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Research article

On new subclasses of bi-starlike functions with bounded boundary rotation

Yumao Li[1], **K. Vijaya**[2], **G. Murugusundaramoorthy**[2] **and Huo Tang**[1,*]

[1] School of Mathematics and Computer Sciences, Chifeng University, Chifeng 024000, People's Republic of China

[2] Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology, Deemed to be University, Vellore-632014, India

* **Correspondence:** Email: thth2009@163.com.

Abstract: In this paper, we introduce two new classes $\mathcal{B}^{\lambda}_{\Sigma}(m,\mu)$ of λ-pseudo bi-starlike functions and $\mathcal{L}^{\eta}_{\Sigma}(m,\beta)$ to determine the bounds for $|a_2|$ and $|a_3|$, where a_2, a_3 are the initial Taylor coefficients of $f \in \mathcal{B}^{\lambda}_{\Sigma}(m,\mu)$ and $f \in \mathcal{L}^{\eta}_{\Sigma}(m,\beta)$. Also, we attain the upper bounds of the Fekete-Szegö inequality by means of the results of $|a_2|$ and $|a_3|$.

Keywords: analytic function; starlike function; convex function; bi-univalent function; bounded boundary rotation
Mathematics Subject Classification: 30C45, 30C50

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}$. Further, denote by $\mathcal{S}$ the class of all functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$ and normalized by the condition $f(0) = 0 = f'(0) - 1$.

One of the important and well examined subclasses of $\mathcal{S}$ is the class $\mathcal{S}^*(\alpha)$ of starlike functions of order $\alpha, (0 \le \alpha < 1)$, defined by the condition

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha$$

and the class $\mathcal{K}(\alpha) \subset \mathcal{S}$ of convex functions of order $\alpha, (0 \le \alpha < 1)$, is defined by the condition

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha.$$

The class $\mathcal{B}_\lambda(\alpha)$ of λ-pseudo-starlike functions of order $\alpha, (0 \le \alpha < 1)$ was introduced and investigated by Babalola [1]. A function f, $f \in \mathcal{A}$ is in the class $\mathcal{B}_\lambda(\alpha)$ if it satisfies

$$\Re\left(\frac{z(f'(z))^\lambda}{f(z)}\right) > \alpha, \qquad (\lambda > 1; z \in \mathbb{U}).$$

In [1] it was showed that all pseudo-starlike functions are Bazilevič functions of type $(1 - 1/\lambda)$ and of order $\alpha^{1/\lambda}$ and univalent in $\mathbb{U}$.

In [13] Padmanabhan and Parvatham defined the classes of functions $\mathcal{P}_m(\beta)$ as follows:

Definition 1.1. [13] Let $\mathcal{P}_m(\beta)$, with $m \ge 2$ and $0 \le \beta < 1$, denote the class of univalent analytic functions P, normalized with $P(0) = 1$, and satisfying

$$\int_0^{2\pi} \left|\frac{\operatorname{Re} P(z) - \beta}{1-\beta}\right| \mathrm{d}\,\theta \le m\pi,$$

where $z = re^{i\theta} \in \mathbb{U}$.

For $\beta = 0$, we denote $\mathcal{P}_m := \mathcal{P}_m(0)$, hence the class $\mathcal{P}_m$ represents the class of functions p analytic in $\mathbb{U}$, normalized with $p(0) = 1$, and having the representation

$$p(z) = \int_0^{2\pi} \frac{1 - ze^{it}}{1 + ze^{it}} \mathrm{d}\,\mu(t),$$

where μ is a real-valued function with bounded variation, which satisfies

$$\int_0^{2\pi} d\mu(t) = 2\pi \quad \text{and} \quad \int_0^{2\pi} |d\mu(t)| \le m,\ m \ge 2.$$

Details referring the above integral representation could be found in [13, Lemma 1]. Remark that $\mathcal{P} := \mathcal{P}_2$ is the well-known class of *Carathéodory functions*, i.e. the normalized functions with positive real part in $\mathbb{U}$.

Lemma 1.1. ([6, Lemma 2.1]) *Let the function* $\Phi(z) = 1 + \sum_{n=1}^{\infty} h_n z^n$, $z \in \mathbb{U}$, *be such that* $\Phi \in \mathcal{P}_m(\beta)$. *Then,*

$$|h_n| \le m(1-\beta),\ n \ge 1.$$

Supposing that the functions $p, q \in \mathcal{P}_m(\beta)$, with

$$p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k \quad \textit{and} \quad q(z) = 1 + \sum_{k=1}^{\infty} q_k z^k,$$

from Lemma 1.1 it follows that

$$|p_k| \le m(1-\beta), \tag{1.2}$$

$$|q_k| \le m(1-\beta), \quad \text{for all} \quad k \ge 1. \tag{1.3}$$

It is well known that every univalent function $f \in \mathcal{S}$ of the form (1.1), has an inverse $f^{-1}(w)$ defined in $\left(|w| < r_0(f); r_0(f) \geq \frac{1}{4}\right)$, where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \ldots \tag{1.4}$$

A function $f \in \mathcal{S}$ is said to be bi-univalent in $\mathbb{U}$ if there exists a function $g \in \mathcal{S}$ such that $g(z)$ is an univalent extension of f^{-1} to $\mathbb{U}$. Let Σ denote the class of bi-univalent functions in $\mathbb{U}$. The functions $\frac{z}{1-z}$, $-\log(1-z)$, $\frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$ are in the class Σ [14]. However, the familiar Koebe function is not bi-univalent. Lewin [8] investigated the class of *bi-univalent* functions Σ and obtained a bound $|a_2| \leqq 1.51$. Further Brannan and Clunie [3], Brannan and Taha [4] also worked on certain subclasses of the bi-univalent function class Σ and obtained estimates for their initial coefficients. Various classes of bi-univalent functions were introduced and studied in recent times, the study of *bi-univalent* functions gained momentum mainly due to the work of Srivastava et al. [14]. Motivated by this, many researchers [2, 5, 11, 14–20] recently investigated several interesting subclasses of the class Σ and found non-sharp estimates on the first two Taylor-Maclaurin coefficients.

Motivated by the aforementioned work on bi-univalent functions and recent works in [7, 10] ,in this paper we define two new subclasses $\mathcal{B}_\Sigma^\lambda(m,\mu)$, λ-bi-pseudo-starlike functions and $\mathcal{L}_\Sigma^\eta(m,\beta)$ of Σ and determine the bounds for the initial Taylor-Maclaurin coefficients of $|a_2|$ and $|a_3|$ for $f \in \mathcal{B}_\Sigma^\lambda(m,\mu)$ and $f \in \mathcal{L}_\Sigma^\eta(m,\beta)$.

Definition 1.2. Assume that $f \in \Sigma$, $\lambda \geq 1$ and $(f'(z))^\lambda$ is analytic in $\mathbb{U}$ with $(f'(0))^\lambda = 1$. Furthermore, assume that $g(z)$ is an univalent extension of f^{-1} to $\mathbb{U}$, and $(g'(z))^\lambda$ is analytic in $\mathbb{U}$ with $(g'(0))^\lambda = 1$. Then $f(z)$ is said to be in the class $\mathcal{B}_\Sigma^\lambda(m,\mu)$ of λ-bi-pseudo-starlike functions if the following conditions are satisfied:

$$\frac{z(f'(z))^\lambda}{(1-\mu)z + \mu f(z)} \in \mathcal{P}_m(\beta) \quad (z \in \mathbb{U}) \tag{1.5}$$

and

$$\frac{w(g'(w))^\lambda}{(1-\mu)w + \mu g(w)} \in \mathcal{P}_m(\beta) \quad (w \in \mathbb{U}), \tag{1.6}$$

where $0 \leq \mu \leq 1$.

Remark 1.1. For $\lambda = 1$, a function $f \in \Sigma$ is in the class $\mathcal{B}_\Sigma^1(m,\mu) \equiv \mathcal{M}_\Sigma(m,\mu)$ if the following conditions are satisfied:

$$\frac{zf'(z)}{(1-\mu)z + \mu f(z)} \in \mathcal{P}_m(\beta) \quad \text{and} \quad \frac{wg'(w)}{(1-\mu)w + \mu g(w)} \in \mathcal{P}_m(\beta), \tag{1.7}$$

where $z, w \in \mathbb{U}$ and the function g is described in (1.4).

Remark 1.2. For $\lambda = 1; \mu = 1$, a function $f \in \Sigma$ is in the class $\mathcal{B}_\Sigma^1(m,1) \equiv \mathcal{S}_\Sigma^*(m)$ if the following conditions are satisfied:

$$\frac{zf'(z)}{f(z)} \in \mathcal{P}_m(\beta) \quad \text{and} \quad \frac{wg'(w)}{g(w)} \in \mathcal{P}_m(\beta), \tag{1.8}$$

where $z, w \in \mathbb{U}$ and the function g is described in (1.4).

Remark 1.3. For $\lambda = 2; \mu = 1$, a function $f \in \Sigma$ is in the class $\mathcal{B}_{\Sigma}^{2}(m, 1) \equiv \mathcal{G}_{\Sigma}(m)$ if the following conditions are satisfied:

$$f'(z)\frac{zf'(z)}{f(z)} \in \mathcal{P}_m(\beta) \quad \text{and} \quad g'(w)\frac{wg'(w)}{g(w)} \in \mathcal{P}_m(\beta), \tag{1.9}$$

where $z, w \in \mathbb{U}$ and the function g is described in (1.4).

Remark 1.4. For $\mu = 0$, a function $f \in \Sigma$ is in the class $\mathcal{B}_{\Sigma}^{\lambda}(m, 0) \equiv \mathcal{R}_{\Sigma}^{\lambda}(m)$ if the following conditions are satisfied:

$$(f'(z))^{\lambda} \in \mathcal{P}_m(\beta) \quad \text{and} \quad (g'(w))^{\lambda} \in \mathcal{P}_m(\beta), \tag{1.10}$$

where $z, w \in \mathbb{U}$ and the function g is described in (1.4).

Remark 1.5. For $\lambda = 1; \mu = 0$, a function $f \in \Sigma$ is in the class $\mathcal{B}_{\Sigma}^{1}(m, 0) \equiv \mathcal{N}_{\Sigma}(m)$ if the following conditions are satisfied:

$$f'(z) \in \mathcal{P}_m(\beta) \quad \text{and} \quad g'(w) \in \mathcal{P}_m(\beta), \tag{1.11}$$

where $z, w \in \mathbb{U}$ and the function g is described in (1.4).

2. Coefficient estimates for $f \in \mathcal{B}_{\Sigma}^{\lambda}(m, \mu)$

Theorem 2.1. *Let $f(z)$ given by* (1.1) *be in the class $\mathcal{B}_{\Sigma}^{\lambda}(m, \mu)$, then*

$$|a_2| \quad \leq \quad \min\left\{\frac{m(1-\beta)}{2\lambda-\mu}; \sqrt{\frac{m(1-\beta)}{2\lambda^2+\lambda(1-2\mu)-\mu(1-\mu)}}\right\}, \tag{2.1}$$

$$\begin{aligned} |a_3| \leq \min\Bigg\{ & \frac{m(1-\beta)}{3\lambda-\mu} + \frac{m(1-\beta)}{[2\lambda^2+\lambda(1-2\mu)-\mu(1-\mu)]}; \\ & \frac{m(1-\beta)}{3\lambda-\mu}\left(1+\frac{m(1-\beta)\left(2\lambda^2-2\lambda(\mu+1)+\mu^2\right)}{(2\lambda-\mu)^2}\right); \\ & \frac{m(1-\beta)}{3\lambda-\mu}\left(1+\frac{m(1-\beta)\left(2\lambda^2+(2\lambda-\mu)(2-\mu)\right)}{(2\lambda-\mu)^2}\right)\Bigg\}, \end{aligned} \tag{2.2}$$

and

$$|a_3 - \delta a_2^2| \leq \frac{m(1-\beta)}{3\lambda-\mu},$$

where

$$\delta = \frac{2\lambda^2+(2\lambda-\mu)(2-\mu)}{3\lambda-\mu}.$$

Proof. It is known that g has the form

$$g(w) = w - a_2w^2 + \left(2a_2^2 - a_3\right)w^3 - \left(5a_2^3 - 5a_2a_3 + a_4\right)w^4 + \cdots .$$

Since $f \in \mathcal{B}_{\Sigma}^{\lambda}(m,\mu)$, there exists two analytic functions

$$p(z) := 1 + p_1 z + p_2 z^2 + \cdots \tag{2.3}$$

and

$$q(w) := 1 + q_1 w + q_2 w^2 + \cdots , \tag{2.4}$$

then

$$\frac{z[f'(z)]^{\lambda}}{(1-\mu)z + \mu f(z)} = p(z), \tag{2.5}$$

$$\frac{w[g'(w)]^{\lambda}}{(1-\mu)w + \mu g(w)} = q(w). \tag{2.6}$$

On the other hand, we have

$$\frac{z[f'(z)]^{\lambda}}{(1-\mu)z + \mu f(z)} = 1 + (2\lambda - \mu)a_2 z + [\left(2\lambda^2 - 2\lambda(\mu+1) + \mu^2\right)a_2^2 + (3\lambda - \mu)a_3]z^2 + \cdots , \tag{2.7}$$

$$\frac{w[g'(w)]^{\lambda}}{(1-\mu)w + \mu g(w)} = 1 - (2\lambda - \mu)a_2 w + [\left(2\lambda^2 + (2\lambda - \mu)(2-\mu)\right)a_2^2 - (3\lambda - \mu)a_3]w^2 + \cdots . \tag{2.8}$$

Using (2.3), (2.4), (2.7) and (2.8) and comparing the like coefficients of z and z^2 , we get

$$(2\lambda - \mu)a_2 \;=\; p_1, \tag{2.9}$$

$$\left(2\lambda^2 - 2\lambda(\mu+1) + \mu^2\right)a_2^2 + (3\lambda - \mu)a_3 = p_2, \tag{2.10}$$

$$-(2\lambda - \mu)a_2 \;=\; q_1, \tag{2.11}$$

$$\left(2\lambda^2 + (2\lambda - \mu)(2-\mu)\right)a_2^2 - (3\lambda - \mu)a_3 = q_2. \tag{2.12}$$

From (2.9) and (2.11), we find that

$$a_2 = \frac{p_1}{2\lambda - \mu} = \frac{-q_1}{2\lambda - \mu}; \tag{2.13}$$

from Lemma 1.1 it follows that

$$|a_2| \le \frac{m(1-\beta)}{2\lambda - \mu}. \tag{2.14}$$

Adding (2.10) and (2.12), we have

$$\left[4\lambda^2 + 2\lambda(1-2\mu) - 2\mu(1-\mu)\right]a_2^2 = p_2 + q_2, \tag{2.15}$$

$$a_2^2 = \frac{p_2 + q_2}{4\lambda^2 + 2\lambda(1-2\mu) - 2\mu(1-\mu)}.$$

Hence by Lemma 1.1

$$|a_2|^2 \leq \frac{2m(1-\beta)}{2\left[2\lambda^2+\lambda(1-2\mu)-\mu(1-\mu)\right]},$$

$$|a_2| \leq \sqrt{\frac{m(1-\beta)}{2\lambda^2+\lambda(1-2\mu)-\mu(1-\mu)}}. \tag{2.16}$$

Subtracting (2.10) from (2.12), we obtain

$$\begin{aligned} a_3 &= \frac{(p_2-q_2)}{2(3\lambda-\mu)}+a_2^2, \\ |a_3| &\leq \frac{m(1-\beta)}{3\lambda-\mu}+|a_2|^2 \\ &= \frac{m(1-\beta)}{3\lambda-\mu}+\frac{m(1-\beta)}{\left[2\lambda^2+\lambda(1-2\mu)-\mu(1-\mu)\right]}. \end{aligned}$$

By using (2.9) and (2.10) and by simple computation, we get

$$|a_3| \leq \frac{m(1-\beta)}{3\lambda-\mu}\left(1+\frac{m(1-\beta)\left(2\lambda^2-2\lambda(\mu+1)+\mu^2\right)}{(2\lambda-\mu)^2}\right). \tag{2.17}$$

Again by using (2.9) and (2.12)

$$|a_3| \leq \frac{m(1-\beta)}{3\lambda-\mu}\left(1+\frac{m(1-\beta)\left(2\lambda^2+(2\lambda-\mu)(2-\mu)\right)}{(2\lambda-\mu)^2}\right). \tag{2.18}$$

From (2.12) we have

$$\frac{\left(2\lambda^2+(2\lambda-\mu)(2-\mu)\right)}{3\lambda-\mu}a_2^2-a_3=\frac{q_2}{3\lambda-\mu}.$$

Furthermore by

$$|a_3-\delta a_2^2|=\frac{|q_2|}{3\lambda-\mu} \leq \frac{m(1-\beta)}{3\lambda-\mu},$$

where

$$\delta=\frac{2\lambda^2+(2\lambda-\mu)(2-\mu)}{3\lambda-\mu}.$$

This completes the proof of Theorem 2.1. □

Remark 2.1. Specializing λ, μ suitably as mentioned in Remarks 1.1 to 1.5 we can state the initial Taylor coefficients $|a_2|$, $|a_3|$ and the inequality $|a_3-\delta a_2^2|$ for the function classes defined in Remarks 1.1 to 1.5.

3. Coefficient estimates for $f \in \mathcal{L}_{\Sigma}^{\eta}(m,\beta)$

In [12], Obradovic et al. gave some criteria for univalence expressing by $\Re(f'(z)) > 0$, for the linear combinations

$$\eta\left(1+\frac{zf''(z)}{f'(z)}\right)+(1-\eta)\frac{1}{f'(z)}, \qquad (\eta \geq 1, z \in \mathbb{U}).$$

Based on the above definition recently, in [9], Lashin introduced and studied the new subclass of bi-univalent functions. We define the following new bi-univalent function class:

Definition 3.1. A function $f(z) \in \Sigma$ given by (1.1) is said to be in the class $\mathcal{L}_{\Sigma}^{\eta}(m,\beta)$ if it satisfies the following conditions :

$$\eta\left(1+\frac{zf''(z)}{f'(z)}\right)+(1-\eta)\frac{1}{f'(z)} \in \mathcal{P}_m(\beta) \tag{3.1}$$

and

$$\eta\left(1+\frac{wg''(z)}{g'(w)}\right)+(1-\eta)\frac{1}{g'(w)} \in \mathcal{P}_m(\beta), \tag{3.2}$$

where $\eta \geq 1, z, w \in \mathbb{U}$ and the function g is given by (1.4).

Theorem 3.1. *Let $f(z)$ be given by* (1.1) *be in the class $\mathcal{L}_{\Sigma}^{\eta}(m,\beta)$, $\eta \geq 1$. Then*

$$|a_2| \quad \leq \quad \min\left\{\frac{m(1-\beta)}{2(2\eta-1)}; \sqrt{\frac{m(1-\beta)}{\eta+1}}\right\}, \tag{3.3}$$

$$\begin{aligned}|a_3| \leq \min\Bigg\{&\frac{m(1-\beta)}{3(3\eta-1)}+\frac{m(1-\beta)}{1+\eta};\ \frac{m(1-\beta)}{3(3\eta-1)}\left(1-\frac{m(1-\beta)}{2\eta-1}\right);\\ &\frac{m(1-\beta)}{3(3\eta-1)}\left(1+\frac{m(1-\beta)(5\eta-1)}{2(1-2\eta)^2}\right)\Bigg\},\end{aligned} \tag{3.4}$$

and

$$|a_3-\rho a_2^2| = \frac{|q_2|}{3(3\eta-1)} \leq \frac{m(1-\beta)}{3(3\eta-1)},$$

where

$$\rho = \frac{2(5\eta-1)}{3(3\eta-1)}.$$

Proof. It follows from (3.1) and (3.2) that

$$\eta\left(1+\frac{zf''(z)}{f'(z)}\right)+(1-\eta)\frac{1}{f'(z)} \in \mathcal{P}_m(\beta) \tag{3.5}$$

and

$$\eta\left(1+\frac{wg''(z)}{g'(w)}\right)+(1-\eta)\frac{1}{g'(w)} \in \mathcal{P}_m(\beta). \tag{3.6}$$

From (3.5) and (3.6), we have

$$1 + 2(2\eta - 1)a_2 z + \left[3(3\eta - 1)a_3 - 4(2\eta - 1)a_2^2\right] z^2 + \cdots$$
$$= 1 + p_1 z + p_2 z^2 + \cdots$$

and

$$1 - 2(2\eta - 1)a_2 w + \left[(10\eta - 2)a_2^2 - 3(3\eta - 1)a_3\right] w^2 - \cdots$$
$$= 1 + q_1 w + q_2 w^2 + \cdots .$$

Now, equating the coefficients, we get

$$(2\eta - 1)a_2 = p_1, \tag{3.7}$$

$$3(3\eta - 1)a_3 + 4(1 - 2\eta)a_2^2 = p_2, \tag{3.8}$$

$$-2(2\eta - 1)a_2 = q_1 \tag{3.9}$$

and

$$(10\eta - 2)a_2^2 - 3(3\eta - 1)a_3 = q_2. \tag{3.10}$$

From (3.7) and (3.9), we get

$$a_2 = \frac{p_1}{2(2\eta - 1)} = \frac{-q_1}{2(2\eta - 1)}; \tag{3.11}$$

it follows that

$$|a_2| \leq \frac{m(1 - \beta)}{2(2\eta - 1)}. \tag{3.12}$$

Now by adding (3.8) and (3.10), we obtain

$$2(\eta + 1)a_2^2 = p_2 + q_2, \tag{3.13}$$

$$a_2^2 = \frac{p_2 + q_2}{2(\eta + 1)},$$

which, by virtue of Lemma 1.1, implies that

$$|a_2|^2 \leq \frac{m(1 - \beta)}{\eta + 1}.$$

Hence

$$|a_2| \leq \sqrt{\frac{m(1 - \beta)}{\eta + 1}}. \tag{3.14}$$

Subtracting (3.10) from (3.8), we obtain

$$a_3 = \frac{(p_2 - q_2)}{6(3\eta - 1)} + a_2^2,$$
$$|a_3| \leq \frac{m(1 - \beta)}{3(3\eta - 1)} + |a_2|^2$$

$$= \frac{m(1-\beta)}{3(3\eta-1)} + \frac{m(1-\beta)}{1+\eta}.$$

By using (3.7) and (3.8) and by simple computation, we get

$$|a_3| \leq \frac{m(1-\beta)}{3(3\eta-1)}\left(1 - \frac{m(1-\beta)}{2\eta-1}\right). \tag{3.15}$$

Again by using (3.7) in (3.10)

$$|a_3| \leq \frac{m(1-\beta)}{3(3\eta-1)}\left(1 + \frac{m(1-\beta)(5\eta-1)}{2(1-2\eta)^2}\right). \tag{3.16}$$

From (3.10) we have

$$\frac{2(5\eta-1)}{3(3\eta-1)}a_2^2 - a_3 = \frac{q_2}{3(3\eta-1)}.$$

Furthermore by

$$|a_3 - \rho a_2^2| = \frac{|q_2|}{3(3\eta-1)} \leq \frac{m(1-\beta)}{3(3\eta-1)},$$

where

$$\rho = \frac{2(5\eta-1)}{3(3\eta-1)}.$$

This completes the proof of Theorem 3.1. □

Corollary 3.2. *Let $f(z)$ be given by* (1.1) *be in the class $\mathcal{L}_\Sigma^\eta(m,\beta)$, $\eta = 1$. Then*

$$|a_2| \leq \min\left\{\frac{m(1-\beta)}{2}; \sqrt{\frac{m(1-\beta)}{2}}\right\},$$

$$|a_3| \leq \min\left\{\frac{3m(1-\beta)}{2}; \frac{m(1-\beta)}{6}(1 - m(1-\beta)); \frac{m(1-\beta)}{6}(1 + 2m(1-\beta))\right\}$$

and

$$|a_3 - \rho a_2^2| = \frac{|q_2|}{6} \leq \frac{m(1-\beta)}{6},$$

where

$$\rho = \frac{4}{3}.$$

4. Conclusion

In this paper, we introduce two new classes $\mathcal{B}_\Sigma^\lambda(m,\mu)$ of λ-pseudo bi-starlike functions and $\mathcal{L}_\Sigma^\eta(m,\beta)$ and obtain the estimates of $|a_2|$, $|a_3|$ and the upper bounds of the Fekete-Szegö inequality, where a_2 and a_3 belong to $f \in \mathcal{B}_\Sigma^\lambda(m,\mu)$ and $f \in \mathcal{L}_\Sigma^\eta(m,\beta)$, respectively. In addition, we observe that, if we choose some suitable parameters λ, μ, η and m in the results involved, we can get some corresponding bounds.

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Conflict of interest

The authors declare no conflicts of interest.

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