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ON NONSMOOTH MULTIOBJECTIVE FRACTIONAL PROGRAMMING PROBLEMS INVOLVING $(p,r)-\rho-(\eta,\theta)$ - INVEX FUNCTIONS

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Abstract: A class of multiobjective fractional programming problems (MFP) is considered where the involved functions are locally Lipschitz. In order to deduce our main results, we introduce the definition of $(p,r)-\rho-(\eta,\theta)$-invex class about the Clarke generalized gradient. Under the above invexity assumption, sufficient conditions for optimality are given. Finally, three types of dual problems corresponding to (MFP) are formulated, and appropriate dual theorems are proved.

Keywords: Multiobjective fractional programming, Clarke gradient, $(p,r)-\rho-(\eta,\theta)$-invexity, efficiency, sufficient optimality conditions, duality theorems.

MSC: 90C32, 90C46, 49N15.

1. INTRODUCTION

In recent past, optimality conditions and duality results have been of much interest for a class of multiobjective fractional programming problems, where the involved functions are locally Lipschitz and have Clarke differentiability. Many researchers have studied this matter in the presence of various assumptions. See, for example, [4, 5, 10, 11, 13-16] and the references cited therein.

Jeyakumar [9] gave the optimality and duality for nondifferentiable nonconvex program under the ρ-invexity assumptions. Chen [4] studied the optimality and duality aspects of a class of nonsmooth multiobjective fractional programming problems expressed in terms of the Clarke generalized gradient under certain generalized (F,ρ)-convexity assumptions.

In particular, using parametric approach, Bector *et al.* [2] derived Fritz John and Karush-Kuhn-Tucker necessary and sufficient optimality conditions for a class of nonsmooth (nondifferentiable) convex multiobjective fractional optimization problems, and established some duality theorems. Following the approaches of Bector *et al*. [2], Liu [11] obtained necessary and sufficient optimality conditions and derived duality theorems for a class of nonsmooth multiobjective fractional programming problems involving nonsmooth (F,ρ)-convex functions.

Recently, Mandal and Nahak [12] introduced a new class of $(p,r)-\rho-(\eta,\theta)$-invex functions by combining the concepts of (p,r)-invexity [1] and the notions of $\rho-(\eta,\theta)$-invex functions [17] and they established symmetric duality results. Jayswal *et al*. [8] considered a class of functions called $(p,r)-\rho-(\eta,\theta)$-invex functions for a multiobjective fractional programming problem with inequality constraints in the differentiable case and derived sufficient conditions and duality theorems. However, the corresponding conclusions cannot be obtained for nondifferentiable programming with the help of generalized $(p,r)-\rho-(\eta,\theta)$-invex functions because the derivative is required in the definitions of such functions. There exists a generalization of convexity to locally Lipschitz functions, with derivative replaced by the Clarke generalized gradient.

Consequently, in the present paper, we are concerned with the following nondifferentiable multiobjective fractional programming problem:

(MFP) Minimize $\left(\frac{f_1(x)}{g_1(x)},\ldots,\frac{f_k(x)}{g_k(x)}\right)$

subject to

$$\Omega=\left\{x\in X: h(x)=(h_1,h_2,\ldots,h_m)(x)\in -R_+^m\right\}, \tag{1}$$

$$x\in X\subset R^n,$$

where X is a separable reflexive Banach space in the n-dimensional Euclidean space R^n. $f_i,\ g_i : X \to R,\ i = 1,2,...,k$ and $h : X \to R^m$ are locally Lipschitz functions on X. Without loss of generality, we can assume that $f_i(x) \geqq 0,\ g_i(x) > 0$, for all $i = 1,2,...,k$ and $x \in X$.

Definition 1.1 *A feasible solution* x^* *of* (MFP) *is said to be an efficient solution of* (MFP) *if and only if there exist no other feasible solution* $x \in \Omega$ *such that*

$$\frac{f_i(x)}{g_i(x)} \leqq \frac{f_i(x^*)}{g_i(x^*)} \text{ for all } i = 1,2,...,k,$$

and

$$\frac{f_t(x)}{g_t(x)} < \frac{f_t(x^*)}{g_t(x^*)} \text{ for some } t \in \{1,2,...,k\}.$$

We extend the $(p,r)-\rho-(\eta,\theta)$-invex functions to the case of nondifferentiable functions. In other words, we define a kind of $(p,r)-\rho-(\eta,\theta)$-invexity about the Clarke generalized gradient. Based upon these generalized invex functions, we derive sufficient optimality conditions for multiobjective fractional programming problems, formulate three different types of dual models, and establish weak, strong and strict converse duality theorems.

2. NOTATIONS AND PRELIMINARIES

Throughout the paper, let R^n be n-dimensional Euclidean space and R_+^n denote the order cone. For cone partial order, if $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n) \in R^n$, we define

$x \leqq y$ if and only if $x_i \leq y_i$ for all $i = 1,2,...,n$;

$x \leq y$ if and only if $x_i \leqq y_i$ for all $i = 1,2,...,n$ and $x \neq y$;

$x < y$ if and only if $x_i < y_i$ for all $i = 1,2,...,n$;

$x \nless y$ is the negation of $x < y$.

Definition 2.1 [6] *Let X be an open subset of* R_+^n. *The function* $f : X \to R$ *is said to be locally Lipschitz (of rank K) at* $x \in X$, *if there exist a positive constant* K *and a neighbourhood N of x such that, for any* $y, z \in N$,

$$|f(y) - f(z)| \leqq K\|y - z\|,$$

where $\|\ \|$ *denote any norm of X.*

Definition 2.2 [6] *If* $f : X \to R$ *is locally Lipschitz at* $x \in X$, *the generalized derivative (in the sense of Clarke) of* f *at* $x \in X$ *in the direction* d, *denoted by* $f^0(x;d)$, *is given by*

$$f^0(x;d) = \limsup_{\substack{y \to x \\ t \downarrow 0}} \frac{f(y+td) - f(y)}{t}.$$

Definition 2.3 [6] *The generalized gradient of* f *at* $x \in X$, *denoted by* $\partial f(x)$, *is defined as follows:*

$$\partial f(x) = \left\{ \xi \in X^* : f^0(x;d) \geq \langle \xi, d \rangle \ \forall d \in X \right\},$$

where X^* *is the dual space of* X *and* $\langle .,. \rangle$ *stands for the dual pair of* X *and* X^*.

Let $h : X \to R^m$ *be a locally Lipschitz function. For* $x_0 \in X$, *we define*

$$J(x_0) = \left\{ j \in J : h_j(x_0) = 0 \right\},\ J = \{1, 2, ..., m\},$$

$$\Lambda = \left\{ v \in X : h_j^0(x_0, v) < 0,\ j \in J(x_0) \right\}.$$

If $\Lambda \neq \phi$, *we say that the problem* (MFP) *has constraint qualification at* x_0 (cf. [7]).

In the following, we introduce a definition which generalizes to nondifferentiable case the concepts of $(p,r)-\rho-(\eta,\theta)$-invex functions.

Definition 2.4 *Let* $f : X \to R$ *be a locally Lipschitz function and let* p, r *be arbitrary real numbers. If there exist* $\eta, \theta : X \times X \to X$ *and* $\rho \in R$ *such that the relations hold:*

$$\frac{1}{r}(e^{r(f(x)-f(u))} - 1) \geqq \frac{1}{p}\xi^T(e^{p\eta(x,u)} - \mathbf{1}) + \rho \left\| \theta(x,u) \right\|^2$$

$$(> \ with \ x \neq u) \text{ for } p \neq 0,\ r \neq 0,$$

$$\frac{1}{r}(e^{r(f(x)-f(u))} - 1) \geqq \xi^T \eta(x,u) + \rho \left\| \theta(x,u) \right\|^2$$

$$(> \ with \ x \neq u) \text{ for } p = 0,\ r \neq 0,$$

$$f(x) - f(u) \geqq \frac{1}{p}\xi^T(e^{p\eta(x,u)} - \mathbf{1}) + \rho \left\| \theta(x,u) \right\|^2$$

$$(> \ with \ x \neq u) \text{ for } p \neq 0,\ r = 0,$$

$$f(x) - f(u) \geqq \xi^T \eta(x,u) + \rho \left\| \theta(x,u) \right\|^2$$

$$(> \ with \ x \neq u) \text{ for } p = 0,\ r = 0,$$

$\forall \xi \in \partial f(u)$, *then* f *is said to be (strictly)* $(p,r)-\rho-(\eta,\theta)$*-invex at the point* $u \in X$ *with respect to* η *and* θ.

Remark 2.1 If the above inequalities are satisfied at any point $u \in X$, then f is said to be (strictly) $(p,r)-\rho-(\eta,\theta)$-invex on X with respect to η and θ.

Remark 2.2 It should be noted that the exponentials appearing on the right-hand sides of inequalities above are understood to be taken componentwise and $\mathbf{1} = (1,1,...,1) \in R^n$.

In the sequel, we need the following results.

Lemma 2.1 [3] *The point* $\bar{x}$ *is an optimal solution to problem* (MFP) *if and only if* $\bar{x}$ *solves* (SFP$_i$), *where* (SFP$_i$) *is given as the following problems:*

$$\textbf{(SFP}_\textbf{i}\textbf{)} \qquad \text{Minimize} \quad \frac{f_i(x)}{g_i(x)}$$

subject to

$$x \in M_i = \left\{ x \in X : \frac{f_p(x)}{g_p(x)} \leqq \frac{f_p(\bar{x})}{g_p(\bar{x})} \underset{\equiv}{\text{notation}}\ \varphi_p(\bar{x}),\ p \neq i,\ p = 1,2,...,k,\ h(x) \in -R_+^m \right\}$$

$$= \left\{ x \in X : f_p(x) - \varphi_p(\bar{x}) g_p(x) \leqq 0,\ p \neq i,\ p = 1,2,...,k,\ h(x) \in -R_+^m \right\}.$$

Theorem 2.1 (Necessary optimality conditions) [11]. *If* x^* *is an optimal solution of* (MFP) *and satisfies a constraint qualification* [7, *Theorem* 12] *for* (SFP$_i$), $i = 1,2,...,k$, *then there exist* $y^* \in R_+^k$, $z^* \in R^m$, *such that*

$$0 \in \sum_{i=1}^{k} y_i^* \left[\partial f_i(x^*) + \varphi_i(x^*) \partial(-g_i)(x^*) \right] + \sum_{j=1}^{m} z_j^* \partial h_j(x^*), \tag{2}$$

$$f_i(x^*) - \varphi_i(x^*) g_i(x^*) = 0, \text{ for all } i = 1,2,...,k, \tag{3}$$

$$z_j^* h_j(x^*) = 0, \text{ for all } j = 1,2,...,m, \tag{4}$$

$$h_j(x^*) \leqq 0, \text{ for all } j = 1,2,...,m, \tag{5}$$

$$y^* \in I,\ z^* \in R_+^m, \tag{6}$$

where $I = \left\{ y^* \in R^k : y^* = (y_1^*, y_2^*, ..., y_k^*) > 0 \text{ and } \sum_{i=1}^{k} y_i^* = 1 \right\}$.

Remark 2.3 All the theorems in the subsequent parts of this paper will be proved only in the case when $p \neq 0$, $r \neq 0$. The proofs in other cases are easier than in this one, since the differences arise only from the form of inequality. Moreover, without loss of generality, we assume that $p>0$ and $r>0$. Furthermore, we assume that ρ, ρ^1 and ρ^2 are all elements of R.

3. SUFFICIENT OPTIMALITY CONDITIONS

Now we establish sufficient optimality conditions under introduced classes of functions defined in the previous section.

Theorem 3.1 (Sufficiency). *Let* $x^* \in \Omega$ *be a feasible solution for* (MFP) *to which conditions* (2) *to* (6) *are satisfied. Moreover, we assume that any one of the following conditions holds:*

a) $\rho \geqq 0$ *and* $A(.) = \sum_{i=1}^{k} y_i^* \left[f_i(.) - \varphi_i(x^*) g_i(.) \right] + \sum_{j=1}^{m} z_j^* h_j(.)$ *is* $(p,r)-\rho\ -(\eta,\theta)$*- invex at* x^* *with respect to* η *and* θ;

b) $\rho^1 + \rho^2 \geqq 0$ *and* $B(.) = \sum_{i=1}^{k} y_i^* \left[f_i(.) - \varphi_i(x^*) g_i(.) \right]$ *is* $(p,r)-\rho^1-(\eta,\theta)$*- invex at* x^* *with respect to* η *and* θ *and* $C(.) = \sum_{j=1}^{m} z_j^* h_j(.)$ *is* $(p,r)-\rho^2-(\eta,\theta)$*-invex at* x^* *with respect to same* η *and* θ.

Then x^* *is an efficient solution to* (MFP).

Proof. Let $x^* \in \Omega$ be a feasible solution to (MFP). By the relation (2), there exist $\xi_i \in \partial f_i(x^*)$, $\zeta_i \in \partial(-g_i)(x^*)$, $i=1,2,...,k$ and $\psi_j \in \partial h_j(x^*)$, $j=1,2,...,m$, such that

$$\sum_{i=1}^{k} y_i^* \left[\xi_i + \varphi_i(x^*)\zeta_i \right] + \sum_{j=1}^{m} z_j^* \psi_j = 0. \tag{7}$$

If condition (a) holds, then

$$\frac{1}{r}\left(e^{r\left(A(x)-A(x^*)\right)} - 1 \right) \geqq \frac{1}{p}\left[\sum_{i=1}^{k} y_i^* \left[\xi_i + \varphi_i(x^*)\zeta_i \right] + \sum_{j=1}^{m} z_j^* \psi_j \right]\left(e^{p\eta(x,x^*)} - \mathbf{1} \right) + \rho \left\| \theta(x,x^*) \right\|^2$$

The above inequality together with (7) and the assumption that $\rho \geqq 0$, imply

$$\frac{1}{r}\left(e^{r\left(A(x)-A(x^*)\right)} - 1 \right) \geqq 0.$$

Using fundamental property of the exponential function, we get

$$A(x) \geqq A(x^*). \tag{8}$$

On the other hand, suppose contrary to the result that x^* is not an efficient solution of (MFP). Then, there exists a feasible solution x of (MFP) such that

$$\frac{f_i(x)}{g_i(x)} \leqq \frac{f_i(x^*)}{g_i(x^*)} \text{ for all } i = 1,2,...,k,$$

$$\frac{f_t(x)}{g_t(x)} < \frac{f_t(x^*)}{g_t(x^*)} \text{ for some } t \in \{1,2,...,k\},$$

that is,

$$f_i(x) - \varphi_i(x^*) g_i(x) \leqq 0 = f_i(x^*) - \varphi_i(x^*) g_i(x^*) \text{ for all } i = 1,2,...,k,$$

$$f_t(x) - \varphi_t(x^*) g_t(x) < 0 = f_t(x^*) - \varphi_t(x^*) g_t(x^*) \text{ for some } t \in \{1,2,...,k\}.$$

By (6) and the above inequalities, we have

$$\sum_{i=1}^{k} y_i^* \left[f_i(x) - \varphi_i(x^*) g_i(x) \right] < \sum_{i=1}^{k} y_i^* \left[f_i(x^*) - \varphi_i(x^*) g_i(x^*) \right]. \tag{9}$$

From the relation $h(x) \in -R_+^m$, (4) and (6), we get

$$\sum_{j=1}^{m} z_j^* h_j(x) \leqq \sum_{j=1}^{m} z_j^* h_j(x^*). \tag{10}$$

On adding (9) and (10), we obtain

$$\sum_{i=1}^{k} y_i^* \left[f_i(x) - \varphi_i(x^*) g_i(x) \right] + \sum_{j=1}^{m} z_j^* h_j(x) < \sum_{i=1}^{k} y_i^* \left[f_i(x^*) - \varphi_i(x^*) g_i(x^*) \right] + \sum_{j=1}^{m} z_j^* h_j(x^*),$$

i.e.,

$$A(x) < A(x^*),$$

which contradicts (8).

If condition (b) holds, from the $(p,r) - \rho^2 - (\eta,\theta)$-invexity of $C(.)$ we get

$$\frac{1}{r}\left(e^{r\left(C(x) - C(x^*)\right)} - 1 \right) \geqq \frac{1}{p}\left[\sum_{j=1}^{m} z_j^* \psi_j \right]\left(e^{p\eta(x,x^*)} - \mathbf{1} \right) + \rho^2 \left\| \theta(x,x^*) \right\|^2, \ \forall \psi_j \in \partial h_j(x^*). \tag{11}$$

From (10) and (11) and the fundamental property of the exponential function, we obtain

$$\frac{1}{p}\left[\sum_{j=1}^{m} z_j^* \psi_j \right]\left(e^{p\eta(x,x^*)} - \mathbf{1} \right) + \rho^2 \left\| \theta(x,x^*) \right\|^2 \leqq 0, \ \forall \psi_j \in \partial h_j(x^*). \tag{12}$$

By (2), (12) and the assumption that $\rho^1 + \rho^2 \geqq 0$, we obtain

$$\frac{1}{p}\left[\sum_{i=1}^{k} y_i^*\left[\xi_i+\varphi_i(x^*)\zeta_i\right]\right]\left(e^{p\eta\left(x,x^*\right)}-\mathbf{1}\right)+\rho^1\left\|\theta\left(x,x^*\right)\right\|^2 \geqq 0 . \tag{13}$$

From the $(p,r)-\rho^1-(\eta,\theta)$ -invexity of $B(.)$, we have

$$\frac{1}{r}\left(e^{r\left(B(x)-B\left(x^*\right)\right)}-1\right) \geqq \frac{1}{p}\left[\sum_{i=1}^{k} y_i^*\left[\xi_i+\varphi_i(x^*)\zeta_i\right]\right]\left(e^{p\eta\left(x,x^*\right)}-\mathbf{1}\right)+\rho^1\left\|\theta\left(x,x^*\right)\right\|^2 . \tag{14}$$

From (13) and (14), we obtain

$$B(x) \geqq B(x^*),$$

i.e.,

$$\sum_{i=1}^{k} y_i^*\left[f_i(x)-\varphi_i(x^*)g_i(x)\right] \geqq \sum_{i=1}^{k} y_i^*\left[f_i(x^*)-\varphi_i(x^*)g_i(x^*)\right]. \tag{15}$$

If x^* is not an efficient solution of (MFP) then, we get (9) in the same way. But (9) contradicts (15). Therefore, x^* is an efficient solution of (MFP). This completes the proof. □

4. PARAMETRIC DUALITY

We consider the following form of parametric dual as follows:

(PD) Maximize $v=(v_1,v_2,...,v_k)$

subject to

$$0 \in \sum_{i=1}^{k} y_i\left[\partial f_i(u)+v_i\partial(-g_i)(u)\right]+\sum_{j=1}^{m} z_j\partial h_j(u), \tag{16}$$

$$f_i(u)-v_i g_i(u)=0, \quad \text{for all } i=1,2,...,k, \tag{17}$$

$$\sum_{j=1}^{m} z_j h_j(u)=0, \tag{18}$$

$$u\in X,\ y\in I,\ z\in R_+^m,\ v \geqq 0. \tag{19}$$

Let $\Gamma=\Big\{(u,y,z,v)\in X\times I\times R_+^m\times R^k : 0\in \sum_{i=1}^{k} y_i\left[\partial f_i(u)+v_i\partial(-g_i)(u)\right]+\sum_{j=1}^{m} z_j\partial h_j(u),$

$f_i(u)-v_i g_i(u)=0,\ \text{for all } i=1,2,...,k,\ \sum_{j=1}^{m} z_j h_j(u)=0\Big\}$

denote the set of all feasible solutions of (PD). Moreover, we denote by $pr_X\Gamma$ the projection of the set Γ on X , i.e. $pr_X\Gamma = \{u \in X : (u, y, z, v) \in \Gamma\}$.

Theorem 4.1 (Weak duality). *Let* x *be a feasible solution for* (MFP), *and let* (u, y, z, v) *be a feasible solution for* (PD). *Moreover, we assume that any of the following condition holds:*

a) $\rho \geqq 0$ *and* $O(.) = \sum_{i=1}^{k} y_i \left[f_i(.) - v_i g_i(.) \right] + \sum_{j=1}^{m} z_j h_j(.)$ is $(p,r)-\rho-(\eta,\theta)$-*invex at* $u \in \Omega \cup pr_X\Gamma$ *with respect to* η *and* θ;

b) $\rho^1 + \rho^2 \geqq 0$ *and* $P(.) = \sum_{i=1}^{k} y_i \left[f_i(.) - v_i g_i(.) \right]$ *is* $(p,r)-\rho^1-(\eta,\theta)$-*invex and* $Q(.) = \sum_{j=1}^{m} z_j h_j(.)$ *is* $(p,r)-\rho^2-(\eta,\theta)$-*invex at* $u \in \Omega \cup pr_X\Gamma$ *with respect to same* η *and* θ.

Then $\varphi(x) \nleq y$.

Proof. Let x and (u, y, z, v) *be* feasible solution to (MFP) and (PD), respectively. Then there exist $\xi_i \in \partial f_i(u),\ \zeta_i \in \partial(-g_i)(u),\ i = 1,2,...,k$ and $\psi_j \in \partial h_j(u),\ j = 1,2,...,m$, such that

$$\sum_{i=1}^{k} y_i \left[\xi_i + v_i \zeta_i \right] + \sum_{j=1}^{m} z_j \psi_j = 0 . \tag{20}$$

If condition (a) holds, then

$$\frac{1}{r}\left(e^{r(O(x)-O(u))} - 1\right) \geqq \frac{1}{p}\left[\sum_{i=1}^{k} y_i \left[\xi_i + v_i \zeta_i\right] + \sum_{j=1}^{m} z_j \psi_j\right]\left(e^{p\eta(x,u)} - \mathbf{1}\right) + \rho \left\|\theta(x,u)\right\|^2 .$$

The above inequality together with (20) and the assumption that $\rho \geqq 0$, imply

$$\frac{1}{r}\left(e^{r(O(x)-O(u))} - 1\right) \geqq 0.$$

Using fundamental property of the exponential function, we get

$$O(x) \geqq O(u). \tag{21}$$

On the other hand, suppose contrary to the result that $\varphi(x) \leq v$. Then

$$\frac{f_i(x)}{g_i(x)} \leqq v_i \text{ for all } i = 1,2,...,k,$$

$$\frac{f_t(x)}{g_t(x)} < v_t \text{ for some } t \in \{1,2,...,k\},$$

that is,

$$f_i(x)-v_i g_i(x) \leqq 0 = f_i(u)-v_i g_i(u) \quad \text{for all } i=1,2,...,k,$$

$$f_t(x)-v_t g_t(x) < 0 = f_t(u)-v_t g_t(u) \quad \text{for some } t \in \{1,2,...,k\}.$$

By (19) and the above inequalities, we have

$$\sum_{i=1}^{k} y_i\left[f_i(x)-v_i g_i(x)\right] < \sum_{i=1}^{k} y_i\left[f_i(u)-v_i g_i(u)\right]. \tag{22}$$

From the relation $h(x)\in -R_+^m$, (18) and (19), yield

$$\sum_{j=1}^{m} z_j h_j(x) \leqq \sum_{j=1}^{m} z_j h_j(u). \tag{23}$$

Adding (22) and (23), we obtain

$$\sum_{i=1}^{k} y_i\left[f_i(x)-v_i g_i(x)\right]+\sum_{j=1}^{m} z_j h_j(x) < \sum_{i=1}^{k} y_i\left[f_i(u)-v_i g_i(u)\right]+\sum_{j=1}^{m} z_j h_j(u),$$

i.e.,

$$O(x) < O(u),$$

which contradicts (21).

If condition (b) holds, from the $(p,r)-\rho^2-(\eta,\theta)$-invexity of $Q(.)$ we get

$$\frac{1}{r}\left(e^{r(Q(x)-Q(u))}-1\right) \geqq \frac{1}{p}\left[\sum_{j=1}^{m} z_j \psi_j\right]\left(e^{p\eta(x,u)}-\mathbf{1}\right)+\rho^2\left\|\theta(x,u)\right\|^2, \ \forall \psi_j \in \partial h_j(u). \tag{24}$$

From (23) and (24) and the fundamental property of the exponential function, we obtain

$$\frac{1}{p}\left[\sum_{j=1}^{m} z_j \psi_j\right]\left(e^{p\eta(x,u)}-\mathbf{1}\right)+\rho^2\left\|\theta(x,u)\right\|^2 \leqq 0, \ \forall \psi_j \in \partial h_j(u). \tag{25}$$

By (20), (25) and the assumption that $\rho^1+\rho^2 \geqq 0$, we obtain

$$\frac{1}{p}\left[\sum_{i=1}^{k} y_i\left[\xi_i+v_i\zeta_i\right]\right]\left(e^{p\eta(x,u)}-\mathbf{1}\right)+\rho^1\left\|\theta(x,u)\right\|^2 \geqq 0. \tag{26}$$

From the $(p,r)-\rho^1-(\eta,\theta)$-invexity of $P(.)$, we have

$$\frac{1}{r}\left(e^{r(P(x)-P(u))}-1\right) \geqq \frac{1}{p}\left[\sum_{i=1}^{k} y_i\left[\xi_i+v_i\zeta_i\right]\right]\left(e^{p\eta(x,u)}-\mathbf{1}\right)+\rho^1\left\|\theta(x,u)\right\|^2. \tag{27}$$

From (26) and (27), we obtain

$$P(x) \geqq P(u),$$

i.e.,

$$\sum_{i=1}^{k} y_i \left[f_i(x) - v_i g_i(x) \right] \geqq \sum_{i=1}^{k} y_i \left[f_i(u) - v_i g_i(u) \right]. \tag{28}$$

Again if $\varphi(x) \leq v$, then we get (22) in the same way. But (22) contradicts (28). Therefore, $\varphi(x) \nleq v$. This completes the proof. □

Theorem 4.2 (Strong duality). *Let x^* be an efficient solution for* (MFP) *and let h satisfy the constraints qualification at x^*. Then there exist $y^* \in I$, $z^* \in R^m$ and $v^* \in R^k$ such that (x^*, y^*, z^*, v^*) is feasible for* (PD). *Also, if the weak duality theorem* 4.1 *holds for all feasible solutions of the problems* (MFP) *and* (PD), *then (x^*, y^*, z^*, v^*) is an efficient solution for* (PD).

Proof. Since x^* is an efficient solution for (MFP) and h satisfy the constraints qualification at x^*, by Theorem 2.1, there exist $y^* \in I$, $z^* \in R^m$ and $v^* \in R^k$ such that (x^*, y^*, z^*, v^*) satisfies (2) - (6). This, in turn, implies that (x^*, y^*, z^*, v^*) is feasible for (PD). From the weak duality theorem, for any feasible points (x, y, z, v) to (PD), the inequality $\varphi(x^*) \geq v$ holds. Hence we conclude that (x^*, y^*, z^*, v^*) is an efficient solution to (PD). This completes the proof. □

Theorem 4.3 (Strict converse duality). *Let x^* and (u^*, y^*, z^*, v^*) be efficient solutions for* (MFP) *and* (PD), *respectively with $v_i^* = \dfrac{f_i(x^*)}{g_i(x^*)}$ for all $i = 1, 2, ..., k$. Assume that*

$$\rho > 0 \text{ and } A(.) = \sum_{i=1}^{k} y_i^* \left[f_i(.) - v_i^* g_i(.) \right] + \sum_{j=1}^{m} z_j^* h_j(.)$$

is strictly $(p, r) - \rho - (\eta, \theta)$-invex at $u^* \in \Omega \cup pr_X \Gamma$ with respect to η and θ. Then

$x^* = u^*$; that is, u^* is an efficient solution for (MFP).

Proof. Suppose on contrary that $x^* \neq u^*$. By relation (16), there exist $\xi_i \in \partial f_i(x^*)$, $\zeta_i \in \partial(-g_i)(x^*)$, $i = 1, 2, ..., k$ and $\psi_j \in \partial h_j(x^*)$, $j = 1, 2, ..., m$, such that

$$\sum_{i=1}^{k} y_i^* \left[\xi_i + v_i^* \zeta_i \right] + \sum_{j=1}^{m} z_j^* \psi_j = 0. \tag{29}$$

From (17), (18) and (19), we get

$$A\left(u^*\right)=\sum_{i=1}^{k} y_i^*\left[f_i\left(u^*\right)-v_i^* g_i\left(u^*\right)\right]+\sum_{j=1}^{m} z_j^* h_j\left(u^*\right)=0\,. \tag{30}$$

From the strict $(p,r)-\rho\ -(\eta,\theta)$-invexity of $A(.)$ at u^*, we have

$$\frac{1}{r}\left(e^{r\left(A\left(x^*\right)-A\left(u^*\right)\right)}-1\right)>\frac{1}{p}\left[\sum_{i=1}^{k} y_i^*\left[\xi_i+v_i^*\zeta_i\right]+\sum_{j=1}^{m} z_j^*\psi_j\right]\left(e^{p\eta\left(x^*,u^*\right)}-\mathbf{1}\right)+\rho\left\|\theta\left(x^*,u^*\right)\right\|^2. \tag{31}$$

The above inequality together with (29) and the assumption that $\rho>0$ imply

$$\frac{1}{r}\left(e^{r\left(A\left(x^*\right)-A\left(u^*\right)\right)}-1\right)>0.$$

Using fundamental property of the exponential function, we get

$$A\left(x^*\right)>A\left(u^*\right). \tag{32}$$

Since $v_i^*=\dfrac{f_i\left(x^*\right)}{g_i\left(x^*\right)}$ for all $i=1,2,...,k,$ we have

$$f_i\left(x^*\right)-v_i^* g_i\left(x^*\right)=0 \text{ for all } i=1,2,...,k\,. \tag{33}$$

From the relation $h\left(x\right)\in -R_+^m$ and (19), we have

$$\sum_{j=1}^{m} z_j^* h_j\left(x^*\right)\leqq 0\,. \tag{34}$$

Therefore, from (33) and (34), we conclude that

$$A\left(x^*\right)=\sum_{i=1}^{k} y_i^*\left[f_i\left(x^*\right)-v_i^* g_i\left(x^*\right)\right]+\sum_{j=1}^{m} z_j^* h_j\left(x^*\right)\leqq 0\,. \tag{35}$$

Hence from (32) and (35), we have $A\left(u^*\right)<0$ which contradicts (30). Hence $x^*=u^*$. This completes the proof. □

Remark 4.1 The function $A(.)$ in Theorem 4.3 is expressed by the sum of the modified objective part $B(.)$ of (MFP) and its constraint part $C(.)$. If $B(.)$ is strictly $(p,r)-\rho\ -(\eta,\theta)$-invex and $C(.)$ is $(p,r)-\rho\ -(\eta,\theta)$-invex then the Theorem 4.3 still holds.

5. WOLFE DUALITY

In what follows, we take the following form of theorem 2.1:

Theorem 5.1 *Let* x^* *be an efficient solution to* (MFP). *Assume that* h *satisfies the constraint qualification at* x^*. *Then there exist* $y^* \in R_+^k$ *and* $z^* \in R^m$, *such that*

$$0 \in \sum_{i=1}^{k} y_i^* g_i\left(x^*\right)\left[\partial f_i\left(x^*\right)+\sum_{j=1}^{m} z_j^* \partial h_j\left(x^*\right)\right]$$

$$+\sum_{i=1}^{k} y_i^*\left[f_i\left(x^*\right)+\sum_{j=1}^{m} z_j^* h_j\left(x^*\right)\right]\left(-\partial g_i\left(x^*\right)\right), \tag{36}$$

$$z_j^* h_j\left(x^*\right)=0, \text{for all } j=1,2,\ldots,m, \tag{37}$$

$$h_j\left(x^*\right) \leqq 0, \text{ for all } j=1,2,\ldots,m, \tag{38}$$

$$y^* \in I,\ z^* \in R_+^m. \tag{39}$$

Now we consider the following Wolfe type dual problem to (FP):

(WD) Maximize $\left(\dfrac{f_1(u)+\sum_{j=1}^{m} z_j h_j(u)}{g_1(u)},\ldots,\dfrac{f_k(u)+\sum_{j=1}^{m} z_j h_j(u)}{g_k(u)}\right)$

subject to

$$0 \in \sum_{i=1}^{k} y_i g_i(u)\left[\partial f_i(u)+\sum_{j=1}^{m} z_j \partial h_j(u)\right]$$

$$+\sum_{i=1}^{k} y_i\left[f_i(u)+\sum_{j=1}^{m} z_j h_j(u)\right]\left(-\partial g_i(u)\right), \tag{40}$$

$$u \in X,\ y \in I,\ z \in R_+^m. \tag{41}$$

Let $\tilde{\Gamma}=\left\{(u, y, z) \in X \times I \times R_+^m : 0 \in \sum_{i=1}^{k} y_i g_i(u)\left[\partial f_i(u)+\sum_{j=1}^{m} z_j \partial h_j(u)\right] +\sum_{i=1}^{k} y_i\left[f_i(u)+\sum_{j=1}^{m} z_j h_j(u)\right]\left(-\partial g_i(u)\right)\right\}$

denote the set of all feasible solutions of (WD). Moreover, we denote by $pr_X \tilde{\Gamma}$ the projection of the set $\tilde{\Gamma}$ on X.

Denote $\psi_i(u,z)=\dfrac{f_i(u)+\sum_{j=1}^{m} z_j h_j(u)}{g_i(u)}$ and $\psi(u,z)=(\psi_1(u,z),\psi_2(u,z),...,\psi_k(u,z))$.

Throughout this section, we assume that $f_i(u)+\sum_{j=1}^{m} z_j h_j(u) \geqq 0$ and $g_i(u)>0$, for all $i=1,2,...,k$.

Theorem 5.2 (Weak duality). *Let* x *be a feasible solution for* (MFP), *and let* (u,y,z) *be a feasible solution for* (WD). *Assume that* $\rho \geqq 0$ and

$$S(.)=\sum_{i=1}^{k} y_i g_i(u)\left[f_i(.)+\sum_{j=1}^{m} z_j h_j(.)\right]-\sum_{i=1}^{k} y_i g_i(.)\left[f_i(u)+\sum_{j=1}^{m} z_j h_j(u)\right]$$

is $(p,r)-\rho-(\eta,\theta)$*-invex at* $u\in\Omega\cup pr_X\tilde{\Gamma}$ *with respect to* η *and* θ. *Then* $\varphi(x)\not\leq\psi(u,z)$.

Proof. Let x and (u,y,z) be feasible solution to (MFP) and (WD), respectively. By the relation (40) there exist $\xi_i\in\partial f_i(u)$, $\zeta_i\in\partial(-g_i)(u)$, $i=1,2,...,k$ and $\psi_j\in\partial h_j(u)$, $j=1,2,...,m$ such that

$$\sum_{i=1}^{k} y_i g_i(u)\left[\xi_i+\sum_{j=1}^{m} z_j\psi_j\right]+\sum_{i=1}^{k} y_i\left[f_i(u)+\sum_{j=1}^{m} z_j h_j(u)\right]\zeta_i=0. \tag{42}$$

From $(p,r)-\rho-(\eta,\theta)$-invexity of $S(.)$ at u with respect to η and θ, we have

$$\frac{1}{r}\left(e^{r(S(x)-S(u))}-1\right)\geqq\frac{1}{p}\left[\sum_{i=1}^{k} y_i g_i(u)\left[\xi_i+\sum_{j=1}^{m} z_j\psi_j\right]+\sum_{i=1}^{k} y_i\left[f_i(u)+\sum_{j=1}^{m} z_j h_j(u)\right]\zeta_i\right]$$
$$\left(e^{p\eta(x,u)}-\mathbf{1}\right)+\rho\left\|\theta(x,u)\right\|^2. \tag{43}$$

The above inequality together with (42) and the assumption that $\rho\geqq 0$ imply

$$\frac{1}{r}\left(e^{r(S(x)-S(u))}-1\right)\geqq 0.$$

Using fundamental property of the exponential function, we get

$$S(x)\geqq S(u)=0,$$

i.e.,

$$S(x)\geqq 0. \tag{44}$$

On the other hand, suppose contrary to the result that $\varphi(x)\leq\psi(u,z)$. Then

$$\frac{f_i(x)}{g_i(x)} \leqq \frac{f_i(u)+\sum_{j=1}^{m} z_j h_j(u)}{g_i(u)} \quad \text{for all } i = 1,2,...,k,$$

$$\frac{f_t(x)}{g_t(x)} < \frac{f_t(u)+\sum_{j=1}^{m} z_j h_j(u)}{g_t(u)} \quad \text{for some } t \in \{1,2,...,k\},$$

Since $y^* \in I$, it follows that

$$\sum_{i=1}^{k} y_i \left[f_i(x) g_i(u) \right] < \sum_{i=1}^{k} y_i g_i(x) \left[f_i(u) + \sum_{j=1}^{m} z_j h_j(u) \right],$$

equivalently,

$$\begin{aligned} &\sum_{i=1}^{k} y_i \left[f_i(x) + \sum_{j=1}^{m} z_j h_j(x) \right] g_i(u) - \sum_{i=1}^{k} y_i g_i(x) \left[f_i(u) + \sum_{j=1}^{m} z_j h_j(u) \right] \\ &< \sum_{i=1}^{k} y_i g_i(u) \sum_{j=1}^{m} z_j h_j(x). \end{aligned} \tag{45}$$

From the relation $h(x) \in -R_+^m$, $g_i(u) > 0$ and (41), we have

$$\sum_{i=1}^{k} y_i g_i(u) \sum_{j=1}^{m} z_j h_j(x) \leqq 0 .$$

Therefore (45), implies

$$\sum_{i=1}^{k} y_i g_i(u) \left[f_i(x) + \sum_{j=1}^{m} z_j h_j(x) \right] - \sum_{i=1}^{k} y_i g_i(x) \left[f_i(u) + \sum_{j=1}^{m} z_j h_j(u) \right] < 0.$$

i.e.,

$$S(x) < 0,$$

which contradicts (44). Therefore, $\varphi(x) \not\leq \psi(u,z)$. This completes the proof. □

Theorem 5.3 (Strong duality). *Let x^* be an efficient solution for* (MFP) *and let h satisfy the constraints qualification at x^*. Then there exist $y^* \in I$ and $z^* \in R^m$ such that (x^*, y^*, z^*) is feasible to* (WD). *Also, if the weak duality theorem* 5.2 *holds for all feasible solutions of the problems* (MFP) *and* (WD), *then (x^*, y^*, z^*) is an efficient solution for* (WD).

Proof. Since x^* is an efficient solution for (MFP) and h satisfy the constraints qualification at x^*, by Theorem 5.1, there exist $y^* \in I$ and $z^* \in R^m$ such that (x^*, y^*, z^*) satisfies (36)-(39). This, in turn, implies that (x^*, y^*, z^*) is feasible for (WD). From the weak duality theorem, for any feasible points (x, y, z) to (WD), the inequality $\varphi(x^*) \le \psi(x, z)$ holds. Hence, we conclude that (x^*, y^*, z^*) is an efficient solution to (WD). This completes the proof. □

Theorem 5.4 (Strict converse duality). *Let* x^* *and* (u^*, y^*, z^*) *be efficient solutions for* (MFP) *and* (WD), *respectively. Assume that*

$$\rho\left\|\theta\left(x^*,u^*\right)\right\|^2 > 0$$

and

$$T(.) = \sum_{i=1}^{k} y_i^* g_i\left(u^*\right)\left[f_i(.) + \sum_{j=1}^{m} z_j^* h_j(.)\right] - \sum_{i=1}^{k} y_i^* g_i(.)\left[f_i\left(u^*\right) + \sum_{j=1}^{m} z_j^* h_j\left(u^*\right)\right]$$

is strictly $(p,r)-\rho\ -(\eta,\theta)$*-invex at* $u^* \in \Omega \cup pr_X\tilde{\Gamma}$ *with respect to* η *and* θ. *Then* $x^* = u^*$; *that is,* u^* *is an efficient solution for* (MFP).

Proof. Suppose on contrary that $x^* \ne u^*$. Let x^* and (u^*, y^*, z^*) be feasible solution to (MFP) and (WD), respectively. By the relation (36) there exist $\xi_i \in \partial f_i\left(u^*\right),\ \zeta_i \in \partial\left(-g_i\right)\left(u^*\right),\ i = 1,2,...,k$ and $\psi_j \in \partial h_j\left(u^*\right),\ j = 1,2,...,m,$ such that

$$\sum_{i=1}^{k} y_i^* g_i\left(u^*\right)\left[\xi_i + \sum_{j=1}^{m} z_j^* \psi_j\right] + \sum_{i=1}^{k} y_i^*\left[f_i\left(u^*\right) + \sum_{j=1}^{m} z_j^* h_j\left(u^*\right)\right]\zeta_i = 0. \tag{46}$$

From Theorem 5.3, we know that there exist $\bar{y}$ and $\bar{z}$ such that $\left(x^*, \bar{y}, \bar{z}\right)$ is an efficient solution for (WD) and

$$\frac{f_i\left(x^*\right) + \sum_{j=1}^{m} \bar{z}_j h_j\left(x^*\right)}{g_i\left(x^*\right)} = \frac{f_i\left(u^*\right) + \sum_{j=1}^{m} z_j^* h_j\left(u^*\right)}{g_i\left(u^*\right)}. \tag{47}$$

By (37), (39) and (47), we obtain

$$\frac{f_i\left(x^*\right)}{g_i\left(x^*\right)} = \frac{f_i\left(u^*\right) + \sum_{j=1}^{m} z_j^* h_j\left(u^*\right)}{g_i\left(u^*\right)}. \tag{48}$$

Hence

$$f_i\left(x^*\right)g_i\left(u^*\right)=\left[f_i\left(u^*\right)+\sum_{j=1}^{m}z_j^*h_j\left(u^*\right)\right]g_i\left(x^*\right), \tag{49}$$

From (41) and (49), we obtain

$$T\left(x^*\right)=\sum_{i=1}^{k}y_i^*g_i\left(u^*\right)\sum_{j=1}^{m}z_j^*h_j\left(x^*\right).$$

From the relation $h\left(x^*\right)\in -R_+^m,\ g_i\left(u^*\right)>0$, from (41) and the above inequality, we have

$$T\left(x^*\right)\leqq 0.$$

Therefore,

$$T\left(x^*\right)\leqq 0=T\left(u^*\right),$$

i.e.,

$$T\left(x^*\right)\leqq T\left(u^*\right). \tag{50}$$

From the strict $(p,r)-\rho\ -(\eta,\theta)$-invexity of $T(.)$ at u^* in $\Omega\cup pr_X\tilde{\Gamma}$ with respect to η and θ, we have

$$\frac{1}{r}\left(e^{r\left(T\left(x^*\right)-T\left(u^*\right)\right)}-1\right)>\frac{1}{p}\left[\sum_{i=1}^{k}y_i^*g_i\left(u^*\right)\left[\xi_i+\sum_{j=1}^{m}z_j^*\psi_j\right]+\sum_{i=1}^{k}y_i^*\left[f_i\left(u^*\right)+\sum_{j=1}^{m}z_j^*h_j\left(u^*\right)\right]\zeta_i\right]$$
$$\left(e^{p\eta\left(x^*,u^*\right)}-\mathbf{1}\right)+\rho\left\|\theta\left(x^*,u^*\right)\right\|^2.$$

The above inequality together with (50) imply

$$\frac{1}{p}\left[\sum_{i=1}^{k}y_i^*g_i\left(u^*\right)\left[\xi_i+\sum_{j=1}^{m}z_j^*\psi_j\right]+\sum_{i=1}^{k}y_i^*\left[f_i\left(u^*\right)+\sum_{j=1}^{m}z_j^*h_j\left(u^*\right)\right]\zeta_i\right]\left(e^{p\eta\left(x^*,u^*\right)}-\mathbf{1}\right)+\rho\left\|\theta\left(x^*,u^*\right)\right\|^2,$$

which by virtue of (46) imply

$$\rho\left\|\theta\left(x^*,u^*\right)\right\|^2<0,$$

which contradicts the assumptions that $\rho\left\|\theta\left(x^*,u^*\right)\right\|^2>0$. Hence $x^*=u^*$; that is, u^* is an efficient solution for (FP). This completes the proof. □

6. MOND-WEIR DUALITY

In this section, we consider the following Mond-Weir dual to (MFP):

(MWD) Maximize $\left(\frac{f_1(u)}{g_1(u)},...,\frac{f_k(u)}{g_k(u)}\right)$

subject to

$$0 \in \sum_{i=1}^{k} y_i g_i(u)\left[\partial f_i(u)+\sum_{j=1}^{m} z_j \partial h_j(u)\right]+\sum_{i=1}^{k} y_i\left(-\partial g_i(u)\right)\left[f_i(u)+\sum_{j=1}^{m} z_j h_j(u)\right], \quad (51)$$

$$\sum_{j=1}^{m} z_j h_j(u) \geqq 0, \quad (52)$$

$$u \in X,\ y \in I,\ z \in R_+^m. \quad (53)$$

Let

$$\bar{\Gamma} = \left\{(u,y,z) \in X \times I \times R_+^m : 0 \in \sum_{i=1}^{k} y_i g_i(u)\left[\partial f_i(u)+\sum_{j=1}^{m} z_j \partial h_j(u)\right]\right.$$

$$\left. +\sum_{i=1}^{k} y_i\left(-\partial g_i(u)\right)\left[f_i(u)+\sum_{j=1}^{m} z_j h_j(u)\right], \sum_{j=1}^{m} z_j h_j(u) \geqq 0\right\},$$

denote the set of all feasible solutions to (MWD). Moreover, we denote by $pr_X\bar{\Gamma}$ the projection of the set $\bar{\Gamma}$ on X.

Denote $\Phi_i(u)=\frac{f_i(u)}{g_i(u)}$ and $\Phi(u)=\left(\Phi_1(u),\Phi_2(u),...,\Phi_k(u)\right)$.

Now we shall state weak, strong and strict converse duality theorems without proof as they can be proved in light of Theorem 5.2, Theorem 5.3 and Theorem 5.4, proved in previous section.

Theorem 6.1 (Weak duality). *Let* x *be a feasible solution for* (MFP)*, and let* (u,y,z) *be a feasible solution for* (MWD). *Assume that*

$$\rho \geqq 0 \text{ and } S(.)=\sum_{i=1}^{k} y_i g_i(u)\left[f_i(.)+\sum_{j=1}^{m} z_j h_j(.)\right]-\sum_{i=1}^{k} y_i g_i(.)\left[f_i(u)+\sum_{j=1}^{m} z_j h_j(u)\right]$$

is $(p,r)-\rho\ -(\eta,\theta)$*-invex at* $u \in \Omega \cup pr_X\bar{\Gamma}$ *with respect to* η *and* θ.

Then, $\varphi(x) \not\leq \Phi(u)$.

Theorem 6.2 (Strong duality). *Let* x^* *be an efficient solution for* (MFP) *and let* h *satisfy the constraints qualification at* x^*. *Then there exist* $y^* \in I$ *and* $z^* \in R^m$ *such that*

$\left(x^*, y^*, z^*\right)$ *is feasible to* (MWD). *Also, if the weak duality theorem* 6.1 *holds for all feasible solutions to the problems* (MFP) *and* (MWD), *then* $\left(x^*, y^*, z^*\right)$ *is an efficient solution for* (MWD).

Theorem 6.3 (Strict converse duality). *Let* x^* *and* $\left(u^*, y^*, z^*\right)$ *be efficient solutions for* (MFP) *and* (MWD), *respectively. Assume that* $\rho\left\|\theta\left(x^*, u^*\right)\right\|^2 > 0$ *and*

$$T(.) = \sum_{i=1}^{k} y_i^* g_i\left(u^*\right)\left[f_i(.) + \sum_{j=1}^{m} z_j^* h_j(.)\right] - \sum_{i=1}^{k} y_i^* g_i(.)\left[f_i\left(u^*\right) + \sum_{j=1}^{m} z_j^* h_j\left(u^*\right)\right]$$

is strictly $(p,r)-\rho-(\eta,\theta)$*-invex at* $u^* \in \Omega \cup pr_X\overline{\Gamma}$ *with respect to* η *and* θ. *Then* $x^* = u^*$*; that is,* u^* *is an efficient solution for* (MFP).

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