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On the limits of discrete Painlevé equations associated with the affine Weyl group E_8

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We study the discrete Painlevé equations that can be obtained as limits from the equations associated with the affine Weyl group $E_8^{(1)}$. We obtain equations associated with the groups $E_7^{(1)}$ and $E_6^{(1)}$ as well as linearisable systems. In the $E_7^{(1)}$ and $E_6^{(1)}$ cases, we obtain several new discrete Painlevé equations along with equations which can be related to the ones already known. The same is true for linearisable systems. In the case of new linearisable mappings, we present their explicit linearisation. *Published by AIP Publishing.* [http://dx.doi.org/10.1063/1.4978330]

I. INTRODUCTION

The Sakai classification[1] of rational surfaces associated with affine root systems provided the key for the systematic classification of discrete Painlevé equations, the latter being obtained by the translation part of an affine Weyl group. As shown by Sakai, the discrete Painlevé equations can be organised in a degeneration cascade starting from the systems associated with the affine Weyl group $E_8^{(1)}$. Given a group one can derive the generic equation, corresponding to a straight line evolution on a lattice associated with the group. However, many more discrete equations exist, corresponding to more complicated trajectories. This last remark led to a working definition of what is a discrete Painlevé equation. As we noted in Ref. 2, a discrete Painlevé equation is a mapping obtained by the periodic repetition of a non-closed pattern in a lattice associated with an affine Weyl group belonging to the degeneration cascade of $E_8^{(1)}$.

Deriving the discrete Painlevé equations associated with a given affine Weyl group, even if one limits oneself to simple trajectories, is a very tall order. Still, this is a project that the present authors have successfully completed for most of the lower groups of the $E_8^{(1)}$ degeneration cascade.[3,4] On the other hand, when it comes to the $E_8^{(1)}$ group itself, the results have been hard to obtain because of the extreme richness of this group, a fact that makes the calculations extremely bulky from the outset. However, in the last one or two years, a series of breakthrough have rendered the calculations for $E_8^{(1)}$ -related systems more manageable.[5–7]

The basic method for the derivation of discrete Painlevé equations is the one we have dubbed deautonomisation.[8] For the practical implementation of this method, one starts from an integrable autonomous mapping and assumes that the parameters which enter it are functions of the independent variable. Their precise form is then fixed through the application of a discrete integrability criterion. For the latter, singularity confinement or algebraic entropy or a combination thereof[9] are the usual choice.

In the case of $E_8^{(1)}$-related equations, three different types of discrete systems exist, additive, multiplicative and elliptic, in which the independent variable appears linearly, exponentially, or through the argument of an elliptic function. In what follows, we shall focus our analysis on the equations of additive type and explain how the results obtained can be transcribed to the multiplicative and elliptic cases. The generic additive $E_8^{(1)}$-related equation has the form[10]

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$$\frac{(x_n - x_{n+1} + (z_n + z_{n+1})^2)(x_n - x_{n-1} + (z_n + z_{n-1})^2) + 4x_n(z_n + z_{n+1})(z_n + z_{n-1})}{(z_n + z_{n-1})(x_n - x_{n+1} + (z_n + z_{n+1})^2) + (z_n + z_{n+1})(x_n - x_{n-1} + (z_n + z_{n-1})^2)} = 2\frac{x_n^4 + \sigma_2 x_n^3 + \sigma_4 x_n^2 + \sigma_6 x_n + \sigma_8}{\sigma_1 x_n^3 + \sigma_3 x_n^2 + \sigma_5 x_n + \sigma_7}, \tag{1}$$

where z_n is equal to $\alpha n + \beta$ and σ_k are the degree-k elementary symmetric functions of the quantities $z_n + \kappa_n^i$, where κ^i are eight parameters which are, generically, functions of the independent variable. Starting from this equation, one can obtain several more through a simplification process, where one assumes that a common factor drops out from the numerator and denominator of the right-hand side. This simplification is implemented in the autonomous form of the equation whereupon one proceeds to deautonomisation. (This process was called "degeneracy" in older publications of ours,[3] a term we prefer now not to use lest it be confused with the term "degeneration."[11]) What is remarkable is that all the equations obtained through the simplification process are still associated with the group $E_8^{(1)}$, in the sense that the geometry of their transformations is governed by that group. Equations related to $E_8^{(1)}$ and obtained in this way have been the object of Ref. 12.

In this paper we shall concentrate on a different procedure. Here we shall consider the limits of the equations of the form (1) when parameters in the right-hand side are put to infinity. These limiting procedures, which materialise the degeneration process referred to in the previous paragraph, have as a consequence to yield equations associated with groups that lie lower in the degeneration cascade of $E_8^{(1)}$. A rule of thumb concerning the identification of the affine Weyl group a given equation is associated with is to count the free parameters appearing in the coefficients of the various terms. A word of caution is mandatory here. Given the fact that x scales like z^2 one can apply a scaling factor on the secular part of $z_n = \alpha n$ and put the parameter α to 1 (unless α was 0 to begin with, in which case we do not have a discrete Painlevé equation but a mapping with periodic coefficients). More generally one should always take into account the possible gauge freedom and eliminate any superfluous parameter before a definitive count.

When too many parameters disappear because of the limits, the resulting system is not a discrete Painlevé equation any more, but rather a linearisable equation.[13] The linearisability of the equations obtained is assessed through the study of the degree growth of the iterates of the initial condition, the details of the growth providing a hint as to the precise method of linearisation of the equation at hand.

II. A FACTORISED REPRESENTATION OF THE $E_8^{(1)}$-RELATED EQUATIONS

Given the form (1) of the general additive (difference) discrete Painlevé equation associated with the affine Weyl group $E_8^{(1)}$, it is clear that it does not easily lend itself to deautonomisation type calculations using integrability criteria. For instance, in order to perform the singularity analysis of Equation (1), we must solve for x_{n+1} in terms of x_n and x_{n-1} and look for the singularities which are defined as the values of x_n for which x_{n+1} is independent of x_{n-1}. We obtain for the latter an equation of degree 8 which at this level does not appear to be tractable. Similar difficulties arise when one tries to implement an algebraic entropy approach.

However there exists a way to simplify the calculations. It is based on the introduction of an ancillary variable ξ such that

$$x_n = \xi_n^2. \tag{2}$$

Using this variable, we find that we can rearrange Equation (1) and bring it to the form[12]

$$\frac{x_{n+1} - (\xi_n - z_n - z_{n+1})^2}{x_{n+1} - (\xi_n + z_n + z_{n+1})^2}\,\frac{x_{n-1} - (\xi_n - z_n - z_{n-1})^2}{x_{n-1} - (\xi_n + z_n + z_{n-1})^2} = \frac{\prod_{i=1}^8(\kappa_n^i + z_n - \xi_n)}{\prod_{i=1}^8(\kappa_n^i + z_n + \xi_n)}. \tag{3}$$

What is interesting here is that both the left and right hand sides of (3) are expressed in a factorised form. This, very convenient, form was also obtained by Kajiwara, Noumi, and Yamada in Ref. 5, through a different approach, but with the introduction of the same ancillary variable.

In order to show how the form (3) is convenient for singularity analysis, we perform it in the general case assuming that no *a priori* constraints exist on the κ_n^i. We remark that x_{n+1} is independent of x_{n-1} whenever the right-hand side of (3) vanishes or diverges. Assume that the value of ξ_n is such that the right-hand sign vanishes, i.e., $\xi_n = \kappa_n^i + z_n$, something that must be balanced by the vanishing of the left-hand side. This leads to $\xi_{n+1} = \kappa_n^i - z_{n+1}$ (neglecting a choice of sign which is immaterial). Iterating, we find that the denominator of the second factor of the left-hand side vanishes. We balance this by requiring that the denominator of the right-hand side also vanishes. We obtain thus the condition $\kappa_{n+1}^i + \kappa_n^i = 0$, i.e., κ^i change sign at each iteration. The singularity is indeed confined since a computation of the value of ξ_{n+2} shows that it does depend on the initial conditions. We would have obtained the same result by considering the singularities related to the vanishing of the denominator of (3). However this does not exhaust all possible singularities: another singularity exists when ξ_n becomes infinite. Since we are dealing with the general case, we require that the singularity be confined in one step, i.e., that x_{n+1} be obtained from the initial conditions. This leads to the constraint

$$z_{n+1} - 2z_n + z_{n-1} = \frac{1}{2}\sum_{i=1}^{8} \kappa_n^i. \tag{4}$$

Since the κ_n^i change sign at each iteration, it is possible, by introducing the appropriate gauge in z_n, to bring the right-hand side of (4) to zero, i.e., $\sum_{i=1}^{8} \kappa_n^i = 0$ and obtain from (4) the solution $z_n = \alpha n + \beta$.

Having the form (3) for the general $E_8^{(1)}$-related equation, we can use it in order to perform all possible limits and simplifications.

III. A FIRST SERIES OF LIMITS

As already announced, this paper is devoted to the study of the various limits of (1) or equivalently of (3). Clearly the right-hand side of (3) assumes a simpler form if we consider that some of the κ^i go to infinity. The simplest such limit is obtained by assuming that two of the κ, say κ^8 and κ^7, go to infinity in such a way so as to keep their sum finite. In this case Equation (3) takes the form

$$\frac{x_{n+1} - (\xi_n - z_n - z_{n+1})^2}{x_{n+1} - (\xi_n + z_n + z_{n+1})^2}\,\frac{x_{n-1} - (\xi_n - z_n - z_{n-1})^2}{x_{n-1} - (\xi_n + z_n + z_{n-1})^2} = \frac{\prod_{i=1}^{6}(\kappa_n^i + z_n - \xi_n)}{\prod_{i=1}^{6}(\kappa_n^i + z_n + \xi_n)}. \tag{5}$$

Going back to a form (1), we find

$$\frac{(x_n - x_{n+1} + (z_n + z_{n+1})^2)(x_n - x_{n-1} + (z_n + z_{n-1})^2) + 4x_n(z_n + z_{n+1})(z_n + z_{n-1})}{(z_n + z_{n-1})(x_n - x_{n+1} + (z_n + z_{n+1})^2) + (z_n + z_{n+1})(x_n - x_{n-1} + (z_n + z_{n-1})^2)} = 2\frac{x_n^3 + \sigma_2 x_n^2 + \sigma_4 x_n + \sigma_6}{\sigma_1 x_n^2 + \sigma_3 x_n + \sigma_5}, \tag{6}$$

where σ_k are the elementary symmetric functions of the quantities $z_n + \kappa_n^i$, constructed now on the 6 remaining κ^i. As in the general case, the κ^i obey the condition $\kappa_{n+1}^i + \kappa_n^i = 0$, which means that they change sign at each iteration. In order to obtain the precise n-dependence of z_n, we consider the singularity where ξ_n becomes infinite. The minimal singularity pattern here is $\{\infty, \infty\}$, which means that the memory of the initial conditions is restored in x_{n+2}. We perform the calculation in the specific gauge where $\sum_{i=1}^{6} \kappa_n^i = 0$ and find for the confinement condition

$$z_{n+2} - z_{n+1} - z_n + z_{n-1} = 0, \tag{7}$$

the solution of which is $z_n = \alpha n + \beta + \gamma(-1)^n$. We remark that the equation has exactly 7 degrees of freedom and it is associated with the affine Weyl group $E_7^{(1)}$.

Next we proceed to consider the equations obtained from (5) through simplifications. As already explained in Ref. 3, the simplifications must be implemented on the autonomous mapping. We start thus from Equation (5) with constant z and κ^i. A simplification in the right-hand side of the relevant equation occurs when two of the κ^i are related through

$$\kappa^i + \kappa^j + 2z = 0. \tag{8}$$

Further simplifications are of course possible and they also lead to equations associated with the affine Weyl group $E_7^{(1)}$. Once the desired simplifications of the right-hand side of (5) have been performed, one can proceed to the deautonomisation of the remaining equation assuming that z and κ^i are functions of the independent variable. The integrability constraints can be obtained by the application of the singularity confinement criterion.

A. A single simplification

We start with the case of a single simplification (by imposing a constraint of the form (8) on κ^6 an κ^5). Before proceeding further and in order to minimise the use of upper indices, we use the letters a, b, c, d *in lieu* of the remaining κ^i. We start by studying the constraint of the confinement of the singularity pattern $\{\infty, \infty\}$. We find readily

$$a_n + b_n + c_n + d_n + a_{n+1} + b_{n+1} + c_{n+1} + d_{n+1} = 2(z_{n+2} + z_{n-1}). \tag{9}$$

The singularity analysis can then be pursued along the lines described in Ref. 12. We assume that we enter a singularity by $\xi_n = \kappa^i + z$ and exit through $-\kappa^j - z$ after M steps and similarly for the remaining parameters after N, P, Q steps respectively. Combining the constraints we find that we must have $M + N + P + Q = 6$. This means that only two distinct integrable classes may exist corresponding to $(M, N, P, Q) = (3, 1, 1, 1), (2, 2, 1, 1)$.

Class $(M, N, P, Q) = (3, 1, 1, 1)$

Two cases can be distinguished here.

In the first one, we enter the singularity with $\xi_n = a_n + z_n$ and exit through $-a_{n+3} - z_{n+3}$, i.e., after three steps, while all remaining singularities entering through $b_n + z_n$, $c_n + z_n$, $d_n + z_n$ exit through $-b_{n+1} - z_{n+1}$, $-c_{n+1} - z_{n+1}$, $-d_{n+1} - z_{n+1}$ after one step. (In fact the three singularities entering through $b_n + z_n$, $c_n + z_n$, $d_n + z_n$ may exit through any permutation of the three exiting values without this changing the conclusions. Thus the choice made here does not lead to loss of generality. Such generality preserving choices will be tacitly made in the analyses that follow in this section and in the next one.) For the singularities related to b, c, d, the constraint is the same as in the general case, i.e., $b_n + b_{n+1} = 0$, $c_n + c_{n+1} = 0$, $d_n + d_{n+1} = 0$. The singularity involving a_n leads to the following confinement constraint $a_n + a_{n+3} = 2(z_{n+1} + z_{n+2})$. Combining the constraints on the a, b, c, d with (9), we obtain for z_n the condition

$$z_{n+4} - z_{n+3} - z_n + z_{n-1} = 0 \tag{10}$$

with solution $z_n = \alpha n + \beta + \phi_4(n)$. Here and in what follows, we use the symbol ϕ_m to denote a periodic function $\phi_m(n + m) = \phi_m(n)$ with period m given by

$$\phi_m(n) = \sum_{\ell=1}^{m-1} \epsilon_\ell^{(m)} \exp\left(\frac{2i\pi\ell n}{m}\right). \tag{11}$$

Notice that the summation starts at 1 instead of 0, since, given the expressions below, the constant term can be absorbed through a redefinition of the secular term $\alpha n + \beta$, and thus, ϕ_m introduces $(m - 1)$ parameters. Using (9), we can express a_n in terms of the z_n as $a_n = 2(z_{n+1} - z_n + z_{n-1})$, up to a gauge choice.

In the second case, we enter the singularity at $a_n + z_n$ and exit through $-b_{n+3} - z_{n+3}$ after three steps. Moreover the singularities $b_n + z_n$, $c_n + z_n$, $d_n + z_n$ exit through $-c_{n+1} - z_{n+1}$, $-a_{n+1} - z_{n+1}$, $-d_{n+1} - z_{n+1}$ after one step. We have thus the confinement constraints $a_n + b_{n+3} = 2(z_{n+1} + z_{n+2})$ and $b_n + c_{n+1} = 0$, $c_n + a_{n+1} = 0$, $d_n + d_{n+1} = 0$. We find thus $a_n = b_{n-2}$, $c_n = -b_{n-1}$ and find for b_n the relation $b_{n+3} - b_{n+1} = 2(z_{n+1} - z_{n-1})$. We obtain for z_n the equation

$$z_{n+3} - z_{n+2} - z_{n-3} + z_{n-4} = 0 \tag{12}$$

the solution of which is $z_n = \alpha n + \beta + \phi_6(n)$. The equation for b_n can be integrated to $b_n = z_{n-2} + z_{n+4} + \phi_2(n)$. Note that the term $\phi_2(n)$ in b_n can be eliminated by a gauge choice that redefines the $(-1)^n$ term in $\phi_6(n)$ and d_n.

Class $(M, N, P, Q) = (2, 2, 1, 1)$

Here we have four distinct cases.

In the first one, we enter the singularity at $a_n + z_n$ and exit through $-a_{n+2} - z_{n+2}$ after two steps, and similarly for b, while we enter at $c_n + z_n$ and $d_n + z_n$ to exit after just one step through

$-c_{n+1} - z_{n+1}$ and $-d_{n+1} - z_{n+1}$, respectively. Hence the constraints $a_n + a_{n+2} = 2z_{n+1}$, $b_n + b_{n+2} = 2z_{n+1}$, $c_n + c_{n+1} = 0$, $d_n + d_{n+1} = 0$. We find $a_n = b_n + \chi_4(n)$ where we have introduced the periodic function χ_{2m} which obeys the equation $\chi_{2m}(n+m) + \chi_{2m}(n) = 0$. It has a period $2m$ while involving only m parameters and can be expressed in terms of roots of unity as

$$\chi_{2m}(n) = \sum_{\ell=1}^{m} \eta_\ell^{(m)} \exp\left(\frac{i\pi(2\ell-1)n}{m}\right). \tag{13}$$

Using (9) we find for b_n the expression $2b_n = z_{n+2} + z_{n+1} - 2z_n + z_{n-1} + z_{n-2} - \chi_4(n)$ leading to the equation for z_n

$$z_{n+4} - z_{n+2} - z_{n+1} + z_{n-1} = 0. \tag{14}$$

Its integration gives $z_n = \alpha n + \beta + \phi_3(n) + \phi_2(n)$. Again the $\phi_2(n)$ term can be eliminated by the proper choice of gauge, which redefines b, c, and d.

In the second case, we enter the singularity at $a_n + z_n$ and exit through $-a_{n+2} - z_{n+2}$ after two steps, while the singularity $b_n + z_n$ exits through $-c_{n+2} - z_{n+2}$ and the singularities $c_n + z_n$, $d_n + z_n$ exit through $-d_{n+1} - z_{n+1}$, $-b_{n+1} - z_{n+1}$ after one step. We have thus the constraints $a_n + a_{n+2} = 2z_{n+1}$, $b_n + c_{n+2} = 2z_{n+1}$, $c_n + d_{n+1} = 0$, $d_n + b_{n+1} = 0$. This means that $c_n = -d_{n+1}$ and $b_n = -d_{n-1}$ while the second constraint can be rewritten as $d_{n+2} + d_{n-2} + 2z_n = 0$. Using (9) and an appropriate gauge, we find $a_n = 2(z_{n+1} - z_n + z_{n-1}) + d_{n+1} - d_n + d_{n-1}$ and a second equation for d_n and z_n only

$$d_{n+1} - 2d_n + d_{n-1} = 2(z_{n+2} - z_{n+1} - z_{n-1} + z_{n-2}). \tag{15}$$

Using the two equations relating d_n to z_n, we find finally

$$z_{n+4} - z_{n+3} - z_{n-3} + z_{n-4} = 0. \tag{16}$$

The latter can be integrated to $z_n = \alpha n + \beta + \phi_7(n)$. Solving Equation (15) we find $d_n = -z_n + 3\phi_7(n) + 2(\phi_7(n+1) + \phi_7(n-1))$.

In the third case, the singularities $a_n + z_n$, $b_n + z_n$ exit through $-b_{n+2} - z_{n+2}$, $-c_{n+2} - z_{n+2}$ after two steps, while the singularities $c_n + z_n$, $d_n + z_n$ exit through $-a_{n+1} - z_{n+1}$, $-d_{n+1} - z_{n+1}$ after one step. We have thus $a_n + b_{n+2} = 2z_{n+1}$, $b_n + c_{n+2} = 2z_{n+1}$, $c_n + a_{n+1} = 0$, $d_n + d_{n+1} = 0$. We find readily that $a_n = -c_{n-1}$ and $b_n = 2z_{n-1} + c_{n-3}$. This gives two equations for c_n and z_n and eliminating c we find for z_n the equation

$$z_{n+4} - z_{n+3} - z_{n+2} + z_{n+1} + z_n - z_{n-1} - z_{n-2} + z_{n-3} = 0 \tag{17}$$

the solution of which, up to a gauge, is $z_n = \alpha n + \beta + \chi_8(n)$ and find for c_n the expression $c_n = 2\alpha + \phi_2(n) + 2(\chi_8(n-1) - \chi_8(n+2))$.

The fourth and last case again has singularities $a_n + z_n$, $b_n + z_n$ exiting through $-b_{n+2} - z_{n+2}$, $-c_{n+2} - z_{n+2}$ after two steps but now the singularities $c_n + z_n$, $d_n + z_n$ exit through $-d_{n+1} - z_{n+1}$, $-a_{n+1} - z_{n+1}$ after one step. We have now the constraints $a_n + b_{n+2} = 2z_{n+1}$, $b_n + c_{n+2} = 2z_{n+1}$, $c_n + d_{n+1} = 0$, $d_n + a_{n+1} = 0$. Thus $a_n = -d_{n-1}$, $c_n = -d_{n+1}$ and $b_n = 2z_{n+1} + d_{n+3} = 2z_{n-1} + d_{n-3}$. Using (9) we find for z_n the equation

$$z_{n+4} - z_{n+2} - z_{n-1} + z_{n-3} = 0 \tag{18}$$

the solution of which, with an appropriate gauge, is $z_n = 3\alpha n + \beta + \phi_5(n)$ and we find for d_n the expression $d_n = -2\alpha n + \gamma + \phi_2(n) - 2\phi_5(n+2) - 2\phi_5(n-2)$.

B. Double simplification

Next we consider the case of two simplifications, which is the maximum possible here. Again in order to minimise the use of upper indices, we use the letters a, b for the remaining κ^i. Studying the constraint of the confinement of the singularity pattern $\{\infty, \infty\}$, we find

$$a_n + b_n + a_{n+1} + b_{n+1} = 2(z_{n+2} + z_{n+1} + z_n + z_{n-1}). \tag{19}$$

Following the same logic as in the case of a single simplification above, we find the constraint $M + N = 6$ and three distinct integrable classes, corresponding to $(M, N) = (5, 1), (4, 2), (3, 3)$.
Class $(M, N) = (5, 1)$

Here we enter the singularity at $a_n + z_n$ and exit through $-a_{n+5} - z_{n+5}$ after 5 steps, while the singularity $b_n + z_n$ exits through $-b_{n+1} - z_{n+1}$ in just one step. This leads to $a_n + a_{n+5} = 2(z_{n+4} + z_{n+3} + z_{n+2} + z_{n+1})$ and $b_n + b_{n+1} = 0$. Using (19) together with the constraints for a_n, b_n, we obtain for z_n the equation

$$z_{n+6} - z_{n+3} - z_{n+2} + z_{n-1} = 0. \tag{20}$$

Its solution is $z_n = \alpha n + \beta + \phi_3(n) + \phi_4(n)$ and since $b_n = \phi_2(n)$ we can obtain a_n as $a_n = 2(z_{n+1} + z_{n-1}) - b_n$.

Class $(M, N) = (4, 2)$

In this class, we exit the singularity $a_n + z_n$ at $-a_{n+4} - z_{n+4}$ after four steps, while the singularity $b_n + z_n$ exits through $-b_{n+2} - z_{n+2}$ in two steps. We have now $a_n + a_{n+4} = 2(z_{n+3} + z_{n+2} + z_{n+1})$ and $b_n + b_{n+2} = 2z_{n+1}$. Combining these constraints with (19), we find the equation for z_n,

$$z_{n+5} + z_{n+4} - z_{n+2} - z_{n+1} - z_n - z_{n-1} + z_{n-3} + z_{n-4} = 0. \tag{21}$$

The solution is $z_n = \alpha n + \beta + \phi_3(n) + \phi_5(n)$, up to an unimportant gauge, while a_n, b_n are given by $a_n = z_{n+4} + z_{n+3} - z_n + z_{n-3} + z_{n-4}$ and $b_n = -z_{n+3} + z_{n+1} + z_n + z_{n-1} - z_{n-3}$.

Class $(M, N) = (3, 3)$

Here both singularities $a_n + z_n$, $b_n + z_n$ exit after three steps through $-a_{n+3} - z_{n+3}$, $-b_{n+3} - z_{n+3}$. We have $a_n + a_{n+3} = 2(z_{n+2} + z_{n+1})$ and $b_n + b_{n+3} = 2(z_{n+2} + z_{n+1})$ which means that $b_n = a_n + 2\chi_6(n)$. We obtain for z_n the equation

$$z_{n+4} - z_{n+2} - z_{n+1} + z_{n-1} = 0 \tag{22}$$

the solution of which, up to gauge, is $z_n = \alpha n + \beta + \phi_3(n)$ while a_n is given by $a_n = z_{n+1} + z_{n-1} + \phi_2(n) - \chi_6(n)$.

C. Linearisable cases

Having exhausted the cases obtained from (5) by simplification, we proceed now to a second limit taking another pair of κ^i to infinity in which case we have just a ratio of four products at the right-hand side. The corresponding form (1) is now

$$\frac{(x_n - x_{n+1} + (z_n + z_{n+1})^2)(x_n - x_{n-1} + (z_n + z_{n-1})^2) + 4x_n(z_n + z_{n+1})(z_n + z_{n-1})}{(z_n + z_{n-1})(x_n - x_{n+1} + (z_n + z_{n+1})^2) + (z_n + z_{n+1})(x_n - x_{n-1} + (z_n + z_{n-1})^2)} = 2\frac{x_n^2 + \sigma_2 x_n + \sigma_4}{\sigma_1 x_n + \sigma_3}, \tag{23}$$

where, again, the symmetric functions are constructed using only the four remaining κ. The constraints of the parameters κ^i remain the same, i.e., they change sign at each iteration the singularity. It turns out that the resulting equation is linearisable and thus it is not necessary to implement additional confinement constraints.[14] As a consequence, the singularity $\xi_n = \infty$ need not confine and thus z_n remains an arbitrary function of n. Equation (23) was first identified in Refs. 15 and 16 and its detailed integration was given in Ref. 17 (Equation (83) in that reference). Since the right-hand side of (23) is a ratio of a quadratic over a linear polynomial, a simplification is possible, leading to a linear right-hand side. Again a linearisable equation is obtained. We give directly its final form

$$\begin{aligned}\frac{(x_n - x_{n+1} + (z_n + z_{n+1})^2)(x_n - x_{n-1} + (z_n + z_{n-1})^2) + 4x_n(z_n + z_{n+1})(z_n + z_{n-1})}{(z_n + z_{n-1})(x_n - x_{n+1} + (z_n + z_{n+1})^2) + (z_n + z_{n+1})(x_n - x_{n-1} + (z_n + z_{n-1})^2)} \\ = \zeta_{n+1} + 2\zeta_n + \zeta_{n-1} + \frac{x_n + a + b(-1)^n}{\zeta_{n+1} + 2\zeta_n + \zeta_{n-1}},\end{aligned} \tag{24}$$

where ζ_n is a free function of n and $z_n = \zeta_{n+1} + \zeta_{n-1}$. The detailed integration of (24) was given in Ref. 17 (Equation (90) in that reference). Both Equations (23) and (24) belong to the class that we have dubbed linearisable equations of the third kind.

The final limit consists in taking another pair of κ^i to infinity. We find readily

$$\frac{(x_n - x_{n+1} + (z_n + z_{n+1})^2)(x_n - x_{n-1} + (z_n + z_{n-1})^2) + 4x_n(z_n + z_{n+1})(z_n + z_{n-1})}{(z_n + z_{n-1})(x_n - x_{n+1} + (z_n + z_{n+1})^2) + (z_n + z_{n+1})(x_n - x_{n-1} + (z_n + z_{n-1})^2)} = z_n + \frac{x_n + a}{z_n}, \tag{25}$$

where a is constant and z_n is again a free function of n. It was first identified in Ref. 16 where it was shown that it is linearisable of Gambier type. Its transformation to the "standard" Gambier equation $(x_n + x_{n+1})(x_n + x_{n-1}) = f_n(x_n^2 - 1)$, where f_n is a free function of n, was also presented in Ref. 16.

IV. A SECOND SERIES OF LIMITS

In Sec. III, we have considered limits which can be obtained systematically from the general form of (3) by taking an even number of the parameters κ to infinity. This resulted always to equations of the form (3) where the right-hand side was a ratio of products of an even number of terms. However the possibility of taking an odd (but greater than 1) number of κ to infinity while ensuring that their sum is finite should also be considered. The simplest case is when three of the κ are chosen in this way in which case Equation (3) becomes

$$\frac{x_{n+1} - (\xi_n - z_n - z_{n+1})^2}{x_{n+1} - (\xi_n + z_n + z_{n+1})^2}\,\frac{x_{n-1} - (\xi_n - z_n - z_{n-1})^2}{x_{n-1} - (\xi_n + z_n + z_{n-1})^2} = \frac{\prod_{i=1}^{5}(\kappa_n^i + z_n - \xi_n)}{\prod_{i=1}^{5}(\kappa_n^i + z_n + \xi_n)}. \tag{26}$$

Written in the form of (1) this equation becomes

$$\frac{(x_n - x_{n+1} + (z_n + z_{n+1})^2)(x_n - x_{n-1} + (z_n + z_{n-1})^2) + 4x_n(z_n + z_{n+1})(z_n + z_{n-1})}{(z_n + z_{n-1})(x_n - x_{n+1} + (z_n + z_{n+1})^2) + (z_n + z_{n+1})(x_n - x_{n-1} + (z_n + z_{n-1})^2)} = 2\frac{\sigma_1 x_n^2 + \sigma_3 x_n + \sigma_5}{x_n^2 + \sigma_2 x_n + \sigma_4}, \tag{27}$$

where σ_k are now the elementary symmetric functions of the quantities $z_n + \kappa_n^i$, constructed now on the 5 remaining κ^i, the latter obeying again the condition $\kappa_{n+1}^i + \kappa_n^i = 0$. The singularity at when ξ_n becomes infinite must also be considered resulting to a minimal singularity pattern $\{\infty, \infty\}$. Working in the gauge where $\sum_{i=1}^{5} \kappa_n^i = 0$ and find the confinement condition

$$z_{n+2} - z_{n+1} - z_n + z_{n-1} = 0 \tag{28}$$

with solution $z_n = \alpha n + \beta + \gamma(-1)^n$. Thus Equation (26), or equivalently (27), has exactly 6 degrees of freedom and it is associated with the affine Weyl group $E_6^{(1)}$ (and the same is true for all equations obtained below by deautonomisation after simplifications).

A. A single simplification

Just as in Section III we start by considering a single simplification in the right-hand side of (26). Again we introduce the letters a, b, c for the remaining parameters κ^i. The confinement constraint for the singularity $\{\infty, \infty\}$ is simply

$$a_n + b_n + c_n + a_{n+1} + b_{n+1} + c_{n+1} = 2(z_{n+2} + z_{n-1}). \tag{29}$$

The singularity analysis is performed, schematically, as before. We assume that the singularities of the form $\kappa^i + z$ are exited through $-\kappa^j - z$ after M, N, or P steps. This leads to the constraint $M + N + P = 5$ which means that two distinct integrable classes do exist with $(M, N, P) = (3, 1, 1), (2, 2, 1)$.
Class $(M, N, P) = (3, 1, 1)$
Two distinct cases exist here.

In the first one, we enter the singularity with $\xi_n = a_n + z_n$ and exit through $-a_{n+3} - z_{n+3}$, i.e., after three steps, while the remaining singularities entering through $b_n + z_n$, $c_n + z_n$ exit through $-b_{n+1} - z_{n+1}$, $-c_{n+1} - z_{n+1}$ after one step. We find the constraints $a_n + a_{n+3} = 2(z_{n+1} + z_{n+2})$, $b_n + b_{n+1} = 0$, and $c_n + c_{n+1} = 0$. We obtain thus for z_n the equation

$$z_{n+4} - z_{n+3} - z_n + z_{n-1} = 0 \tag{30}$$

with solution $z_n = \alpha n + \beta + \phi_4(n)$ and $a_n = 2(z_{n+1} - z_n + z_{n-1})$.

In the second case, we enter the singularity at $a_n + z_n$ and exit by $-b_{n+3} - z_{n+3}$ after three steps while the singularities entering through $b_n + z_n$, $c_n + z_n$ exit through $-c_{n+1} - z_{n+1}$, $-a_{n+1} - z_{n+1}$ after one step. The confinement constraints become $a_n + b_{n+3} = 2(z_{n+1} + z_{n+2})$, $b_n + c_{n+1} = 0$, and $c_n + a_{n+1} = 0$. Combining these constraints with (29) we find for z_n the equation

$$z_{n+6} - z_{n+5} - z_n + z_{n-1} = 0 \tag{31}$$

the solution of which is $z_n = \alpha n + \beta + \phi_6(n)$, and for b_n the expression $b_n = z_{n+4} + z_{n-2}$.
Class $(M, N, P) = (2, 2, 1)$
Again two distinct cases exist here.

In the first case, we enter the singularities $a_n + z_n$, $b_n + z_n$ and exit through $-a_{n+2} - z_{n+2}$, $-b_{n+2} - z_{n+2}$ after two steps while $c_n + z_n$ exits through $-c_{n+1} - z_{n+1}$ after one step. The confinement constraints are now $a_n + a_{n+2} = 2z_{n+1}$, $b_n + b_{n+2} = 2z_{n+1}$, and $c_n + c_{n+1} = 0$. Combining these constraints with (29), we find to z_n the equation

$$z_{n+4} - z_{n+2} - z_{n+1} + z_{n-1} = 0 \tag{32}$$

the solution of which, up to a gauge, is $z_n = \alpha n + \beta + \phi_3(n)$. For the remaining parameters, we find $a_n = z_{n+1} + z_{n-1} - z_n + \chi_4(n)$ and $b_n = z_{n+1} + z_{n-1} - z_n - \chi_4(n)$.

In the second case, we enter the singularities $a_n + z_n$, $b_n + z_n$ and exit through $-b_{n+2} - z_{n+2}$, $-c_{n+2} - z_{n+2}$ after two steps while $c_n + z_n$ exits through $-a_{n+1} - z_{n+1}$ after one step. The constraints are now $a_n + b_{n+2} = 2z_{n+1}$, $b_n + c_{n+2} = 2z_{n+1}$, and $c_n + a_{n+1} = 0$. We find for z_n the equation

$$z_{n+6} - z_{n+5} - z_{n+4} + z_{n+3} + z_{n+2} - z_{n+1} - z_n + z_{n-1} = 0 \tag{33}$$

with solution (up to a gauge) $z_n = \alpha n + \beta + \chi_8(n)$. We have for b_n the expression $b_n = z_{n+4} + z_{n-4} + \phi_2(n)$, whereupon a_n, c_n can be computed from the confinement constraints.

B. Double simplification

In the case of a double, i.e., maximal, simplification, we have necessarily $M = 5$. Here we exit the singularity $a_n + z_n$ through $-a_{n+5} - z_{n+5}$, i.e., after 5 steps, the confinement constraint being $a_n + a_{n+5} = 2(z_{n+4} + z_{n+3} + z_{n+2} + z_{n+1})$. On the other hand from the confinement of the singularity $\{\infty, \infty\}$, we have $a_n + a_{n+1} = 2(z_{n+2} + z_{n+1} + z_n + z_{n-1})$ or, by the proper gauge choice, $a_n = 2(z_{n+1} + z_{n-1})$. We obtain thus the equation for z_n,

$$z_{n+6} - z_{n+3} - z_{n+2} + z_{n-1} = 0 \tag{34}$$

the solution of which is $z_n = \alpha n + \beta + \phi_3(n) + \phi_4(n)$.

C. Linearisable cases

Next we take a further limit by assuming that 5 of the κ^i are infinite. The resulting equation in form (1) is now

$$\frac{(x_n - x_{n+1} + (z_n + z_{n+1})^2)(x_n - x_{n-1} + (z_n + z_{n-1})^2) + 4x_n(z_n + z_{n+1})(z_n + z_{n-1})}{(z_n + z_{n-1})(x_n - x_{n+1} + (z_n + z_{n+1})^2) + (z_n + z_{n+1})(x_n - x_{n-1} + (z_n + z_{n-1})^2)} = 2\frac{\sigma_1 x_n + \sigma_3}{x_n + \sigma_2}, \tag{35}$$

where, again, the symmetric functions are constructed using only the three remaining κ^i (which obey always the condition $\kappa^i_{n+1} + \kappa^i_n = 0$). Here we are in the presence of a linearisable equation, which in the gauge $\sum_{i=1}^{3} \kappa^i_n = 0$ can be written in a more convenient form as

$$\frac{(x_n - x_{n+1} + (z_n + z_{n+1})^2)(x_n - x_{n-1} + (z_n + z_{n-1})^2) + 4x_n(z_n + z_{n+1})(z_n + z_{n-1})}{(z_n + z_{n-1})(x_n - x_{n+1} + (z_n + z_{n+1})^2) + (z_n + z_{n+1})(x_n - x_{n-1} + (z_n + z_{n-1})^2)} = \frac{6z_n x_n + 2z_n^3 + 2az_n + b(-1)^n}{x_n + 3z_n^2 + a}. \tag{36}$$

This is an equation which has not been previously encountered. A study of its degree growth suggests that (36) belongs to the family of linearisable equations of third kind. The linearisation of the latter type equations can be obtained by noticing that their discrete derivative with respect to some constant has a common factor with the discrete derivative of a linear equation with respect to some other constant. It turns out that this is indeed the case for (36). We find that when we take the discrete derivative of (36) with respect to b, i.e., eliminating b between (36) and its up-shift, the resulting third-order mapping has a common factor with the discrete derivative of the linear equation

$$(k(-1)^n - z_n)\left(\frac{x_{n+1} - x_n}{z_{n+1} + z_n} + \frac{x_{n-1} - x_n}{z_{n-1} + z_n} - z_{n+1} - 2z_n - z_{n-1}\right) - (x_n + 3z_n^2 + a) = 0 \tag{37}$$

with respect to k. Following the procedure detailed in Ref. 18, it is possible to construct the solution of (36) starting from the solution of the linear Equation (37). The way to do this is quite simple. One starts from an initial condition where two x's are given, say x_{n-1} and x_n. Using (36) one obtains x_{n+1} which allows one to compute the value of k in (37). It suffices now to solve the linear equation in order to obtain x_n for all n, constructing thus the solution of (36).

Since the right-hand side of (36) is homographic, a simplification is possible, leading to a right-hand side equal simply to $6z$. Once the equation is simplified, we proceed to its deautonomisation. Requiring that the degree growth of the iterates be linear leads to constraints which can best be expressed if we introduce an auxiliary function q_n, which is an arbitrary function of n. We find thus

$$\frac{(x_n - x_{n+1} + (z_n + z_{n+1})^2)(x_n - x_{n-1} + (z_n + z_{n-1})^2) + 4x_n(z_n + z_{n+1})(z_n + z_{n-1})}{(z_n + z_{n-1})(x_n - x_{n+1} + (z_n + z_{n+1})^2) + (z_n + z_{n+1})(x_n - x_{n-1} + (z_n + z_{n-1})^2)} = 2(q_{n+1} + q_n + q_{n-1}), \tag{38}$$

where z_n is related to q_n through $z_n = q_{n+1} + q_{n-1} - q_n$. We can linearise (38) just as we did in the case of (36). We introduce the linear equation

$$(k(-1)^n + q_{n-1})\left(\frac{x_{n+1} - x_n}{z_{n+1} + z_n} - z_{n+1} - z_n\right) + (k(-1)^n + q_{n+1})\left(\frac{x_{n-1} - x_n}{z_{n-1} + z_n} - z_{n-1} - z_n\right)$$
$$+ x_n - (q_{n+1} + q_n - q_{n-1})(q_{n-1} + q_n - q_{n+1}) = 0 \tag{39}$$

and take its discrete derivative with respect to k obtaining a third order mapping for x_n. Since (38) does not possess any visible parameter which would have allowed us to take a discrete derivative, we simply obtain x_{n-1} and x_{n+2} from (38) and its up-shift respectively, expressed in terms of x_{n+1}, x_n and show that the third-order equation obtained from (39) is identically satisfied. This allows us to construct the solution of (38) once the solution of the linear Equation (39) is known. However it is interesting at this point to propose a slightly different method of linearisation. We start by introducing the Miura transformation

$$x_n = \frac{1}{4}(y_n + 4q_{n+1})(y_{n-1} - 4q_{n-1}) + (q_{n+1} + q_n + q_{n-1})^2, \tag{40a}$$

$$y_n = \frac{x_{n+1} - x_n}{z_{n+1} + z_n} - q_{n+2} - 2q_{n+1} + 2q_n + q_{n-1} \tag{40b}$$

and obtain from (38) the following equation for y_n:

$$y_{n+1} = y_{n-1}\frac{y_n + 4q_{n+1}}{y_n - 4q_n}. \tag{41}$$

This is an equation first obtained in Ref. 19 where it was shown that it is linearisable and belongs to the family of Gambier mappings. The linearisation of the non-autonomous case was given in Ref. 20.

The last equation is obtained by assuming that 7 among the κ^i go to infinity. The resulting equation is then

$$\frac{(x_n - x_{n+1} + (z_n + z_{n+1})^2)(x_n - x_{n-1} + (z_n + z_{n-1})^2) + 4x_n(z_n + z_{n+1})(z_n + z_{n-1})}{(z_n + z_{n-1})(x_n - x_{n+1} + (z_n + z_{n+1})^2) + (z_n + z_{n+1})(x_n - x_{n-1} + (z_n + z_{n-1})^2)} = 2(z_n + \kappa_n^1). \tag{42}$$

However since $\kappa_{n+1}^1 + \kappa_n^1 = 0$ and on the left-hand side z_n appears on through sums $z_n + z_{n\pm1}$, the parameter κ^1 can be absorbed into z_n which means that, without loss of generality, we can put $\kappa^1 = 0$ in the right-hand side of (42). Next we introduce the ancillary variable y_n through $y_n = (-1)^n(x_n - z_n^2)$ and find for y_n the equation

$$(y_{n+1} + y_n)(y_n + y_{n-1}) = (-1)^n y_n(z_{n+1} + z_n)(z_n + z_{n-1}). \tag{43}$$

This is a linearisable equation of Gambier type identified in Ref. 15. Its integration was presented in Ref. 21.

V. CONCLUSIONS

The exploration of the domain of discrete Painlevé equations associated with the affine Weyl group $E_8^{(1)}$ using the deautonomisation method was first made possible through the introduction of what we called the trihomographic representation. The advantage of the latter resided in the fact that it was well adapted to singularity analysis. However when it came to studying the general $E_8^{(1)}$-related equation, the trihomographic representation started showing its limit, leading to prohibitively cumbersome calculations. This led us to the proposal of a new representation which, using an ancillary dependent variable, allows us to write the $E_8^{(1)}$ discrete Painlevé equations in a factorised form perfectly suitable for singularity analysis. The usefulness of this method was materialised in Ref. 12 where we derived the $E_8^{(1)}$ -related equations obtained from the general one through a process of successive simplifications.

In this paper, we decided to proceed in a different direction and study the equations obtained from the general $E_8^{(1)}$ one by taking some of the parameters to infinity. This limiting procedure led us to the equations that are not associated with $E_8^{(1)}$ any more, but rather with groups lying lower in the $E_8^{(1)}$ degeneration cascade. Thus in Section III we obtained equations associated with $E_7^{(1)}$ while in Section IV the equations obtained were associated with $E_6^{(1)}$. All discrete Painlevé equations obtained in this paper are new with a minor exception concerning two equations, namely, the first case of the class (2,2,1,1) and the single case of the class (3,3), for which we have derived in Ref. 16 a non-autonomous form, limiting ourselves to the secular dependence of the parameters (Equations (170) and (51) in that paper). Here we have extended that result obtaining the full freedom of the parameters. Implementing a second or a third limit to (1) led to linearisable equations. Typically the equations obtained after a second limit belonged to the class of linearisable mappings of the third kind, according to our nomenclature, while a third limit resulted in equations of Gambier type. The third-kind linearisable equations obtained in Section IV were never derived before and thus we confirmed their linearisability by showing how they can be effectively linearised.

While the present paper focused on the equations of additive type, extending the results to multiplicative equations is tractable as explained in Ref. 22. (It goes without saying that none of the results presented here can be extended to the elliptic case, the latter being exclusively associated with $E_8^{(1)}$). Moreover our paper focused on mappings written as a single equation involving one dependent variable, i.e., "symmetric" systems in the QRT[23] terminology. It would be interesting, in the light of our results in Refs. 21 and 24, to study the case of QRT asymmetric forms, in particular the ones we have dubbed "strongly" asymmetric. While this is definitely an interesting future project, it will have to wait till the analogous studies for equations associated with the affine Weyl group $E_7^{(1)}$ are completed.

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