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Performance characteristics of a batch service queueing system with functioning server failure and multiple vacations

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Performance characteristics of a batch service queueing system with functioning server failure and multiple vacations

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Abstract. This paper examines bulk arrival and batch service queueing system with functioning server failure and multiple vacations. Customers are arriving into the system in bulk according to Poisson process with rate λ. Arriving customers are served in batches with minimum of 'a' and maximum of 'b' number of customers according to general bulk service rule. In the service completion epoch if the queue length is less than 'a' then the server leaves for vacation (secondary job) of random length. After a vacation completion, if the queue length is still less than 'a' then the server leaves for another vacation. The server keeps on going vacation until the queue length reaches the value 'a'. The server is not stable at all the times. Sometimes it may fails during functioning of customers. Though the server fails service process will not be interrupted.It will be continued for the current batch of customers with lower service rate than the regular service rate. The server will be repaired after the service completion with lower service rate. The probability generating function of the queue size at an arbitrary time epoch will be obtained for the modelled queueing system by using supplementary variable technique. Moreover various performance characteristics will also be derived with suitable numerical illustrations.

1. Introduction

Mathematical modelling and analysis of queueing system plays a vital role in many real time systems such as communication networks, production line systems, manufacturing industries, etc. Mathematical analysis of queueing systems helps to reduce the congestion existed in many real time systems. Also it helps to reduce the total average cost of the system.

Queueing system with vacation has been analysed by many researchers.Some of them which includes a single server queueing models with vacations by Takagi [12]and vacation queueing models by Tian and Zhang [13].Neuts[9] introduced general bulk service rule for batch arrival queueing systems. Arumuganathan and Jeyakumar[1] analysed $M^X/G(a,b)/1$ queueing system with multiple vacations, setup times, closedown times and N-policy. Lee et al.[7]studied $M^X/G/1$queue with N-policy and multiple vacations. Recently Jeyakumar and Senthilnathan [6] introduced multiple working vacations for $M^X/G(a,b)/1$ queueing system. Niranjan et. al. [11] presented secondary service for bulk arrival and batch service queueing system with vacation and breakdown.

It is clear that if breakdown occurs the server is allowed to interrupt immediately. But in most of the situations it is not possible to disturb the server before completing the batch of service. Breakdown without service interruption in a bulk arrival and batch service queueing model are also studied by Jeyakumar and Senthilnathan[5]. They modelled with closedown time and derived probability generating function of service completion epoch, vacation completion epoch and renovation completion epoch. Wu et al. [14]analysed an M/G/1 queue with N-policy, single vacation, unreliable service station and replaceable repair facility. Niranjan et al.[10] used supplementary variable technique to derive probability generating function of queue size at an arbitrary timefor state dependent service in bulk arrival queueing system with server loss and vacation break-off.

In all the above queueing systems, when the server gets breakdown it will be send to repair station immediately or after the service. But in this model though the server gets breakdown, service process will not be interrupted it will be continued for current batch of customers by lower service rate than the regular service rate. Addressing this the authors modelled $M^X/G(a,b)/1$ queueing system with functioning server failure and multiple vacations.

2. Model description

In this paper bulk arrival and batch service queueing system with functioning server failure and multiple vacations are considered. An arrival of customers into the system in bulk according to Poisson process with rate λ. Server provides service in batches according to general bulk service rule introduced by Neuts[9].Server capacity ranges from minimum of 'a' and maximum of 'b' number of customers. During service completionif the queue lengthτ (say) is less than $'b'$ then the server picks only 'b' customers for service. Remaining $\tau - b$ customers will be served in succeedingbatch of service. On the other hand if $a \leq \tau \leq b$ then the server will take entire customers for service. The server may fails at any time. When the server gets failure while serving customers, the service process will not be stopped because it is not possible to repair the server in the small period, and therefore service process will be continued for current batch with lower service rate than the regular service rate by doing some technical precaution arrangements. The process of maintaining the server or identifyingthe server failure and repairing of the server is called renewal of service station. The server will be repaired after the service process during renewal of service station. At a service completion or renewal time completion epoch if the queue length is less than 'a' then the server leaves for vacation (secondary job)of random length. If the queue length is still less than 'a' even after a vacation then the server leaves for another vacation (multiple vacations). Likewise the vacation period will be continued until the queue length attains the value 'a'. The pictorial representation of the proposed queueing system is depicted below.

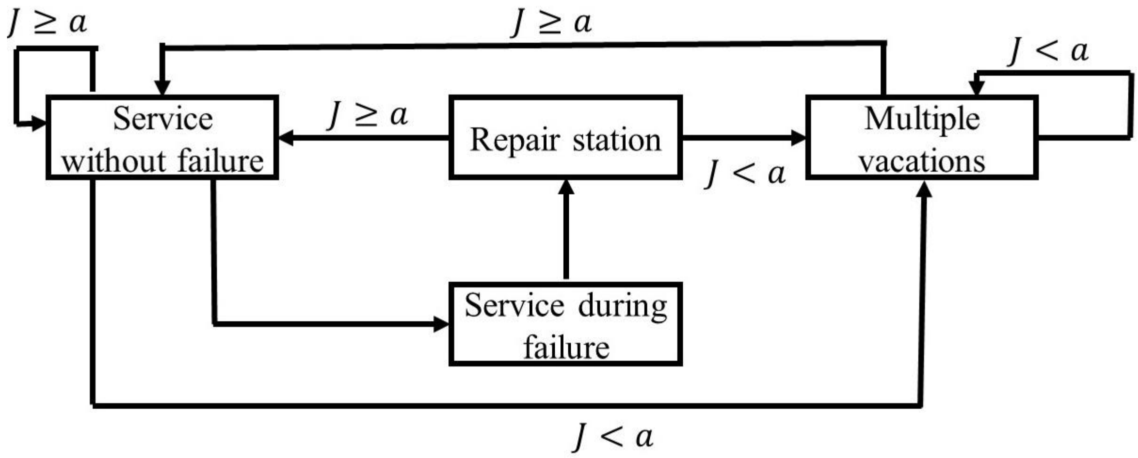

Figure 1. Schematic representation of the queueing model: J - Queue length

2.1. Notations

Let λ be the Poisson arrival rate, X be the group size random variable of the arrival, g_k be the probability that 'k' customers arrive in a batch, X (z) be the probability generating function of X, N_q(t) be the number of customers waiting for service at time t, N_s(t) be the number of customers under the service at time t.γ be the failure rate.

$$C(t)=\begin{cases}0, \text{ when the server is busy with regular service}\\ 1, \text{ when the server is busy during server failure}\\ 2, \text{ when the server is on vacation}\\ 3, \text{ when the server is on renewal process}\end{cases}$$

Y (t) =j. if the server is onj^{th}vacation

The state probabilities are defined as follows:

$$A_{ij}(x,t)dt = P_r\{N_s(t)=i, N_q(t)=j, x \le S_1{}^0(t) \le x+dt,\ C(t)=0\}\ a\le i\le b; j\ge 0$$

$$G_{ij}(x,t)dt = P_r\{N_s(t)=i, N_q(t)=j, \quad x \le S_2{}^0(t) \le x+dt,\ C(t)=1\}$$

$$H_n(x,t)dt = P_r\{ \quad N_q(t)=n, x \le T^0(t) \le x+dt,\ C(t)=2, Y(t)=j)\}; 1\le n\le a-1$$

$$F_n(x,t)dt = P_r\{ \quad N_q(t)=n, \quad x \le F^0(t) \le x+dt,\ C(t)=3\}; n\ge 0$$

	Cumulative distribution function	**Probability density Function**	**Laplace-Stieltjes transform**	**Remaining time**
Regular service	$S_1(x)$	$s_1(x)$	$\tilde{S}_1(\theta)$	$S_1{}^0(x)$
Service during server failure	$S_2(x)$	$s_2(x)$	$\tilde{S}_2(\theta)$	$S_2{}^0(x)$
Vacation	T(x)	t(x)	$\tilde{T}(\theta)$	$T^0(x)$
Renewal time	$T(x)$	$f(x)$	$\tilde{F}(\theta)$	$U^0(x)$

3. Steady state queue size distribution

$$-\frac{d}{dx}A_{i0}(x) = -\lambda A_{i0}(x) + \left(\sum_{m=c}^{b} A_{mi}(0) + \sum_{\beta=1}^{\infty} H_{\beta i}(0) + F_i(0)\right)s_1(x) a\le i\le b \qquad (1)$$

$$-\frac{d}{dx}A_{ij}(x) = -\lambda A_{1a-1}(x) + \sum_{k=1}^{j} A_{i\,j-k}(x)\lambda g_k \qquad a\le i\le b-1 j\ge 1 \qquad (2)$$

$$-\frac{d}{dx}A_{bj}(x) = -\lambda A_{1a-1}(x) + \left(\sum_{m=c}^{b} A_{mb+j}(0) + \sum_{\beta=1}^{\infty} H_{\beta b+j}(0) + F_{b+j}(0)\right)s_1(x) j\ge 1 \qquad (3)$$

$$-\frac{d}{dx}G_{ij}(x) = -\lambda G_{ij}(x) + \sum_{i=a}^{b} A_{ij}(x)s_2(x) \qquad (4)$$

$$-\frac{d}{dx}F_n(x) = -(\lambda+\gamma)F_n(x) + \sum_{i=a}^{b} G_{ij}(0)f(x) \qquad (5)$$

$$-\frac{d}{dx}H_{10}(x) = -\lambda H_{10}(x) + \sum_{i=a}^{b} A_{i0}(0)t(x) + F_0(0)t(x) \qquad (6)$$

$$-\frac{d}{dx}H_{1j}(x) = -\lambda H_{1j}(x) + \sum_{k=1}^{j} H_{1\ j-k}(x)\,\lambda g_k + \sum_{i=a}^{b} A_{ij}(0)t(x) + F_j(0)t(x) \quad 1 \le j \le a-1 \tag{7}$$

$$-\frac{d}{dx}H_{l0}(x) = -\lambda H_{l0}(x) + H_{l-1\ 0}(0)t(x) \qquad l \ge 2 \tag{8}$$

$$-\frac{d}{dx}H_{lj}(x) = -\lambda H_{lj}(x) + \sum_{k=1}^{j} H_{l\ j-k}(x)\,\lambda g_k + H_{l-1\ j}(0)t(x) \quad j = 1,2,\dots a-1 \tag{9}$$

$$-\frac{d}{dx}H_{lj}(x) = -\lambda H_{lj}(x) + \sum_{k=1}^{j} H_{l\ j-k}(x)\,\lambda g_k \qquad j \ge a \quad l \ge 2 \tag{10}$$

Laplace-Stieltjes transform

$$\tilde{A}_{in}(\theta) = \int_0^\infty e^{-\theta x} A_{in}(x)dx \qquad \tilde{G}_{in}(\theta) = \int_0^\infty e^{-\theta x} G_{in}(x)dx \tag{11}$$
$$\tilde{H}_{ln}(\theta) = \int_0^\infty e^{-\theta x} H_{ln}(x)dx \qquad \tilde{F}_n(\theta) = \int_0^\infty e^{-\theta x} F_{in}(x)dx$$

$$\theta\tilde{A}_{i0}(\theta) - A_{i0}(0) = \lambda\tilde{A}_{i0}(\theta) - \left(\sum_{m=c}^{b} A_{mi}(0) + \sum_{\beta=1}^{\infty} H_{\beta i}(0) + F_i(0)\right)\tilde{S}_1(\theta) a \le i \le b \tag{12}$$

$$\theta\tilde{A}_{ij}(\theta) - A_{ij}(0) = \lambda\tilde{A}_{ij}(\theta) - \sum_{k=1}^{j} \tilde{A}_{i\ j-k}(\theta)\lambda g_k \qquad a \le i \le b-1 j \ge 1 \tag{13}$$

$$\theta\tilde{A}_{bj}(\theta) - A_{bj}(0) = \lambda\tilde{A}_{bj}(\theta) - \left(\sum_{m=c}^{b} A_{mb+j}(0) + \sum_{\beta=1}^{\infty} H_{\beta b+j}(0) + F_{b+j}(0)\right)\tilde{S}_1(\theta) \tag{14}$$

$$\theta\tilde{G}_{ij}(\theta) - G_{ij}(0) = \lambda\tilde{G}_{ij}(\theta) - \sum_{i=a}^{b} \tilde{A}_{ij}(\theta)\tilde{S}_2(\theta) \tag{15}$$

$$\theta\tilde{F}_n(\theta) - F_n(0) = (\lambda+\gamma)\tilde{F}_n(\theta) - \sum_{i=a}^{b} G_{ij}(0)\tilde{F}(\theta) \tag{16}$$

$$\theta\tilde{H}_{10}(\theta) - H_{10}(0) = \lambda\tilde{H}_{10}(\theta) - \left\{\sum_{i=a}^{b} A_{i0}(0) + F_0(0)\right\}\tilde{T}(\theta) \tag{17}$$

$$\theta\tilde{H}_{1j}(\theta) - H_{1j}(0) = \lambda\tilde{H}_{1j}(\theta) - \sum_{k=1}^{j} H_{1\ j-k}(x)\,\lambda g_k - \left(\sum_{i=a}^{b} A_{ij}(0) + F_j(0)\right)\tilde{T}(\theta) \quad 1 \le j \le a-1 \tag{18}$$

$$\theta\tilde{H}_{l0}(\theta) - H_{l0}(0) = \lambda\tilde{H}_{l0}(\theta) - H_{l-1\ 0}(0)\tilde{T}(\theta) l \ge 2 \tag{19}$$

$$\theta\tilde{H}_{lj}(\theta) - H_{lj}(0) = \lambda\tilde{H}_{lj}(\theta) - \sum_{k=1}^{j} \tilde{H}_{l\ j-k}(\theta)\,\lambda g_k - H_{l-1\ j}(0)\tilde{T}(\theta) \quad j = 1,2,\dots a-1 \tag{20}$$

$$\theta\tilde{H}_{lj}(\theta) - H_{lj}(0) = \lambda\tilde{H}_{lj}(\theta) - \sum_{k=1}^{j} \tilde{H}_{l\ j-k}(\theta)\,\lambda g_k \qquad \boldsymbol{j \ge a} \tag{21}$$

4. Probability generating function

$$\tilde{A}_i(z,\theta) = \sum_{j=0}^{\infty} \tilde{A}_{ij}(\theta)z^j \qquad A_i(z,0) = \sum_{j=0}^{\infty} A_{ij}(0)z^j \qquad a \le i \le b \tag{22}$$

$$\tilde{G}_i(z,\theta) = \sum_{j=0}^{\infty} \tilde{G}_{ij}(\theta)z^j \qquad G_i(z,0) = \sum_{j=0}^{\infty} G_{ij}(0)z^j$$

$$\tilde{H}_j(z,\theta) = \sum_{l=1}^{\infty} \tilde{H}_{1j}(0)z^j \qquad H_j(z,0) = \sum_{l=1}^{\infty} H_{lj}(0)z^j \qquad j \ge 1$$

$$\tilde{F}(z,\theta)=\sum_{n=0}^{\infty}\tilde{F}_n(\theta)\,z^n F(z,0)=\sum_{n=0}^{\infty}R_n\,(0)\,z^n$$

$$(\theta-\lambda+\lambda X(Z))\tilde{A}_i(z,\theta)=A_i(z,0)-\left(\sum_{m=c}^{b}A_{mi}(0)+\sum_{\beta=1}^{\infty}H_{\beta i}(0)+F_i(0)\right)\tilde{S}_1(\theta) \quad (23)$$

$$a\le i\le b-1$$

$$z^b\big(\theta-\lambda+\lambda X(Z)\big)\tilde{A}_b(z,\theta)=A_b(z,0)z^b-\begin{pmatrix}\left(\sum_{m=c}^{b}A_m(z,0)-\sum_{j=0}^{b-1}A_{mj}(0)z^j\right)\\+F(z,0)-\sum_{j=0}^{b-1}F_j(0)z^j\\+\left(\sum_{\beta=1}^{\infty}\left(H_\beta(z,0)-\sum_{j=0}^{b-1}H_{\beta j}(0)z^j\right)\right)\end{pmatrix} \quad (24)$$

$$(\theta-\lambda+\lambda X(Z))\tilde{G}_i(z,\theta)=G_i(z,0)-\sum_{i=a}^{b}\tilde{A}_{ij}\,(\theta)\tilde{S}_2(\theta) \quad (25)$$

$$(\theta-\lambda+\lambda X(Z))\widetilde{H}_\beta(z,\theta)=H_\beta(z,0)-\tilde{T}(\theta)\sum_{n=0}^{a-1}H_{j-1n}\,(0)z^n j\ge 2 \quad (26)$$

$$(\theta-(\lambda+\gamma)+\lambda X(Z))\tilde{F}(z,\theta)=F(z,0)-\tilde{F}(\theta)\left(\sum_{m=a}^{b}G_m\,(z,0)\right) \quad (27)$$

$$(\theta-\lambda+\lambda X(z))\widetilde{H}_1(z,\theta)=H_1(z,0)-\left(\sum_{j=0}^{a-1}\sum_{i=a}^{b}A_{ij}(0)+R_j(0)\right)\tilde{T}(\theta) \quad (28)$$

4.1. Probability generating function of the queue size at an arbitrary time epoch

Let P (z) be the probability generating function of the queue size at an arbitrary time epoch, then

$$P(z)=\sum_{m=a}^{b-1}\tilde{A}_m(z,0)+\tilde{A}_b(z,0)+\sum_{m=a}^{b}\tilde{G}_m(z,0)+\sum_{\beta=1}^{\infty}\widetilde{H}_\beta(z,0)+\tilde{F}(z,0) \quad (29)$$

Substituting $\theta=\lambda-\lambda X\,(z)$ in equations from (23) to (27) and $\theta=(\lambda+\gamma)-\lambda X(z)$in equation (28), after doing some algebra, the PGF of the queue size is defined in (29) is simplified as

$$P(z)=\frac{M_3\sum_{i=a}^{b-1}(A_i+F_i+H_i)+\ M_2\sum_{n=0}^{b-1}(A_n+F_n+H_n)z^n\\ \left(\left(\tilde{T}\big(\lambda-\lambda X(z)\big)-1\right)M_1+M_2\right)\sum_{n=0}^{a-1}(A_n+F_n+H_n)z^n}{\boldsymbol{M_1}\big(-\lambda+\lambda X(z)\big)}$$

where

$$M_1=\left(z^b-\tilde{S}_1\big(\lambda-\lambda X(z)\big)-\tilde{S}_1\big(\lambda-\lambda X(z)\big)\tilde{F}\big((\lambda+\gamma)-\lambda X(z)\big)\tilde{S}_2\big(\lambda-\lambda X(z)\big)\right)$$

$$M_2=\left(\tilde{S}_1\big(\lambda-\lambda X(z)\big)-1\right)+\left[\left(\tilde{S}_2\big(\lambda-\lambda X(z)\big)-1\right)+\left(\tilde{F}\big((\lambda+\gamma)-\lambda X(z)\big)-1\right)\right]\tilde{S}_1\big(\lambda-\lambda X(z)\big)$$

$$M_3=\left(\tilde{S}_1\big(\lambda-\lambda X(z)\big)-1\right)\left(\boldsymbol{M_1}+\tilde{S}_1\big(\lambda-\lambda X(z)\big)\left(1+\tilde{S}_2\big(\lambda-\lambda X(z)\big)+\tilde{F}\big((\lambda+\gamma)-\lambda X(z)\big)\right)\right)\\+\left(\tilde{F}\big((\lambda+\gamma)-\lambda X(z)\big)-1\right)$$

4.2. Computational aspects of unknown probabilities

To find the unknown constants, Rouches's theorem of complex variables can be used. It follows that the expression $\left(z^b-\tilde{S}_1\big(\lambda-\lambda X(z)\big)-\tilde{S}_1\big(\lambda-\lambda X(z)\big)\tilde{F}\big((\lambda+\gamma)-\lambda X(z)\big)\tilde{S}_2\big(\lambda-\lambda X(z)\big)\right)$has b-1 zeros inside and one on the unit circle$|z|$=1. "Since P (z) is analytic within and on the unit circle, the numerator of P (z) must vanish at these points, which gives ‘b’ equations and ‘b’ unknowns”. These equations can be solved by suitable numerical techniques. MATLAB 2015a is used for programming.

4.3. Steady state condition

The probability generating function P (z) has to satisfy P (1) = 1. “In order to satisfy this condition, applying Hospital’s rule and evaluating $\lim_{z\to1} P(z)$ and equating the expression to 1, it is derived that, $\rho < 1$, is the condition to be satisfied for the existence of steady state for the model under consideration”, where

$$\rho = \frac{\lambda E(X)[2E(S_1) + E(S_2) + E(F)]}{b}$$

5. Performance measures

In a waiting line, it is customary to access the mean number of waiting units and mean waiting time. In this section, the system’s measures of effectiveness are derived from the steady-state probability distribution function, which are useful to find the total average cost of the system.

5.1. Expected queue length

$$E(Q) = \lim_{z\to1} P'(z)$$

$$E(Q) = \frac{\begin{matrix} 4M_1'\big(\lambda E(X)\big)\big(\Psi(X,\lambda) + M_3''' \sum_{i=a}^{b-1}\omega_i + 3M_2'' \sum_{i=0}^{b-1} i\omega_i + M_3''' \sum_{i=0}^{b-1}\omega_i\big) \\ 6\big(M_1'\lambda X''(1) + M_1''\lambda E(X)\big)\{2M_2' \sum_{n=0}^{a-1} n\omega_n + (2T_1M_1' + M_2'') \sum_{n=0}^{a-1}\omega_n + \Omega(X,\lambda)\} \\ 3\big(4M_1'\lambda X'''(1) + 6M_1''\lambda X''(1) + 4M_1'''\lambda E(X)\big)\big(M_3' \sum_{i=0}^{b-1}\omega_i + M_2' \sum_{i=0}^{b-1}\omega_i\big) \end{matrix}}{24\lambda E(X)(b - 2C_1 - F_1 - D_1)^2}$$

$$C_1 = \lambda E(S_1)E(X) \qquad C_2 = E(S_1)\lambda X''(1) + \lambda^2 E({S_1}^2)(E(X))^2$$

$$C_3 = E(S_1)\lambda X'''(1) + 3\lambda^2 E({S_1}^2)E(X)X''(1) + \lambda^3 E({S_1}^3)(E(X))^3$$

$$D_1 = \lambda E(S_2)E(X) \qquad D_2 = E(S_2)\lambda X''(1) + \lambda^2 E({S_2}^2)(E(X))^2$$

$$D_3 = E(S_2)\lambda X'''(1) + 3\lambda^2 E({S_2}^2)E(X)X''(1) + \lambda^3 E({S_2}^3)(E(X))^3 \omega_i = A_i + F_i + H_i$$

$$F_1 = (\lambda+\gamma)E(F)E(X) \qquad F_2 = E(F)(\lambda+\gamma)X''(1) + (\lambda+\gamma)^2 E(F^2)\big(E(X)\big)^2$$

$$F_3 = E(F)(\lambda+\gamma)X'''(1) + 3(\lambda+\gamma)^2 E(F^2)E(X)X''(1) + (\lambda+\gamma)^3 E(F^3)(E(X))^3$$

$$T_1 = \lambda E(T)E(X) \qquad T_2 = E(T)\lambda X''(1) + \lambda^2 E(T^2)\big(E(X)\big)^2$$

$$T_3 = E(T)\lambda X'''(1) + 3\lambda^2 E(T^2)E(X)X''(1) + \lambda^3 E(T^3)(E(X))^3$$

$$M_1' = b - 2C_1 - F_1 - D_1 M_1'' = b(b-1) - 2C_2 - F_2 - D_2 - 2C_1(F_1 + D_1) - 2D_1F_1$$

$$M_1''' = b(b-1)(b-2) - 2C_3 - F_3 - D_3 - 2C_2(F_1 + D_1) - 2C_1(F_2 + D_2) - 2D_1F_2 - 2D_2F_1$$

$$M_2' = C_1 + D_1 + F_1 M_2'' = C_2 + D_2 + F_2 + 2C_1(D_1 + F_1)$$

$$M_2''' = C_3 + D_3 + F_3 + C_1(D_2+F_2) + 3C_2(D_1+F_1) + 2C_1(D_2+F_2)$$

$$M_3' = C_1 + F_1 M_3'' = 2C_1(M_1' + D_1 + F_1 + C_1) + C_2 + F_2$$

$$M_3''' = 3C_1(M_1'' + D_2 + F_2 + 2C_1(D_1+F_1) + C_2) + 3C_2(M_1' + D_1 + F_1 + C_1) + C_3 + F_1$$

$$\Psi(X,\lambda) = 3(2T_1 M_1' + M_2'')\sum_{n=0}^{a-1} n\omega_n + M_2'\sum_{n=0}^{a-1} n(n-1)\omega_n$$

$$+(3T_1 M_1'' + 3T_2 M_1' + M_2''')\sum_{n=0}^{a-1}\omega_n + 3M_2'\sum_{i=0}^{b-1} i(i-1)\omega_i$$

$$\Omega(X,\lambda) = M_3''\sum_{i=a}^{b-1}\omega_i + 2M_2'\sum_{n=0}^{b-1} n\omega_n + M_2''\sum_{i=0}^{b-1}\omega_i$$

5.2. Expected length of busy period

Let 'S' be the busy period random variable and N be the residence time that the server is rendering service or under renewal

$$\text{E (N)} = E(S_1) + E(S_2) + \text{E(F)}$$

where E(S_1) is the expected regular service time

E(S_2) is the expected service time during server failure

E(F) is the expected renewal time

We define a random variable J as

$$\text{I} = \begin{cases} 0, \text{ if the server finds less than 'a' customers after service} \\ 1, \text{ if the server finds atleast 'a' customers after service} \end{cases}$$

Then expected length of busy period is given by

$$E(S) = E(S/I = 0)P(I = 0) + E(S/I = 1)P(I = 1)$$

$$= E(N)P(I = 0) + \big(E(N) + E(S)\big)P(I = 1)$$

$$E(S)\big(1 - P(I = 1)\big) = E(N)\big(P(I = 0) + P(I = 1)\big)$$

Since P (I=0) + P (I=1) = 1, solving for E (S), we get

$$\text{E (S)} = \frac{E(N)}{P(I=0)}$$

$$E(S) = \frac{E(S_1) + E(S_2) + E(F)}{\sum_{i=0}^{a-1}(A_i + G_i + F_i)}$$

$$= \qquad \text{E (Y)} = \frac{E(T)}{\big(P(J=0) + P(J=2)\big)}$$

5.3. Expected length of idle period

Idle period is defined as “the time period between the first service completion epochand succeeding service initiation epoch.”
Let ‘L’ be the random variable defined for ‘idle Period and the randomvariable ‘J’ is defined as follows

$$J = \begin{cases} 0, if\ the\ server\ finds\ at\ least\ 'a'\ customers\ after\ the\ first\ vacation \\ 1, if\ the\ server\ finds\ less\ than\ 'a'\ customers\ after\ the\ first\ vacation \end{cases}$$

Then the expected length of idle period E (L) is given by

E(L) = E(L / J = 0) P(J = 0) + E(L / J = 1) P(J = 1)

$$= \mathrm{E(V)\,P(U = 0) + (E(V) + E(L))\,P(J = 1)}$$

where E(V) is the average vacation time.
Solving for E (L), we get

$$E(L) = \frac{E(V)}{1 - \sum_{n=0}^{a-1} \sum_{i=0}^{n} \gamma_i A_{n-i}}$$

Where γ_i is the “probability that ‘i’ customers arrive during a vacation and A_n is the probability of ‘n’ customers being in the queue at a departure epoch.”

5.4. Expected waiting time in the queue

“The mean waiting time of the customers in the queue E (W) can be easily obtained using Little’s formula”

$$\mathrm{E\,(W)} = \frac{E(Q)}{\lambda E(X)}$$

5.5 Probability that the server is busy with regular service

$$P(B) = \lim_{z\to1}\left(\sum_{i=0}^{b-1} \tilde{A}_m(z,0) + \tilde{A}_b(z,0)\right)$$

$$P(B) = \frac{\begin{matrix}\sum_{i=a}^{b-1} \omega_i\,[2C_1M_1' + 24^2 + 2C_1(C_1 + 4D_1) + 3C_2 + 2D_2 + 2F_1(3C_1 + D_1)] + 2C_1 \sum_{j=0}^{a-1} j\omega_j \\ +(C_1T_1 + 2C_2)\sum_{j=0}^{a-1}\omega_j - \sum_{j=0}^{b-1}\omega_j\left(2C_2 + 3C_1D_1 + 2D_2 + 2{C_1}^2\right) - 2C_1\sum_{j=0}^{b-1} j\omega_j\end{matrix}}{2M_1'\lambda E(X)}$$

5.5. Probability that the server is on vacation

$$P(V) = \lim_{z\to1} \sum_{\beta=1}^{\infty} \widetilde{H}_\beta(z,0)$$

$$P(V) = E(T)\sum_{j=0}^{a-1} A_j + R_j$$

6. Special cases.

Case. 1: Single server batch arrival queue with hyper exponential vacation time

Now, the case of hyper-exponential vacation time random variable is considered. The probability density function of hyper- exponential vacation time is given as follows

$$t(x) = kpe^{-px} + (1-k)\,qe^{-qx}$$

where 'p' and 'q' are the parameters.

Then, $\tilde{T}(\lambda - \lambda X(z)) = \left(\frac{pk}{p+\lambda(1-\lambda X(z))}\right) + \left(\frac{q(1-k)}{q+\lambda(1-\lambda X(z))}\right)$

Hence, the PGF of the queue size distribution of this model hyper exponential vacation time can be obtained as follows

$$P(z) = \frac{\boldsymbol{M_3}\sum_{i=a}^{b-1}(A_i+F_i+H_i) + \boldsymbol{M_2}\sum_{n=0}^{b-1}(A_n+F_n+H_n)z^n \\ \left(\left(\left(\left(\frac{pk}{p+\lambda(1-\lambda X(z))}\right) + \left(\frac{q(1-k)}{q+\lambda(1-\lambda X(z))}\right)\right)-1\right)\boldsymbol{M_1}+\boldsymbol{M_2}\right)\sum_{n=0}^{a-1}(A_n+F_n+H_n)z^n}{\boldsymbol{M_1}(-\lambda+\lambda X(z))}$$

Case. 2: Single server batch arrival queue with exponential service time

Now, the case of exponential service time random variable is considered. The probability density function of exponential service time is given as follows:

$s_1(x) = s_1(x) = \mu e^{-\mu x}$, where μ is the parameter. Then,

$$\tilde{S}_1(\lambda - \lambda X(z)) = \left(\frac{\mu}{\mu+\lambda(1-X(z))}\right) = \tilde{S}_2(\lambda - \lambda X(z))$$

Hence, the PGF of the queue size distribution of this model with exponential service time can be obtained as follows

$$(z) = \frac{M_3\sum_{i=a}^{b-1}(A_i+F_i+H_i) + M_2\sum_{n=0}^{b-1}(A_n+F_n+H_n)z^n \\ \left(\left(\tilde{T}(\lambda-\lambda X(z))-1\right)M_1+M_2\right)\sum_{n=0}^{a-1}(A_n+F_n+H_n)z^n}{M_1(-\lambda+\lambda X(z))}$$

where

$$M_1 = \left(z^b - \left(\frac{\mu}{\mu+\lambda(1-X(z))}\right) - \left(\frac{\mu}{\mu+\lambda(1-X(z))}\right)^2 \tilde{F}\left((\lambda+\gamma)-\lambda X(z)\right)\right)$$

$$M_2 = \left(\left(\frac{\mu}{\mu+\lambda(1-X(z))}\right)-1\right) + \left[\left(\left(\frac{\mu}{\mu+\lambda(1-X(z))}\right)-1\right) + \left(\tilde{F}\left((\lambda+\gamma)-\lambda X(z)\right)-1\right)\right]\left(\frac{\mu}{\mu+\lambda(1-X(z))}\right)$$

$$M_3 = \left(\left(\frac{\mu}{\mu+\lambda(1-X(z))}\right)-1\right)\left(M_1 + \left(\frac{\mu}{\mu+\lambda(1-X(z))}\right)\left(1+\left(\frac{\mu}{\mu+\lambda(1-X(z))}\right)+\tilde{F}\left((\lambda+\gamma)-\lambda X(z)\right)\right)\right) \\ +\left(\tilde{F}\left((\lambda+\gamma)-\lambda X(z)\right)-1\right)$$

7. Numerical illustrations

The following assumptions are made to obtain the numerical results

Regular service time distribution is 4-Erlang with parameter	μ
Service time distribution during server failure is 2-Erlang with parameter	$\mu^1 (\mu > \mu^1)$
Batch size distribution of the arrival is geometric with mean	2
Vacation time is exponential with parameter	ε
Renewal time is exponential with parameter	η
Failure rate	γ
Minimum server capacity	$a = 3$
Maximum server capacity	$b = 7$

7.1. Effects of failure rate on the performance measures

In this section an effect of failure rate on performance measures is given in table 1. It is clear that when failure rate increases expected queue length, expected length of idle period and expected waiting time in the queue are increases, whereas expected length of busy period decreases.

7.2. Effects of failure rate on the performance measures

An effect of renewal rate on performance measures is presented in table 2. It can be seen that when renewal rate increases expected queue length and expected length of idle period increases, whereas expected length of busy period and expected waiting time in the queue are decreases.

Table 1. Failure rate Vs performance measures
($\boldsymbol{For}\ \lambda = 4, \mu = 3, \mu^1 = 2,$ ε $= 3, \eta = 2$)

Failure rate(γ)	E (Q)	E(S)	E (L)	E (W)
2.0	*4.5281*	*5.2349*	*3.1429*	*4.1291*
2.3	*4.7613*	*4.1195*	*5.0345*	*6.2138*
2.6	*4.8904*	*2.7391*	*6.2392*	*7.8426*
2.9	*5.1163*	*1.4729*	*8.2329*	*9.3361*
3.2	*7.2974*	*0.9542*	*11.4268*	*10.6541*
3.5	*8.6192*	*0.2741*	*12.9341*	*13.0462*

Table 2. Renewal rate Vs performance measures
($\lambda = 4.5, \mu = 3, \mu^1 = 2, \gamma = 3, \eta = 2$)

Renewal rate (η)	***E(Q)***	***E(S)***	***E(L)***	***E(W)***
2	*5.932*	*2.912*	*0.324*	*1.732*
3	*5.134*	*2.516*	*0.336*	*1.662*
4	*4.513*	*2.089*	*0.343*	*1.554*
5	*3.927*	*1.736*	*0.379*	*1.332*
6	*2.623*	*1.223*	*0.396*	*1.247*

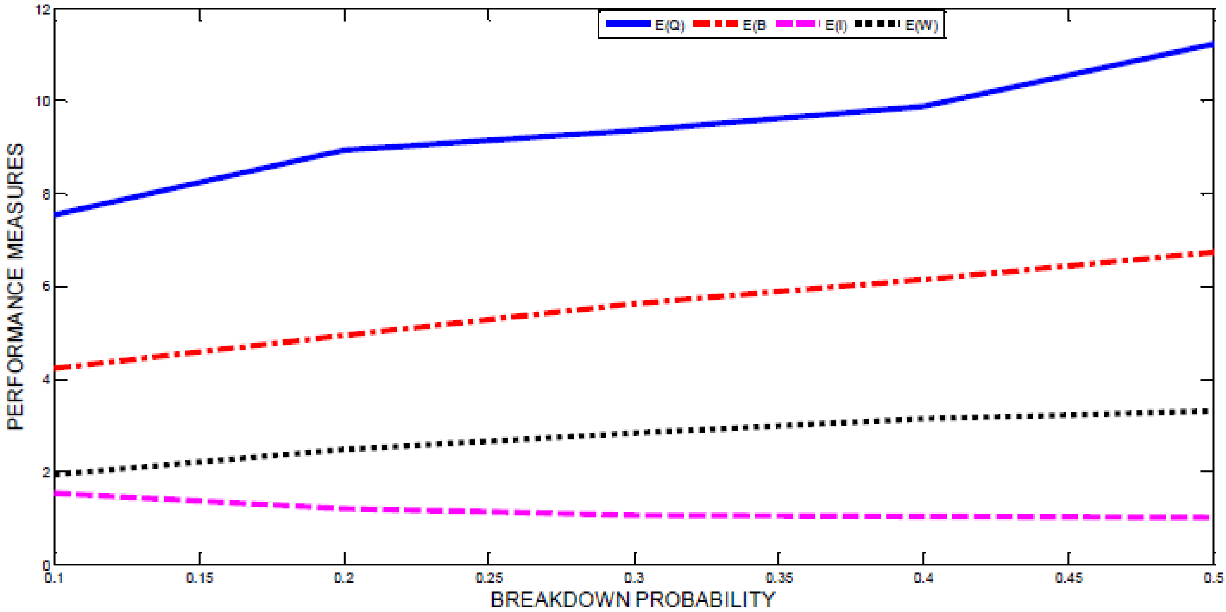

Figure 2. Arrival rate Vs Performance measures

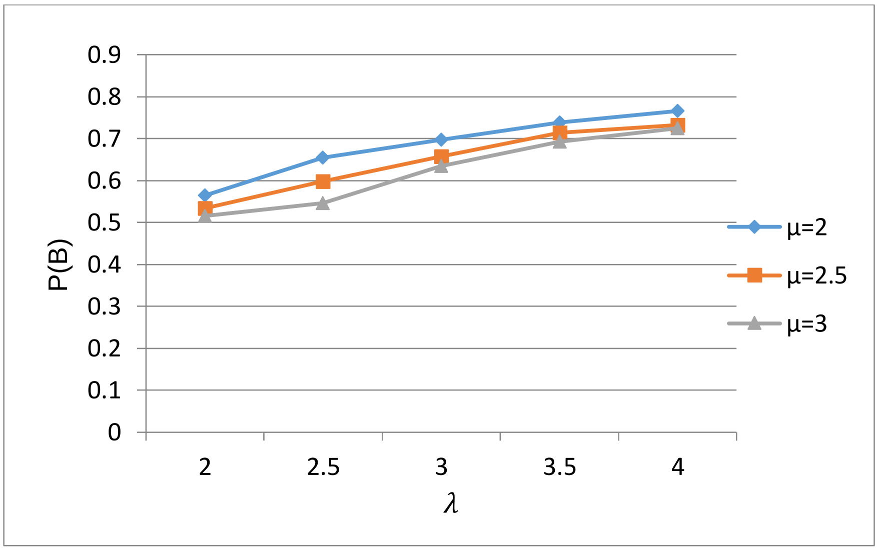

Figure 3. Arrival rate (Vs) Probability of busy period

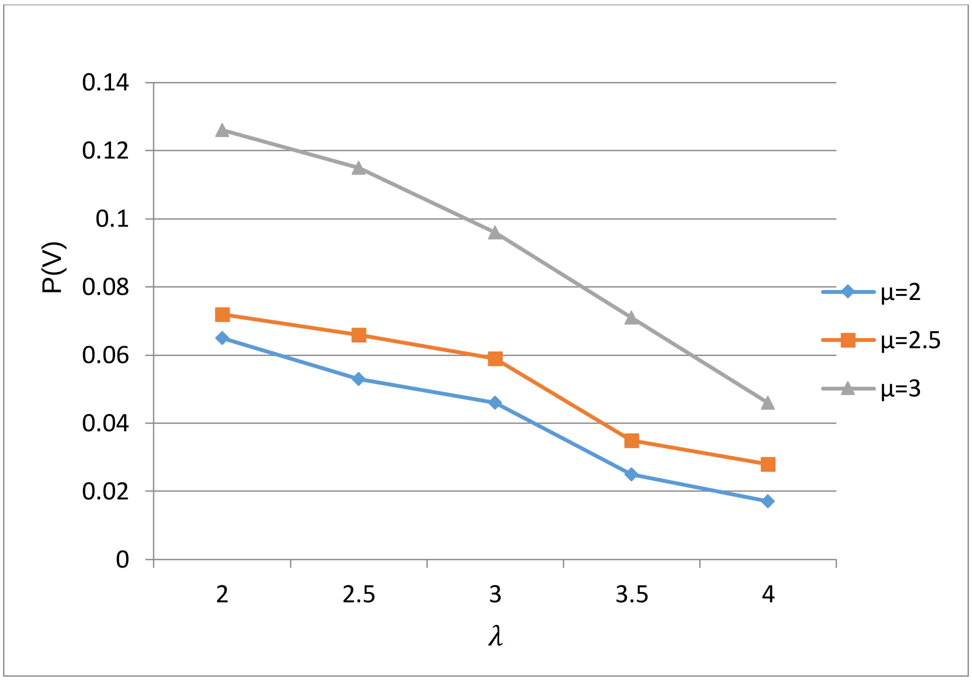

Figure 4. Arrival rate (Vs) Probability of Vacation period

8. Conclusions

In this paper bulk arrival and batch service queuing system with functioning server failure and multiple vacations are analysed. The model taken into consideration is unique because functioning server failure is introduced for $M^X/G(a,b)/1$ queueing system with multiple vacations. For the modelled system probability generating function of the queue size at an arbitrary time is derived by using supplementary variable technique. Various performance characteristics are also derived with suitable numerical illustrations.

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