

$$\frac{1}{1+r} p(x, Tx) \leq p(x, y) \leq \frac{1}{1-r} p(x, Tx) \quad \Longrightarrow \quad p(Tx, Ty) \leq r M_p(x, y)$$

for all $x, y \in X$, where

$$M_p(x, y) = \max\left\{p(x, y), p(x, Tx), p(y, Ty), \frac{p(x, Ty) + p(y, Tx)}{2}\right\}.$$

Then T has a fixed point.

Proof It is easy to prove that for every $u_1 \in X$ the sequence $\{u_n\}$ defined by $u_{n+1} = Tu_n$ satisfies the relationship $p(u_{n+1}, u_{n+2}) \leq rp(u_n, u_{n+1})$. Consequently, the sequence $\{u_n\}$ is Cauchy, and there is some point $z \in X$ such that $\lim_{n\to+\infty} u_n = z$. Proceeding as in the proof of Theorem 2.9, we can show that

$$p(z, z) = \lim_{n\to+\infty} p(u_n, z) = \lim_{n\to+\infty} p(u_n, u_n) = 0.$$

Also in view of Theorem 2.9, for all $n \geq N$, we can assume $p(z, u_n) \leq \frac{1}{1+r} p(u_n, u_{n+1})$ for leading to contradiction. Consequently, there exists a subsequence $\{u_{n(k)}\}$ of $\{u_n\}$ such that

$$p(z, u_{n(k)}) \geq \frac{1}{1+r} p(u_{n(k)}, u_{n(k)+1})$$

holds for every $k \geq N$. Therefore, we obtain that

$$\begin{aligned} &p(u_{n(k)+1}, Tz) \\ &\quad \leq r \max\left\{p(u_{n(k)}, z), p(u_{n(k)}, Tu_{n(k)}), p(z, Tz), \frac{p(u_{n(k)}, Tz) + p(z, Tu_{n(k)})}{2}\right\} \\ &\quad \leq r \max\left\{p(u_{n(k)}, z), p(u_{n(k)}, u_{n(k)+1}), p(z, Tz), \frac{p(u_{n(k)}, z) + p(z, Tz) + p(z, u_{n(k)+1})}{2}\right\}. \end{aligned}$$

On passing to the limit as $k \to +\infty$, we get

$$p(z, Tz) \leq r \max\left\{p(z, Tz), \frac{p(z, Tz)}{2}\right\}.$$

Since $r < 1$, it follows that

$$p(z, Tz) = 0$$

and so $z = Tz$, that is, z is a fixed point of T. □

The following example shows that Theorem 2.9 is proper extension of the respective result in standard metric spaces.

Example 2.11 Let $X = \{0,1,2\}$ and $p : X \times X \to \mathbb{R}^+$ be defined by $p(0,0) = p(1,1) = 0$, $p(2,2) = \frac{1}{4}$, $p(1,0) = \frac{1}{3}$, $p(2,0) = \frac{2}{5}$, $p(2,1) = \frac{11}{15}$ and $p(y,x) = p(x,y)$ for all $x, y \in X$. Then (X,p) is a complete partial metric space. Also define $T : X \to \mathrm{CB}^p(X)$ by

$$Tx = \begin{cases} \{0\} & \text{if } x \neq 2, \\ \{0,1\} & \text{if } x = 2. \end{cases}$$

Therefore, we get

$$\min\{p(x, Tx) : x \in X \setminus \{0\}\} = \min\{p(x,y) : x, y \in X \text{ and } x \neq y\} = \frac{1}{3}$$

and

$$\max\{p(x, Tx) : x \in X\} = \frac{2}{5}.$$

It follows easily that the inequalities

$$\frac{1}{1+r} p(x, Tx) \leq p(x,y) \leq \frac{1}{1-s} p(x, Tx)$$

hold for all $x, y \in X$ with $x \neq y$ and for some $1 > s > r \geq \frac{1}{5}$. Also the above inequalities hold for $x = y = 2$ with $1 > s > r \geq \frac{3}{5}$. On the other hand, the above inequalities are not applicable for $x = y \in \{0,1\}$. Clearly we have

$$\begin{aligned}
&H_p(T0, T0) = H_p(T0, T1) = H_p(T1, T1) = p(0, T0) = p(0, T1) = p(0,0), \\
&H_p(T0, T2) = H_p(T1, T2) = p(1, T0) = p(1, T1) = p(1,0), \\
&p(2, T2) = \min\{p(2,0), p(2,1)\} = p(2,0), \\
&H_p(T2, T2) = p(2,2).
\end{aligned}$$

Finally, by routine calculations and taking $1 > s > r \geq \frac{5}{6}$, one can show that the inequality

$$H_p(Tx, Ty) \leq r M_p(x,y)$$

holds true as for all $x, y \in X$ with $x \neq y$, as for $x = y = 2$. Thus all the conditions of Theorem 2.9 are satisfied and 0 is a fixed point of T.

Next, we consider the metric p^S induced by the partial metric p. Indeed, we have $p^S(x,x) = 0$ for all $x \in X$, $p^S(1,0) = \frac{2}{3}$, $p^S(2,1) = \frac{73}{60}$, $p^S(2,0) = \frac{11}{20}$ and $p^S(x,y) = p^S(y,x)$ for all $x, y \in X$.

We show that Theorem 2.9 is not applicable in this case. Indeed, for $x = 2$ and $y = 0$, the inequalities

$$\frac{1}{1+r}p^S(2,T2) = \frac{1}{1+r}p^S\bigl(2,\{0,1\}\bigr) = \frac{1}{1+r}p^S(2,0) \le p^S(2,0) \le \frac{1}{1-s}p^S(2,T2)$$

hold true for all $r \in [0,1)$ with $r < s$. Therefore, we need to have

$$H_{p^S}(T0,T2) \le rM_{p^S}(0,2).$$

Unfortunately, this is not the case because

$$H_{p^S}(T0,T2) = H_{p^S}\bigl(\{0\},\{0,1\}\bigr) = \frac{2}{3}$$

and

$$\begin{aligned} M_{p^S}(0,2) &= \max\left\{p^S(0,2), p^S(0,T0), p^S(2,T2), \frac{p^S(0,T2)+p^S(2,T0)}{2}\right\} \\ &= \max\left\{\frac{11}{20}, 0, \frac{11}{20}, \frac{0+\frac{11}{20}}{2}\right\} = \frac{11}{20} < \frac{2}{3}. \end{aligned}$$

Consequently, for any $r \in [0,1)$ we have

$$H_{p^S}(T0,T2) \nleq rM_{p^S}(0,2).$$

3 Data dependence theorem in partial Hausdorff metric spaces

The aim of this section is to discuss data dependence of a fixed points set for MWP operators on partial metric spaces. Also, this section is motivated by Popescu [44], see also [4]. Precisely, we will prove a result for $(1,r)$-contractive multivalued operators in partial Hausdorff metric spaces.

First, we need the following auxiliary lemma.

Lemma 3.1 *Let* (X,p) *be a partial metric space and* $T : X \to \mathrm{CB}^p(X)$ *be a* $(1,r)$*-contractive multivalued operator. If* $z \in Tz$*, then* $p(z,z) = 0$.

Proof Since $p(z,Tz) = p(z,z)$, then we have

$$H_p(Tz,Tz) \le r\max\left\{p(z,z), p(z,Tz), p(z,Tz), \frac{p(z,Tz)+p(z,Tz)}{2}\right\} = rp(z,z).$$

Thus, by definition, we write

$$p(z,z) \le p(z,Tz) \le H_p(Tz,Tz) \le rp(z,z) < p(z,z)$$

and hence we deduce that $p(z,z) = 0$. □

We recall the following concept.

Definition 3.2 ([47]) Let (X,p) be a partial metric space and let $\phi : X \to \mathbb{R}^+$ be a function on X. Then the function ϕ is called p-lower semi-continuous on X whenever

$$\lim_{n\to+\infty} p(x_n,x) = p(x,x) \quad \Longrightarrow \quad \phi(x) \le \lim_{n\to+\infty} \inf \phi(x_n) = \sup_{n\ge 1} \inf_{m\ge n} \phi(x_m).$$

Now we state and prove our theorem.

Theorem 3.3 *Let (X,p) be a partial metric space and $T_1, T_2 : X \to \mathrm{CB}^p(X)$ be two multivalued operators. We suppose that*:

(i) *T_i is a $(1,r_i)$-contractive multivalued operator for $i \in 1,2$;*

(ii) *there exists $\lambda > 0$ such that $H_p(T_1x, T_2x) \le \lambda$ for all $x \in X$;*

(iii) *the function $\phi : X \to \mathbb{R}^+$ defined by $\phi(x) = p(x,x)$ is p-lower semi-continuous.*

Then:

(a) $\mathrm{Fix}(T_i) \in \mathrm{CL}^p(X)$, $i \in \{1,2\}$;

(b) *T_1 and T_2 are MWP operators and*

$$H_p\big(\mathrm{Fix}(T_1), \mathrm{Fix}(T_2)\big) \le \frac{\lambda}{1 - \max\{r_1, r_2\}}.$$

Proof (a) From Theorem 2.2 we have that $\mathrm{Fix}(T_i)$ is a nonempty set, $i \in \{1,2\}$. Let us prove that the fixed point set of a $(1,r_i)$-contractive multivalued operator T_i is closed. Let $x_n \in \mathrm{Fix}(T_i)$, with $n \ge 1$, be such that $\lim_{n\to+\infty} x_n = z$ in (X,p). In view of (iii) and Lemma 3.1, we have

$$\lim_{n\to+\infty} p(x_n,z) = p(z,z) \quad \Longrightarrow \quad p(z,z) \le \lim_{n\to+\infty} \inf p(x_n,x_n) = 0.$$

It follows that

$$p(z,z) = \lim_{n\to+\infty} p(x_n,z) = \lim_{n\to+\infty} p(x_n,x_n) = 0.$$

Also, since $x_n \in T_ix_n$, we have $p(z, T_ix_n) \le p(z,x_n)$ and then

$$\begin{aligned} p(z, T_iz) &\le p(z,x_n) + p(x_n, T_iz) - p(x_n,x_n) \\ &\le p(z,x_n) + H_p(T_ix_n, T_iz) \\ &\le p(z,x_n) + r_ip(x_n,z). \end{aligned}$$

Passing to the limit as $n \to +\infty$, we obtain that $p(z, T_iz) = 0$. Therefore

$$p(z, T_iz) = 0 = p(z,z)$$

and hence by Lemma 1.10 and $T_iz \in \mathrm{CL}^p(X)$ we get $z \in T_iz$, that is, $z \in \mathrm{Fix}(T_i)$.

(b) From the proof of Theorem 2.2 we immediately get that a $(1,r_i)$-contractive multivalued operator is an MWP operator. For the second conclusion, let q be a real number such that $q > 1$, and $x_0 \in \mathrm{Fix}(T_1)$ be arbitrary. Then, by Lemma 1.14, there exists $x_1 \in T_2x_0$ such that

$$p(x_0,x_1) \le qH_p(T_1x_0, T_2x_0).$$

Next, for $x_1 \in T_2x_0$, there exists $x_2 \in T_2x_1$ such that

$$p(x_1,x_2) \leq qH_p(T_2x_0, T_2x_1).$$

Since $x_1 \in T_2x_0$, $p(x_1, T_2x_0) = p(x_1,x_1) \leq p(x_0,x_1)$, then we have

$$p(x_1,x_2) \leq qH_p(T_2x_0, T_2x_1) \leq qr_2p(x_0,x_1).$$

Iterating this process allows us to construct a sequence of successive approximations for T_2 starting from x_0, satisfying the following assertions:

$$x_{n+1} \in T_2x_n \quad \text{and} \quad p(x_n,x_{n+1}) \leq (qr_2)^n p(x_0,x_1) \quad \text{for all } n \in \mathbb{N}.$$

Hence, for all $n \geq N$ and $m \geq 1$, we write

$$\begin{aligned} p(x_{n+m},x_n) &\leq p(x_n,x_{n+1}) + p(x_{n+1},x_{n+2}) + \cdots + p(x_{n+m-1},x_{n+m}) - \sum_{i=n+1}^{m-1} p(x_i,x_i) \\ &\leq p(x_n,x_{n+1}) + p(x_{n+1},x_{n+2}) + \cdots + p(x_{n+m-1},x_{n+m}) \\ &\leq \frac{(qr_2)^n}{1-qr_2} p(x_0,x_1). \end{aligned}$$

Consequently, we get

$$p^S(x_n,x_{n+m}) \leq 2p(x_n,x_{n+m}) \leq \frac{2(qr_2)^n}{1-qr_2} p(x_0,x_1). \tag{8}$$

Now, choosing $1 < q < \min\{\frac{1}{r_1}, \frac{1}{r_2}\}$ and passing to the limit as $n \to +\infty$, we deduce easily that the sequence $\{x_n\}$ is Cauchy in (X,p^S) and, by Lemma 1.9, $\{x_n\}$ is Cauchy in (X,p). Then there exists $u \in X$ such that

$$\lim_{n\to+\infty} x_n = u \quad \text{in } (X,p^S).$$

Therefore, (8) and $\lim_{n\to+\infty} p^S(x_n,u) = 0$ imply

$$p(u,u) = \lim_{n\to+\infty} p(x_n,u) = \lim_{n,m\to+\infty} p(x_n,x_m) = 0.$$

We will prove that u is a fixed point for T_2. To this aim, suppose that there exists a positive integer N such that

$$p(u, T_2x_n) > p(u,x_n) \quad \text{for all } n \geq N.$$

This implies that $p(u,x_{n+1}) > p(u,x_n)$ for all $n \geq N$, which leads to contradiction since $x_n \to u$ as $n \to +\infty$. Hence, there exists a subsequence $\{x_{n(k)}\}$ such that

$$p(u, T_2x_{n(k)}) \leq p(u,x_{n(k)}) \quad \text{for all } k \in \mathbb{N}.$$

Next, from

$$p(u, T_2u) \le p(x_{n(k)+1}, u) + H_p(T_2x_{n(k)}, T_2u),$$

on passing to the limit as $k \to +\infty$, we get

$$\begin{aligned} p(u, T_2u) &\le \lim_{k\to+\infty} p(x_{n(k)+1}, u) + \lim_{k\to+\infty} H_p(T_2x_{n(k)}, T_2u) \\ &\le \lim_{k\to+\infty} r_2 p(x_{n(k)}, u) = 0 \end{aligned}$$

and so $u \in \mathrm{Fix}(T_2)$.

By (8), passing to the limit as $m \to +\infty$, we get

$$\begin{aligned} p(x_n, u) &\le \lim_{m\to+\infty} \bigl[p(x_n, x_{n+m}) + p(x_{n+m}, u) - p(u, u)\bigr] \\ &\le \lim_{m\to+\infty} \biggl[\frac{2(qr_2)^n}{1-qr_2} p(x_0, x_1) + p(x_{n+m}, u)\biggr] \\ &\le \frac{(qr_2)^n}{1-qr_2} p(x_0, x_1) \quad \text{for each } n \in \mathbb{N}. \end{aligned}$$

Then

$$p(x_0, u) \le \frac{1}{1-qr_2} p(x_0, x_1) \le \frac{q\lambda}{1-qr_2}.$$

Analogously, one can show that, for each $u_0 \in \mathrm{Fix}(T_2)$, there exists $x \in \mathrm{Fix}(T_1)$ such that

$$p(u_0, x) \le \frac{1}{1-qr_1} p(u_0, u_1) \le \frac{q\lambda}{1-qr_1}$$

and hence

$$H_p\bigl(\mathrm{Fix}(T_1), \mathrm{Fix}(T_2)\bigr) \le \frac{q\lambda}{1-\max\{qr_1, qr_2\}}.$$

Finally, passing to the limit as $q \to 1^+$, we obtain the assertion. This concludes the proof. □

4 Well-posedness of fixed point problems in partial Hausdorff metric spaces

According to [48, 49], we get the notions of well-posedness of a fixed point problem in the setting of partial metric spaces.

Definition 4.1 (see [48, 49]) Let (X, p) be a partial metric space and $T : X \to \mathrm{CB}^p(X)$ be a multivalued operator. Then the fixed point problem is well posed for T with respect to p if:

(a_1) $\mathrm{Fix}(T) = \{z\}$;
(b_1) if $x_n \in X$, $n \in \mathbb{N}$ and $\lim_{n\to+\infty} p(x_n, Tx_n) = 0$, then $\lim_{n\to+\infty} p(x_n, z) = 0$.

Definition 4.2 (see [48, 49]) Let (X, p) be a partial metric space and $T : X \to \mathrm{CB}^p(X)$ be a multivalued operator. Then the fixed point problem is well posed for T with respect to H_p if:

(a_2) $S\,\mathrm{Fix}(T) = \{z\}$;
(b_2) if $x_n \in X$, $n \in \mathbb{N}$ and $\lim_{n\to+\infty} H_p(x_n, Tx_n) = 0$, then $\lim_{n\to+\infty} p(x_n, z) = 0$.

Clearly, (b_2) of Definition 4.2 implies (b_1) of Definition 4.1. Moreover, from (a_1) and (a_2), that is, $\mathrm{Fix}(T) = S\,\mathrm{Fix}(T) = \{z\}$, we deduce that if the fixed point problem is well posed for T with respect to p, then it is well posed for T with respect to H_p.

Motivated by the above facts, we will prove the following theorem for (s,r)-contractive multivalued operators, with $s > 1$, in partial metric spaces.

Theorem 4.3 *Let (X,p) be a partial metric space and $T : X \to \mathrm{CB}^p(X)$ be a multivalued operator. We suppose that*:

(1) *T is an (s,r)-contractive multivalued operator with $s \geq 1$*;
(2) $S\,\mathrm{Fix}(T) \neq \emptyset$.

Then:

(a) $\mathrm{Fix}(T) = S\,\mathrm{Fix}(T) = \{z\}$;
(b) *the fixed point problem is well posed for T with respect to H_p if $s > 1$.*

Proof (a) Suppose $z \in S\,\mathrm{Fix}(T)$. Clearly, $z \in \mathrm{Fix}(T)$. We show that $\mathrm{Fix}(T) = \{z\}$. To this aim, let $u \in \mathrm{Fix}(T)$ with $u \neq z$. Since $p(u, Tz) = p(u,z) \leq sp(u,z)$, we get $H_p(Tz, Tu) \leq rp(z,u)$ and therefore

$$p(z,u) = p(Tz,u) \leq H_p(Tz,Tu) \leq rp(z,u),$$

which leads to contradiction. Thus, $\mathrm{Fix}(T) = \{z\}$ and so the assertion (a) holds true.

(b) Now, let $x_n \in X$, with $n \in \mathbb{N}$, be such that $\lim_{n\to+\infty} p(x_n, Tx_n) = 0$. We have to show that $\lim_{n\to+\infty} p(x_n, z) = 0$. Suppose this is not the case; suppose that $p(x_n, z)$ does not converge to zero. Consequently, there exist $\epsilon > 0$ and a subsequence $\{x_{n(k)}\}$ such that

$$p(x_{n(k)}, z) \geq \epsilon \quad \text{for all } k \in \mathbb{N}.$$

Now, assume that there exists a subsequence $\{x_{n(k(j))}\}$ of $\{x_{n(k)}\}$ with

$$p(z, Tx_{n(k(j))}) \leq sp(z, x_{n(k(j))}).$$

Then we get

$$H_p(Tz, Tx_{n(k(j))}) \leq rp(z, x_{n(k(j))}),$$

and so we write

$$\begin{aligned} p(z, x_{n(k(j))}) &= p(x_{n(k(j))}, Tz) \\ &\leq p(x_{n(k(j))}, Tx_{n(k(j))}) + H_p(Tz, Tx_{n(k(j))}) \\ &\leq p(x_{n(k(j))}, Tx_{n(k(j))}) + rp(z, x_{n(k(j))}). \end{aligned}$$

The above inequality leads to the following:

$$\epsilon \leq p(z, x_{n(k(j))}) \leq \frac{1}{1-r} p(x_{n(k(j))}, Tx_{n(k(j))}).$$

Passing to the limit as $j \to +\infty$, since $\lim_{n\to+\infty} p(x_n, Tx_n) = 0$, we get the contradiction $\epsilon = 0$. Consequently, we deduce that there exists $k_1 \in \mathbb{N}$ such that

$$p(z, Tx_{n(k)}) > sp(z, x_{n(k)}) \quad \text{for all } k \geq k_1.$$

Again, since $\lim_{n\to+\infty} p(x_n, Tx_n) = 0$, there exists $k_2 \geq k_1$ such that

$$p(x_{n(k)}, Tx_{n(k)}) < (s-1)\epsilon \quad \text{for all } k \geq k_2.$$

Finally, for all $k \geq k_2$, we write

$$\begin{aligned}
(s-1)\epsilon &\leq (s-1)p(z, x_{n(k)}) \\
&= sp(z, x_{n(k)}) - p(z, x_{n(k)}) \\
&< p(z, Tx_{n(k)}) - p(z, x_{n(k)}) \\
&\leq p(x_{n(k)}, Tx_{n(k)}) \quad \text{(by (P4) of Definition 1.6)} \\
&< (s-1)\epsilon,
\end{aligned}$$

which leads to contradiction. Consequently, we conclude that $\lim_{n\to+\infty} p(x_n, z) = 0$. □

5 Application to integral equations

The literature is rich with papers focusing on the study of integral operators of various types: Fredholm, Urysohn, Volterra and others. It is well known that integral operators provide an important subject of numerous mathematical investigations and are often applicable in many scientific disciplines as physics, biology and economics. The papers we refer to essentially present a fixed point approach based on the Banach contraction principle and its constructive proof. These results give sufficient conditions for establishing the existence (and uniqueness) of solution of certain integral operators, see [50–52].

Here, following this line of research, we prove an existence theorem for the solution of integral equations by using Corollary 2.6. Precisely, we consider the following integral equation:

$$u(t) = \int_0^{\mathcal{T}} K\big(t, s, u(s)\big)\, ds + g(t), \tag{9}$$

$t \in I = [0, \mathcal{T}]$, where $\mathcal{T} > 0$. Also we denote

$$C(I) := \{u : I \to \mathbb{R} \mid u \text{ is continuous on } I\},$$

and define $d : C(I) \times C(I) \to \mathbb{R}$ by

$$d(u, v) = \max_{t\in I} \big|u(t) - v(t)\big|$$

for all $u, v \in C(I)$, so that $(C(I), d)$ is a complete metric space.

Finally, we define the operator $T : C(I) \to C(I)$ by

$$Tx(t) = \int_0^{\mathcal{T}} K\big(t, s, x(s)\big)\, ds + g(t)$$

for all $x \in C(I)$ and $t \in I$.

Then we consider the following hypotheses:

(i) $K : I \times I \times \mathbb{R} \to \mathbb{R}$ and $g : I \to \mathbb{R}$ are continuous;

(ii) there exist $\alpha \in [0,1)$, $\beta \geq 1$ and a continuous function $G : I \times I \to \mathbb{R}^+$ such that

$$\max_{t\in I} \int_0^{\mathcal{T}} G(t,s)\,ds = \alpha,$$

and the inequality

$$\max_{t\in I} \left| \int_0^{\mathcal{T}} K\bigl(t,s,u(s)\bigr)\,ds + g(t) - v(t) \right| \leq \beta \max_{t\in I} \bigl|u(t) - v(t)\bigr|$$

implies

$$\begin{aligned} &\bigl|K\bigl(t,s,u(t)\bigr) - K\bigl(t,s,v(t)\bigr)\bigr| \\ &\quad \leq G(t,s) \max\biggl\{ \bigl|u(t) - v(t)\bigr|, \\ &\qquad \left| \int_0^{\mathcal{T}} K\bigl(t,s,u(s)\bigr)\,ds + g(t) - u(t) \right|, \left| \int_0^{\mathcal{T}} K\bigl(t,s,v(s)\bigr)\,ds + g(t) - v(t) \right|, \\ &\qquad \frac{1}{2}\left(\left| \int_0^{\mathcal{T}} K\bigl(t,s,v(s)\bigr)\,ds + g(t) - u(t) \right| + \left| \int_0^{\mathcal{T}} K\bigl(t,s,u(s)\bigr)\,ds + g(t) - v(t) \right| \right) \biggr\} \end{aligned}$$

for all $u, v \in C(I)$ and $t, s \in I$.

We will prove the following result.

Theorem 5.1 *Suppose that hypotheses* (i) *and* (ii) *hold. Then the integral equation* (9) *has a unique solution* $x^* \in C(I)$.

Proof First we note that the space $(C(I), d)$ is trivially a complete partial metric space with zero self-distance.

Next, for all $u, v \in C(I)$, by (ii), we deduce that

$$\begin{aligned} &\bigl|Tu(t) - Tv(t)\bigr| \\ &\quad \leq \int_0^{T} \bigl|K\bigl(t,s,u(s)\bigr) - K\bigl(t,s,v(s)\bigr)\bigr|\,ds \\ &\quad \leq \int_0^{T} G(t,s)\,ds \max\biggl\{ \bigl|u(t) - v(t)\bigr|, \bigl|u(t) - T\bigl(u(t)\bigr)\bigr|, \\ &\qquad \bigl|v(t) - T\bigl(v(t)\bigr)\bigr|, \frac{|u(t) - T(v(t))| + |v(t) - T(u(t))|}{2} \biggr\}, \end{aligned}$$

which, on routine calculations, leads to

$$d(Tu, Tv) \leq \alpha M_d(u,v).$$

Therefore, without loss of generality, we can write that

$$d(v, Tu) \leq \beta d(u,v) \quad \Longrightarrow \quad d(Tu, Tv) \leq \alpha M_d(u,v)$$

for all $u, v \in C(I)$. Thus Corollary 2.6 is applicable in this case, and hence the operator T has a unique fixed point $x^* \in C(I)$. Clearly, $x^* \in C(I)$ is the unique solution of (9). □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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