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Cite as: AIP Conference Proceedings **2112**, 020165 (2019); https://doi.org/10.1063/1.5112350
Published Online: 24 June 2019

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Oscillatory Behavior of Perturbed Generalized Second-Order Quasilinear $\alpha-$Difference Equations

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Abstract. This paper deals with oscillatory behavior of solutions of generalized second order quasilinear $\alpha-$difference equation of the form

$$\Delta_{\alpha(\ell)}\left(a((k-1)\ell+j)\left|\Delta_{\alpha(\ell)}v((k-1)\ell+j)\right|^{\gamma-1}\Delta_{\ell}v((k-1)\ell+j)\right)+F(k\ell+j,v(k\ell+j))=G(k\ell+j,v(k\ell+j),\Delta_{\alpha(\ell)}v(k\ell+j)),$$

for $k\in\mathbb{N}_{\ell}(n_0)$, where $\gamma>0$ and $\alpha>1$. Some sufficient conditions for all solutions of the equation to be oscillatory are obtained.

INTRODUCTION

Difference equations represent a fascinating mathematical area on its own as well as a rich field of the applications in such diverse disciplines as population dynamics, operations research, ecology, biology etc. For general background as difference equations with many examples from diverse fields, one can refer to [1].

The theory of difference equations is based on the operator Δ defined as

$$\Delta u(k)=u(k+1)-u(k),k\in\mathbb{N}=\{0,1,2,\ldots\}.$$

Even though many authors [1],[2] have suggested the definition of Δ as

$$\Delta u(k)=u(k+\ell)-u(k),\ k\in[0,\infty),\ \ell\in(0,\infty), \tag{1}$$

no significant progress took place on this line. In [3], they took up the definition of Δ as given in (1), and developed the theory of difference equations in a different direction. For convenience, we labeled the operator Δ defined by (1) as Δ_ℓ and by defining its inverse Δ_ℓ^{-1}, many interesting results in numerical methods were obtained.

Jerzy Popenda, et.al., [4], while discussing the behavior of solutions of a particular type of difference equation, defined Δ_α as $\Delta_\alpha u(k)=u(k+1)-\alpha u(k)$. This definition of Δ_α is being ignored for a long time. Recently, in [5], they generalized the definition of Δ_α by $\Delta_{\alpha(\ell)}$ defined as $\Delta_{\alpha(\ell)}u(k)=u(k+\ell)-\alpha u(k)$ for the real valued function $u(k)$ and $\ell\in(0,\infty)$ and also obtained the solutions of certain types of generalized α-difference equations, in particular, the generalized Clairauts α-difference equation, generalized Euler α-difference equation and the generalized α-Bernoulli polynomial $B_{\alpha(n)}(k,\ell)$, which is a solution of the α-difference equation $u(k+\ell)-\alpha u(k)=nk^{n-1},\ for\ n\in\mathbb{N}(1)$.

Recently, there has been a lot of interest in the study of oscillation and nonoscillation of second order difference equations see, for example [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20] and the references contained therein. Following this trend in this paper, we shall consider the generalized perturbed quasilinear $\alpha-$difference equation

$$\begin{aligned}&\Delta_{\alpha(\ell)}\left(a((k-1)\ell+j)\left|\Delta_{\alpha(\ell)}v((k-1)\ell+j)\right|^{\gamma-1}\Delta_{\ell}v((k-1)\ell+j)\right)+F(k\ell+j,v(k\ell+j))\\&=G(k\ell+j,v(k\ell+j),\Delta_{\alpha(\ell)}v(k\ell+j)),\end{aligned} \tag{2}$$

where $k\in\mathbb{N}_\ell(k_0)=\{k_0,k_0+\ell,\cdots\}$, ($k_0$ is a fixed nonnegative integer), $\Delta_{\alpha(\ell)}$ is the generalized $\alpha-$difference operator defined by $\Delta_{\alpha(\ell)}v(k)=v(k+\ell)-\alpha v(k)$, $\alpha>1$, $a(k\ell+j)$ is a positive real valued function and $\gamma>0$. Moreover, F and G are real valued functions with $v(k\ell+j):\mathbb{N}_\ell(k_0)\to\mathbb{R}$, $F:\mathbb{N}_\ell(k_0)\times\mathbb{R}\to\mathbb{R}$ and $G:\mathbb{N}_\ell(k_0)\times\mathbb{R}^2\to\mathbb{R}$.

The 11th National Conference on Mathematical Techniques and Applications
AIP Conf. Proc. 2112, 020165-1–020165-9; https://doi.org/10.1063/1.5112350
Published by AIP Publishing. 978-0-7354-1844-8/$30.00

By a solution of (2) we mean a real valued function $v(k\ell+j)$ satisfying (2) for $k \in \mathbb{N}_\ell(k_0)$. A nontrivial solution $v(k\ell+j)$ is said to be oscillatory if it is neither eventually positive nor eventually negative, and nonoscillatory otherwise.

The purpose of this paper is to establish some new results on the oscillatory behavior of solutions of (2). The results obtained here differ greatly from those in [21, 17, 18, 22, 19, 20] and other known literature.

Throughout this paper we shall assume that there exist real valued functions $q(k\ell+j)$, $p(k\ell+j)$ and a function $f:\mathbb{R}\to\mathbb{R}$ such that

(c_1) $xf(x) > 0$ for all $x \neq 0$;
(c_2) $f(x) - f(y) = g(x,y)(x-y)$ for $x, y \neq 0$, where g is a nonnegative function; and
(c_3) $\frac{F(k,x)}{f(x)} \geq q(k\ell+j)$, $\frac{G(k,x,y)}{f(y)} \leq p(k\ell+j)$ for $x, y \neq 0$.

MAIN RESULTS

In this section we will establish some oscillation criteria for equation (2).

Theorem 1. *Let $f(x) = |x|^\gamma sgn\ x$ and the condition (c_3) holds. Assume that function $H(k\ell+j, s\ell+j)$ satisfies* $H(k\ell+j, k\ell+j) = 0$ *for* $k \geq k_0 \in \mathbb{N}_\ell(n_0)$, $H(k\ell+j, s\ell+j) > 0$ *for* $k > s \geq k_0$ *and* $\Delta_{2\alpha(\ell)}H(k\ell+j, s\ell+j) \leq 0$ *for* $k > s \geq k_0$ *where* $\Delta_{2\alpha(\ell)}H(k\ell+j, s\ell+j) = H(k\ell+j, (s+1)\ell+j) - \alpha H(k\ell+j, s\ell+j)$. *Suppose there exists a real valued function $h(k\ell+j, s\ell+j)$ such that*

$$-\Delta_{\alpha(\ell)}\alpha^{\left[\frac{k}{\ell}\right]}H(k\ell+j, s\ell+j) = h(k\ell+j, s\ell+j)[H(k\ell+j, s\ell+j)]^{1/\beta} \ for\ all\ k > s > k_0$$

where $\beta = \frac{\gamma+1}{\gamma}$. If sufficiently large $k_1 \in \mathbb{N}_\ell(k_0)$ such that $k \in \mathbb{N}_\ell(k_1)$

$$q(k\ell+j) - p(k\ell+j) \geq 0, \tag{3}$$

$$\sum_{k=k_0+1}^{\infty} \frac{1}{a^{1/\gamma}((k-1)\ell+j)} = \infty, \text{ and} \tag{4}$$

$$\lim_{k\to\infty} \frac{1}{H(k,k_0)} \sum_{s=k_0}^{k-1} \alpha^{\left[\frac{s}{\ell}\right]} H(k,s)(q(s\ell+j) - p(s\ell+j)) - (a(s-1)\ell+j)\frac{h^{\gamma+1}(k,s)}{\gamma+1} = \infty \tag{5}$$

then all solutions of equation (2) are oscillatory.

Proof. Let $v(k\ell+j)$ be a nonoscillatory solution of equation (2). Without loss of generality we may assume that $v(k\ell+j) > 0$ for $k \geq K_0 \in \mathbb{N}_\ell(k_0)$. From condition (3) and the equation (2), we get

$$\Delta_{\alpha(\ell)}\left(a((k-1)\ell+j)\left|\Delta_{\alpha(\ell)}v((k-1)\ell+j)\right|^{\gamma-1}\Delta_{\alpha(\ell)}v((k-1)\ell+j)\right) \leq 0,\ k \geq K_0$$

and hence $a((k-1)\ell+j)\left|\Delta_{\alpha(\ell)}v((k-1)\ell+j)\right|^{\gamma-1}\Delta_{\alpha(\ell)}v((k-1)\ell+j)$ is nonincreasing. We claim that

$$a((k-1)\ell+j)\left|\Delta_{\alpha(\ell)}v((k-1)\ell+j)\right|^{\gamma-1}\Delta_{\alpha(\ell)}v((k-1)\ell+j) > 0.$$

If $a((k-1)\ell+j)\left|\Delta_{\alpha(\ell)}v((k-1)\ell+j)\right|^{\gamma-1}\Delta_{\alpha(\ell)}v((k-1)\ell+j) \leq 0$, then there exists an integer $K_1 \in \mathbb{N}_\ell(k_0)$ such that

$$a((k-1)\ell+j)\left|\Delta_{\alpha(\ell)}v((k-1)\ell+j)\right|^{\gamma-1}\Delta_{\alpha(\ell)}v((k-1)\ell+j) < (\alpha-1)\sum_{i=K_1}^{k-2} a(i\ell+j)\left|\Delta_{\alpha(\ell)}v(i\ell+j)\right|^{\gamma-1}\Delta_{\alpha(\ell)}v(i\ell+j)$$
$$+ a((K_1-1)\ell+j)\left|\Delta_{\alpha(\ell)}v((K_1-1)\ell+j)\right|^{\gamma-1}\Delta_{\alpha(\ell)}v((K_1-1)\ell+j) \leq 0$$

for $k > K_1$. Since

$$a((k-1)\ell+j)\left|\Delta_{\alpha(\ell)}v((k-1)\ell+j)\right|^{\gamma-1}\Delta_{\alpha(\ell)}v((k-1)\ell+j) = -a((k-1)\ell+j)\left(-\Delta_{\alpha(\ell)}v((k-1)\ell+j)\right)^{\gamma},$$

we have

$$-a((k-1)\ell+j)\left(-\Delta_{\alpha(\ell)}v((k-1)\ell+j)\right)^{\gamma} < -\alpha a((K_1-1)\ell+j)\left(-\Delta_{\alpha(\ell)}v((K_1-1)\ell+j)\right)^{\gamma}$$

or

$$\Delta_{\alpha(\ell)}v((k-1)\ell+j) < \frac{\alpha^{\frac{1}{\gamma}}a^{1/\gamma}((K_1-1)\ell+j)\left(\Delta_{\alpha(\ell)}v((K_1-1)\ell+j)\right)}{a^{1/\gamma}((k-1)\ell+j)},\ k > K_1.$$

Summing the above inequality from K_1+1 to k and letting $k\to\infty$, we obtain, in view of condition (4), $v(k\ell+j)\to-\infty$, a contradiction. Hence there exists an integer $K_0\in\mathbb{N}_\ell(k_0)$ such that $a((k-1)\ell+j)\left|\Delta_{\alpha(\ell)}v((k-1)\ell+j)\right|^{\gamma-1}\Delta_{\alpha(\ell)}v((k-1)\ell+j)>0$ or $\Delta_{\alpha(\ell)}v((k-1)\ell+j)^{\gamma-1}>0$ for all $k\geq K_0$.
Define

$$w(k\ell+j) = \frac{\alpha^{\left[\frac{k}{\ell}\right]-1}a((k-1)\ell+j)\left(\Delta_{\alpha(\ell)}v((k-1)\ell+j)\right)^{\gamma}}{v^{\gamma}((k-1)\ell+j)},\ for\ k\geq K_0.$$

For $k\geq K_0$, we get

$$\Delta_{\alpha(\ell)}w(k\ell+j) \leq -(q(k\ell+j)-p(k\ell+j)) - \frac{\alpha^{\left[\frac{k}{\ell}\right]-1}a((k-1)\ell+j)\left(\Delta_{\alpha(\ell)}v((k-1)\ell+j)\right)^{\gamma}\Delta_\ell v^{\gamma}((k-1)\ell+j)}{v^{\gamma}((k-1)\ell+j)v^{\gamma}(k\ell+j)}. \tag{6}$$

By the mean value theorem

$$\Delta^{\gamma}_{\alpha(\ell)}v((k-1)\ell+j) = \gamma t^{\gamma-1}(k)\Delta_{\alpha(\ell)}v((k-1)\ell+j) \tag{7}$$

where $v((k-1)\ell+j)<t(k)<v(k\ell+j)$. Using (7) in (6), we get

$$\begin{aligned}\Delta_{\alpha(\ell)}w(k\ell+j) &< -(q(k\ell+j)-p(k\ell+j))\\ &- \frac{\gamma\alpha^{\left[\frac{k}{\ell}\right]}a((k-1)\ell+j)\left(\Delta_{\alpha(\ell)}v((k-1)\ell+j)\right)^{\gamma}v^{\gamma-1}((k-1)\ell+j)\Delta_\ell v((k-1)\ell+j)}{v^{\gamma}((k-1)\ell+j)v^{\gamma}(k\ell+j)},\ for\ k\geq K_0.\end{aligned} \tag{8}$$

Now using monotonic nature of $v(k\ell+j)$ and $a(k\ell+j)(\Delta_{\alpha(\ell)}v(k\ell+j))^{\gamma}$, in (8), we obtain

$$\Delta_{\alpha(\ell)}w(k\ell+j) < -(q(k\ell+j)-p(k\ell+j)) - \frac{\alpha^{\left[\frac{k}{\ell}\right]}\gamma w^{1+1/\gamma}((k+1)\ell+j)}{a^{1/\gamma}((k-1)\ell+j)},\ for\ all\ k\geq K_0.$$

Then, for all $k\geq K\geq k_0$

$$\begin{aligned}&\sum_{s=K}^{k-1}\alpha^{\left[\frac{s}{\ell}\right]}H(k\ell+j,s)(q(s\ell+j)-p(s\ell+j)) = H(k\ell+j,K\ell+j)w(K\ell+j)\\
&-\sum_{s=K}^{k-1}\Delta_{2\alpha(\ell)}H(k\ell+j,s\ell+j)w((s+1)\ell+j) + \frac{\gamma H(k\ell+j,s\ell+j)w^{\beta}((s+1)\ell+j)}{\alpha^{\left[\frac{k}{\ell}\right]}a^{1/\gamma}((s-1)\ell+j)}\\
&= H(k\ell+j,K\ell+j)w(K\ell+j) - \sum_{s=K}^{k-1}h(k\ell+j,s\ell+j)H^{1/\beta}(k\ell+j,s\ell+j)w((s+1)\ell+j)\\
&+ \frac{\gamma H(k\ell+j,s\ell+j)}{\alpha^{\left[\frac{k}{\ell}\right]}a^{1/\gamma}((s-1)\ell+j)}w^{\beta}((s+1)\ell+j)\\
&= H(k\ell+j,K\ell+j)w(K\ell+j) - \sum_{s=K}^{k-1}h(k\ell+j,s\ell+j)H^{1/\beta}(k\ell+j,s\ell+j)w((s+1)\ell+j)\\
&+ \frac{\gamma H(k\ell+j,s\ell+j)}{\alpha^{\left[\frac{k}{\ell}\right]}a^{1/\gamma}((s-1)\ell+j)}w^{\beta}((s+1)\ell+j)\\
&+ \frac{h(k\ell+j,s\ell+j)^{\gamma+1}}{\gamma+1}a((s-1)\ell+j) + \sum_{s=K}^{k-1}\frac{h(k\ell+j,s\ell+j)^{\gamma+1}}{\gamma+1}a((s-1)\ell+j).\end{aligned}$$

Hence for all $k \geq K \geq K_0$, we have

$$\sum_{s=K}^{k-1} \alpha^{\left[\frac{s}{\ell}\right]} H(k\ell + j, s\ell + j)(q(s\ell + j) - p(s\ell + j)) - a((s-1)\ell + j)\frac{h(k\ell + j, s\ell + j)^{\gamma+1}}{\gamma + 1}$$

$$= H(k\ell + j, K\ell + j)w(K\ell + j) - \sum_{s=K}^{k-1} h(k\ell + j, s\ell + j)H^{1/\beta} w((s+1)\ell + j)$$

$$+ \frac{\gamma H(k\ell + j, s\ell + j)}{\alpha^{\left[\frac{k}{\ell}\right]}(a^{1/\gamma}(s-1)\ell + j)} w^{\beta}((s+1)\ell + j) + \frac{h(k\ell + j, s\ell + j)^{\gamma+1}}{\gamma + 1} a((s-1)\ell + j).$$

Since $\beta > 0$, we have

$$h(k\ell + j, s\ell + j)H^{1/\beta}(k\ell + j, s\ell + j)w((s+1)\ell + j) + \frac{\gamma H(k\ell + j, s\ell + j)}{\alpha^{\left[\frac{k}{\ell}\right]} a^{1/\gamma}((s-1)\ell + j)} w^{\beta}((s+1)\ell + j)$$

$$+ \frac{h(k\ell + j, s\ell + j)^{\gamma+1}}{\gamma + 1} a((s-1)\ell + j) \geq 0$$

for all $k \geq s \geq K_0$. This implies that for all $k \geq K_0$

$$\sum_{s=K_0}^{k-1} \alpha^{\left[\frac{s}{\ell}\right]} H(k\ell + j, s\ell + j)(q(s\ell + j) - p(s\ell + j)) - a((s-1)\ell + j)\frac{h(k\ell + j, s\ell + j)^{\gamma+1}}{\gamma + 1}$$

$$\leq H(k\ell + j, K_0\ell + j)w(K_0\ell + j) \leq H(k\ell + j, k_0\ell + j)w(K_0\ell + j).$$

Therefore,

$$\sum_{s=k_0}^{k-1} \alpha^{\left[\frac{s}{\ell}\right]} H(k\ell + j, s\ell + j)(q(s\ell + j) - p(s\ell + j)) - a((s-1)\ell + j)\frac{h(k\ell + j, s\ell + j)^{\gamma+1}}{\gamma + 1}$$

$$= \sum_{s=k_0}^{K_0-1} \alpha^{\left[\frac{s}{\ell}\right]} H(k\ell + j, s\ell + j)(q(s\ell + j) - p(s\ell + j)) - (a(s-1)\ell + j)\frac{h(k\ell + j, s\ell + j)^{\gamma+1}}{\gamma + 1}$$

$$+ \sum_{s=K_0}^{k-1} \alpha^{\left[\frac{s}{\ell}\right]} H(k\ell + j, s\ell + j)(q(s\ell + j) - p(s\ell + j)) - (a(s-1)\ell + j)\frac{h(k\ell + j, s\ell + j)^{\gamma+1}}{\gamma + 1}$$

$$\leq \alpha^{\left[\frac{k}{\ell}\right]} H(k\ell + j, k_0\ell + j) \sum_{s=k_0}^{K_0-1} (q(s\ell + j) - p(s\ell + j)) + H(k\ell + j, k_0\ell + j)w(K_0\ell + j)$$

$$= \alpha^{\left[\frac{k}{\ell}\right]} H(k\ell + j, k_0\ell + j) \sum_{s=k_0}^{K_0-1} (q(s\ell + j) - p(s\ell + j)) + w(K_0\ell + j)$$

for all $k \geq K_0$. This gives

$$\limsup_{k\to\infty} \frac{1}{H(k\ell + j, k_0\ell + j)} \sum_{s=k_0}^{k-1} \alpha^{\left[\frac{s}{\ell}\right]} H(k\ell + j, s\ell + j)(q(s\ell + j) - p(s\ell + j)) - a((s-1)\ell + j)\frac{h(k\ell + j, s\ell + j)^{\gamma+1}}{\gamma + 1}$$

$$\leq \sum_{s=k_0}^{K_0-1} (q(s\ell + j) - p(s\ell + j)) + w(K_0\ell + j)$$

which contradicts (7). This completes the proof of the theorem. □

In the following theorems, we obtain oscillation criterion for equation (2) by using a generalized Riccati transformation due to Yu [23].

Theorem 2. *Let $\gamma = 1$ and the conditions (c_1),(c_2),(c_3), equation (3) and (4), satisfies with $g(x, y) \geq \lambda > 0$ for $x, y \neq 0$. Let $H(k\ell + j, s\ell + j)$ be such that $H(k\ell + j, k\ell + j) = 0$ for $k \in \mathbb{N}_\ell(k_0)$, $H(k\ell + j, s\ell + j) > 0$ for $k > s \in \mathbb{N}_\ell(k_0)$ and $\Delta_{2\alpha(\ell)} H(k\ell + j, s\ell + j)$ is nonpositive for all $k \geq s \in \mathbb{N}_\ell(k_0)$. Suppose there exists a real valued function $h(k\ell + j, s\ell + j)$ with*

$$-\Delta_{2\alpha(\ell)}\alpha^{\left[\frac{k}{\ell}\right]} H(k\ell + j, s\ell + j) = h(k\ell + j, s\ell + j)\overline{H(k\ell + j, s\ell + j)} \; for \; all \; k \geq s \in \mathbb{N}_\ell(k_0).$$

If there exists a positive real valued function $\phi(k\ell + j)$ with $\phi((k + 1)\ell + j) < \frac{1}{2\lambda}$ and

$$\limsup_{k\to\infty} \frac{1}{H(k\ell + j, k_0\ell + j)} \sum_{s=k_0}^{k-1} \alpha^{\left[\frac{s}{\ell}\right]} H(k\ell + j, s\ell + j)\psi(s\ell + j) - \frac{1}{4\lambda} \frac{a(s\ell + j)\phi^2((s + 1)\ell + j)}{\alpha^{\left[\frac{s}{\ell}\right]}\phi(s\ell + j)} h^2(k\ell + j, s\ell + j) = \infty$$

where $\phi(k\ell + j) = \frac{\phi(k_0\ell + j)}{1 + 2\lambda((k - k_0)\ell + j)\phi(k_0\ell + j)}$ and

$$\psi(k\ell + j) = \phi(k\ell + j)(q(k\ell + j) - p(k\ell + j)) + \lambda a(k\ell + j)\phi^2((k + 1)\ell + j) - \Delta_{\alpha(\ell)} \left(a((k - 1)\ell + j)\phi(k\ell + j)\right)$$

then all solution of equation (2) are oscillatory.

Proof. Let $v(k\ell + j)$ be a nonoscillatory solution of equation (2). Without loss of generality we may assume that $v(k\ell + j) > 0$ for $k \geq K_0 \in \mathbb{N}_\ell(n_0)$. From the proof of theorem 1 and from equation (2). we have $\Delta_{\alpha(\ell)} v((k-1)\ell + j) > 0$ and $a((k - 1)\ell + j)\Delta_{\alpha(\ell)} v((k - 1)\ell + j)$ is noninceasing for all $k \geq K_1 \geq K_0 + \ell$. Define

$$w(k\ell + j) = \alpha^{\left[\frac{k}{\ell}\right]-1}\phi(k\ell + j)\frac{a((k - 1)\ell + j)\Delta_{\alpha(\ell)} v((k - 1)\ell + j)}{f(v(k\ell + j))} + a((k - 1)\ell + j)\phi(k\ell + j) \; for \; k \geq K_1.$$

Then

$$\begin{aligned}
\Delta_{\alpha(\ell)} w(k\ell + j) = {} & \Delta_{\alpha(\ell)}\alpha^{\left[\frac{k}{\ell}\right]-1}\phi(k\ell + j)\frac{a(k\ell + j)\Delta_{\alpha(\ell)} v(k\ell + j)}{f(v(k - 1)\ell + j)} + a(k\ell + j)\phi((k + 1)\ell + j) \\
& + \frac{\alpha^{\left[\frac{k}{\ell}\right]}\Delta_{\alpha(\ell)} \left(a((k - 1)\ell + j)\Delta_{\alpha(\ell)} v((k - 1)\ell + j)\right)}{f(k\ell + j)} \\
& - \frac{\alpha^{\left[\frac{k}{\ell}\right]} a(k\ell + j)\Delta_{\alpha(\ell)} v(k\ell + j) g(v(k\ell + j), v((k - 1)\ell + j))\Delta_{\alpha(\ell)} v(k\ell + j)}{f(v(k\ell + j))f(v((k - 1)\ell + J))} \\
& + \Delta_{\alpha(\ell)}(a((k - 1)\ell + j)\phi(k\ell + j)), \; k \geq K_1.
\end{aligned}$$

In view of condition (c_1) and from the monotonic nature of $v(k\ell + j)$ and $a((k - 1)\ell + j)\Delta_{\alpha(\ell)} v((k - 1)\ell + j)$, we obtain

$$\begin{aligned}
\Delta_{\alpha(\ell)} w(k\ell + j) = {} & \frac{\Delta_{\alpha(\ell)}\alpha^{\left[\frac{k}{\ell}\right]-1}\phi(k\ell + j)z((k + 1)\ell + j)}{(\phi(k + 1)\ell + j)} + \phi(k\ell + j) - (q(k\ell + j) - p(k\ell + j)) \\
& - \frac{\alpha^{\left[\frac{k}{\ell}\right]}\gamma}{a(k\ell + j)} \frac{z((k + 1)\ell + j)}{(\phi(k + 1)\ell + j)} - a(k\ell + j)\phi(k + 1)\ell + j)^2 + \Delta_{\alpha(\ell)}(a((k - 1)\ell + j)\phi(k\ell + j)) \\
= {} & -\psi(k\ell + j) - \frac{\alpha^{\left[\frac{k}{\ell}\right]}\gamma}{a(k\ell + j)} \frac{\phi(k\ell + j)}{\phi^2((k + 1)\ell + j)} w^2((k + 1)\ell + j).
\end{aligned}$$

Hence for all $k \geq K \geq K(1)$, we have

$$\sum_{s=K}^{k-1} \alpha^{\left[\frac{s}{\ell}\right]} H(k\ell + j, s\ell + j)\psi(s\ell + j) \leq H(k\ell + j, K\ell + j)\psi(k\ell + j, K\ell + j)\psi(K\ell + j)$$
$$- \sum_{s=K}^{k-1} \alpha^{\left[\frac{s}{\ell}\right]} \left(-\Delta_{2(\alpha(\ell))} H(k\ell + j, s\ell + j) w((k+1)\ell + j)\right)$$
$$+ \frac{\gamma H(t\ell + j, s\ell + j)}{\alpha^{\left[\frac{s}{\ell}\right]} a(s\ell + j)\phi^2((s+1)\ell + j)} \phi(s\ell + j) w^2((s+1)\ell + j)$$
$$= \alpha^{\left[\frac{k}{\ell}\right]} H(k\ell + j, K\ell + j)\psi(K\ell + j) - \sum_{s=K}^{k-1} h(k\ell + j, s\ell + j)\overline{H(k\ell + j, s\ell + j)} w((s+1)\ell + j)$$
$$+ \frac{\gamma}{\alpha^{\left[\frac{k}{\ell}\right]} a(s\ell + j)} \frac{\phi(s\ell + j)}{\phi((s+1)\ell + j)} H(k\ell + j, s\ell + j) w^2((s+1)\ell + j)$$
$$= \alpha^{\left[\frac{k}{\ell}\right]} H(k\ell + j, K\ell + j)\psi(K\ell + j) - \sum_{s=K}^{k-1} \overline{\frac{\gamma\phi(s\ell + j) H(k\ell + j, s\ell + j)}{a(s\ell + j)\phi^2((s+1)\ell + j)}} w((s+1)\ell + j)$$
$$+ \frac{1}{2} \frac{h(k\ell + j, s\ell + j)\phi((s+1)\ell + j)\sqrt{a(s\ell + j)^2}}{\sqrt{\gamma\phi(s\ell + j)}} + \frac{1}{4\gamma} \sum_{s=K}^{k-1} \frac{h^2(k\ell + j, s\ell + j)\phi^2((s+1)\ell + j) a(s\ell + j)}{\alpha^{\left[\frac{s}{\ell}\right]}\phi(s\ell + j)}.$$

Then, for all $k \geq K \geq K_1$

$$\sum_{s=K}^{k-1} \alpha^{\left[\frac{s}{\ell}\right]} H(k\ell + j, s\ell + j)\psi(s\ell + j) - \frac{1}{4\gamma} \frac{a(s\ell + j)\phi^2((s+1)\ell + j) h^2(k\ell + j, s\ell + j)}{\alpha^{\left[\frac{s}{\ell}\right]}\phi(s\ell + j)}$$
$$\leq \alpha^{\left[\frac{s}{\ell}\right]} H(k\ell + j, K\ell + j)\psi(K\ell + j) - \sum_{s=K}^{k-1} \overline{\frac{\gamma\phi(s\ell + j) H(k\ell + j, s\ell + j)}{a(s\ell + j)\phi^2((s+1)\ell + j)}} w((s+1)\ell + j)$$
$$+ \frac{1}{2} \frac{h(k\ell + j, s\ell + j)\phi((s+1)\ell + j)\sqrt{a(s\ell + j)^2}}{\sqrt{\gamma\phi(s\ell + j)}}$$

This implies that for every $k \geq K_1$

$$\sum_{s=K_1}^{k-1} H(k\ell + j, s\ell + j)\psi(s\ell + j) - \frac{1}{4\gamma} \frac{a(s\ell + j)\phi^2((s+1)\ell + j) h^2(k\ell + j, s\ell + j)}{\alpha^{\left[\frac{s}{\ell}\right]}\phi(s\ell + j)}$$
$$\leq H(k\ell + j, K_1\ell + j)\psi(N_1\ell + j) \leq H(k\ell + j, K_1\ell + j)|\psi(N_1\ell + j)| \leq H(k\ell + j, k_0)|\psi(N_1\ell + j))|.$$

The rest of the proof is similar to that of Theorem 1 and hence the details are omitted.

To prove the next result, we need the following well-known inequality which is due to Hardy, Littlewood and Polya ([24]Theorem 41). □

Lemma 3. *If U and V are nonnegative, then*

$$U^{\beta} + (\beta - 1)V^{\beta} - \beta U V^{\beta - 1} \geq 0, \ \beta > 1,$$

where equality holds if and only if $U = V$.

Theorem 4. *Let conditions (c_1), (c_2), (c_3) with $g(x\ell + j, y\ell + j) \geq 0$ for $x, y \neq 0$, and condition (3) and (4) hold. If there exists a positive (nondecreasing for $\gamma \neq 1$) real valued function $c(k\ell + j)$ such that*

$$\sum_{n=k_1+1}^{\infty} c(k\ell + j)(q(k\ell + j) - p(k\ell + j)) = \infty \tag{9}$$

and

$$\sum_{n=k_1+1}^{\infty}\frac{a((k-1)\ell+j)(\Delta_{\ell}c((k-1)\ell+j)^{\lambda+1}}{c^{\gamma}(n)\sum_{s=k}^{\infty}(q(s\ell+j)-p(s\ell+j))^{\lambda}}<\infty \tag{10}$$

then all solutions of equation (2) are oscillatory.

Proof. Suppose there exists a nonoscillatory solution $v(k\ell+j)$. Without loss of generality, we may assume that $v(k\ell+j)>0$ for $k\in\mathbb{N}_{\ell}(k_0)$. From the proof of Theorem 1,

$$\Delta_{\alpha(\ell)}v((k-1)\ell+j)>0 \text{ and } a((k-1)\ell+j)(\Delta_{\alpha(\ell)}v((k-1)\ell+j))^{\gamma}\le a(k_1\ell+j)(\Delta_{\alpha(\ell)}v(k\ell+j))^{\gamma}=M_0. \tag{11}$$

for all $k\ge k_1\in\mathbb{N}_{\ell}(k_0)$.
Since,

$$\Delta_{\alpha(\ell)}\frac{\alpha^{\left[\frac{k}{\ell}\right]-1}a((k-1)\ell+j)(\Delta_{\alpha(\ell)}v((k-1)\ell+j))^{\gamma}}{f(v(k\ell+j))}<-(q(k\ell+j)-p(k\ell+j)) \text{ for } k\ge k_1.$$

Summing the last inequality from $k\in\mathbb{N}_{\ell}(k_1)$, to $K+\ell$ and letting $K\to\infty$, we have

$$0\le\lim_{K\to\infty}\frac{\alpha^{\left[\frac{K}{\ell}\right]-1}a(K\ell+j)(\Delta_{\alpha(\ell)}v(K\ell+j))^{\gamma}}{f(v(K\ell+j))}\le\frac{\alpha^{\left[\frac{K}{\ell}\right]-1}a((k-1)\ell+j)(\Delta_{\alpha(\ell)}v((k-1)\ell+j))^{\gamma}}{f(v(k\ell+j))}$$
$$-\sum_{s=k}^{\infty}(q(s\ell+j)-p(s\ell+j)).$$

Thus

$$\frac{1}{a((k-1)\ell+j)}\sum_{s=k}^{\infty}(q(s\ell+j)-p(s\ell+j))\le\frac{\alpha^{\left[\frac{k}{\ell}\right]-1}(\Delta_{\alpha(\ell)}v((k-1)\ell+j))^{\gamma}}{f(v(K\ell+j))}\text{for } k\in\mathbb{N}_{\ell}(k_1). \tag{12}$$

Now,

$$\Delta_{\alpha(\ell)}\frac{\alpha^{\left[\frac{k}{\ell}\right]-1}a((k-1)\ell+j)c((k-1)\ell+j)(\Delta_{\alpha(\ell)}v((k-1)\ell+j))^{\gamma}}{f(v((k-1)\ell+j))}$$
$$=c(k\ell+j)\frac{G(k\ell+j,v(k\ell+j),\Delta_{\alpha(\ell)}u(k\ell+j))-F(k\ell+j,v(k\ell+j))}{f(v(k\ell+j))}$$
$$-\frac{c(k\ell+j)a((k-1)\ell+j)(\Delta_{\alpha(\ell)}v((k-1)\ell+j))^{\gamma+1}g(v(k\ell+j),v((k-1)\ell+j))}{f(v(k\ell+j))f(v((k-1)\ell+j))}$$
$$+\frac{a((k-1)\ell+j)(\Delta_{\alpha(\ell)}v((k-1)\ell+j))^{\gamma}\Delta_{\alpha(\ell)}c((k-1)\ell+j)}{f(v((k-1)\ell+j))}$$
$$\le -c(k\ell+j)(q(k\ell+j)-p(k\ell+j))-\frac{\lambda c(k\ell+j)a((k-1)\ell+j)(\Delta_{\alpha(\ell)}v((k-1)\ell+j))^{\gamma+1}}{f(v(k\ell+j))f(v((k-1)\ell+j))}$$
$$+\frac{a((k-1)\ell+j)(\Delta_{\alpha(\ell)}v((k-1)\ell+j))^{\gamma}\Delta_{\alpha(\ell)}c((k-1)\ell+j)}{f(v((k-1)\ell+j))}$$
$$=-c(k\ell+j)(q(k\ell+j)-p(k\ell+j))-\frac{a((k-1)\ell+j)f^{\gamma}(v(k\ell+j))}{f(v((k-1)\ell+j))}\frac{\lambda c(k\ell+j)(\Delta_{\alpha(\ell)}v((k-1)\ell+j))^{\gamma+1}}{(f(v(k\ell+j)))^{\gamma+1}}$$
$$-\frac{(\Delta_{\alpha(\ell)}v((k-1)\ell+j))^{\gamma}\Delta_{\alpha(\ell)}c((k-1)\ell+j)}{f^{\gamma}(v(k\ell+j))}+\frac{\left(\frac{\gamma\Delta_{\alpha(\ell)}c((k-1)\ell+j)}{\gamma+1}\right)^{\gamma+1}}{\gamma\lambda^{\gamma}c^{\gamma}(k\ell+j)}$$
$$+\frac{a((k-1)\ell+j)\frac{\gamma}{\gamma+1}\Delta_{\alpha(\ell)}c((k-1)\ell+j)^{\gamma+1}}{\gamma\lambda^{\gamma}c^{\gamma}(k\ell+j)}\frac{f^{\gamma}(v(k\ell+j))}{f(v((k-1)\ell+j))}.$$

Since by lemma

$$\frac{\lambda c(k\ell+j)\,(\Delta_{\alpha(\ell)}v((k-1)\ell+j))^{\gamma+1}}{f(v(k\ell+j))^{\gamma+1}} - \frac{(\Delta_{\alpha(\ell)}v((k-1)\ell+j))^{\gamma}\Delta_{\alpha(\ell)}c((k-1)\ell+j)}{f^{\gamma}(v(k\ell+j))} + \frac{\frac{\gamma}{\gamma+1}\Delta_{\alpha(\ell)}c((k-1)\ell+j)^{\gamma+1}}{\gamma\lambda^{\gamma}c^{\gamma}(k\ell+j)} \geq 0$$

for all $k \geq k_1$. Then we have

$$\begin{aligned}
&\Delta_{\alpha(\ell)}\frac{\alpha^{\left[\frac{k}{\ell}\right]-1}a((k-1)\ell+j)c((k-1)\ell+j)(\Delta_{\alpha(\ell)}v((k-1)\ell+j))^{\gamma}}{f(v((k-1)\ell+j))}\\
&\leq -c(k\ell+j)(q(s\ell+j)-p(s\ell+j)) + \frac{a((k-1)\ell+j)\frac{\gamma}{\gamma+1}\Delta_{\alpha(\ell)}c((k-1)\ell+j)^{\gamma+1}}{\gamma\lambda^{\gamma}c^{\gamma}(k\ell+j)}\frac{f^{\gamma}(v(k\ell+j))}{f(v((k-1)\ell+j))}\\
&= -c(k\ell+j)(q(k\ell+j)-p(k\ell+j))\\
&+ \frac{a((k-1)\ell+j)\frac{\gamma}{\gamma+1}\Delta_{\alpha(\ell)}c((k-1)\ell+j)^{\gamma+1}(\Delta_{\alpha(\ell)}v((k-1)\ell+j))^{\gamma^2}f^{\gamma}(v(k\ell+j))}{\gamma\lambda^{\gamma}c^{\gamma}(k\ell+j)(\Delta_{\alpha(\ell)}v((k-1)\ell+j))^{\gamma^2}f(v((k-1)\ell+j))}\\
&= -c(k\ell+j)(q(k\ell+j)-p(k\ell+j))\\
&+ \frac{a^{\gamma-1}((k-1)\ell+j)\frac{\gamma}{\gamma+1}\Delta_{\alpha(\ell)}c((k-1)\ell+j)^{\gamma+1}(a((k-1)\ell+j)(\Delta_{\alpha(\ell)}v((k-1)\ell+j))^{\gamma})^{\gamma}}{\gamma\lambda^{\gamma}c^{\gamma}(k\ell+j)\left(\frac{(\Delta_{\alpha(\ell)}v((k-1)\ell+j))^{\gamma}}{f(v(k\ell+j))}\right)^{\gamma}f(v(k_1-1)\ell+j))}.
\end{aligned} \tag{13}$$

Using (11), (12) in (13), we obtain

$$\begin{aligned}
&\Delta_{\alpha(\ell)}\frac{\alpha^{\left[\frac{k}{\ell}\right]-1}a((k-1)\ell+j)c((k-1)\ell+j)(\Delta_{\alpha(\ell)}v((k-1)\ell+j))^{\gamma}}{f(v((k-1)\ell+j))}\\
&\leq -c(k\ell+j)(q(k\ell+j)-p(k\ell+j)) + \frac{Ma((k-1)\ell+j)\,(\Delta_{\alpha(\ell)}c((k-1)\ell+j))^{\gamma+1}}{c^{\gamma}(k\ell+j)\sum_{s=k}^{\infty}(q(s\ell+j)-p(s\ell+j))^{\gamma}}
\end{aligned} \tag{14}$$

where $M = \frac{M_0\frac{\gamma}{\gamma+1}^{\gamma+1}}{\gamma\lambda^{\gamma}f(v((k_1-1)\ell+j))}$. Summing (14) from k_1+1 to k, we have

$$\begin{aligned}
\frac{\alpha^{\left[\frac{k}{\ell}\right]-1}a(k\ell+j)c(k\ell+j)(\Delta_{\alpha(\ell)}v(k\ell+j))^{\gamma}}{f(v(k\ell+j))} &\leq \frac{\alpha^{\left[\frac{k}{\ell}\right]-1}a(k_1\ell+j)c(k_1\ell+j)(\Delta_{\alpha(\ell)}v(k_1\ell+j))^{\gamma}}{f(v(k_1\ell+j))}\\
&- \sum_{n=k_1+1}^{k}c(k\ell+j)(q(k\ell+j)-p(k\ell+j)) + M\sum_{n=k_1+1}^{k}\frac{a((k-1)\ell+j)(\Delta_{\alpha(\ell)}c((k-1)\ell+j))^{\gamma+1}}{c(k\ell+j)\sum_{i=k}^{\infty}(q(i\ell+j)-p(i\ell+j))^{\gamma}}.
\end{aligned}$$

Letting $k \to \infty$ and noting (9) and (10), we get

$$\lim_{k\to\infty}\frac{\alpha^{\left[\frac{k}{\ell}\right]-1}a(k\ell+j)c(k\ell+j)(\Delta_{\alpha(\ell)}v(k\ell+j))^{\gamma}}{f(v(k\ell+j))} = -\infty$$

which is a contradiction. This completes the proof of the theorem. □

Remark 5. When $\gamma = 1$ and $f(x) = x$ Theorem 4 reduces to Theorem 2 of B. Liu and J. Yan [25].

Example 6. *Consider the generalized $\alpha-$difference equation*

$$\Delta_{\alpha(\ell)}\frac{1}{(k\ell+j)^{1+\sigma}}|\Delta_{\alpha(\ell)}v((k-1)\ell+j)|^{\gamma-1}\Delta_{\alpha(\ell)}v(k\ell+j) + \frac{2^{\gamma}v(k\ell+j)}{(k\ell+j)^{1+\delta}} = \frac{-2^{\gamma}v(k\ell+j)}{4((k+1)\ell+j)^{1+\delta}}(\Delta_{\alpha(\ell)}v(k\ell+j))^{2}, \tag{15}$$

for $k \in \mathbb{N}_\ell(1)$, where $\gamma > 0,\ 0 < \delta < 1$. Let $c(k\ell+j) = k\ell+j,\ q(k\ell+j) = \frac{1}{(k\ell+j)^{1+\delta}}$ and $p(k\ell+j) = 0$, $k \in \mathbb{N}_\ell(1)$ and $f(v(k\ell+j)) = v(k\ell+j)$, then we find that all conditions of Theorem 4 are satisfied. Hence all solutions of (15) are oscillatory. In fact $v(k\ell+j) = -1^{k}$ is such a solution.

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