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Planar graph characterization of NDSS graphs

M Yamuna and A Elakkiya
Department of Mathematics, School of Advanced Sciences, VIT University, Vellore-632014, India.

E-mail: myamuna@vit.ac.in

Abstract. Planar graph characterization is always of interest due to its complexity in characterization. In this paper, we obtain a necessary and sufficient condition for a graph to be NDSS and hence characterize the planarity and outer – planarity of its complement $\overline{G}$.

1. Introduction

Dominating sets has been used in graph theory for characterizing graphs based on various properties. In [1], Magda Dettlaff, Joanna Raczek and Jerzy Topp have proved that the decision problem of the domination subdivision number is NP – complete even for bipartite graphs. In [2], B. Sharada et.al have provided the problem of domination subdivision number of grid graphs $P_{m,n}$ and determine the domination subdivision numbers of grid graphs $P_{m,n}$ for m = 2, 3 and $n \geq 2$.

Characterizing planar graphs based on graph properties is a general problem discussed by different authors. In [3], M. Yamuna et al have provided a characterization of planar graphs when G and its complement are γ – stable. In [4], Val Pinciu showed that for outer planar graphs where all bounded regions are 3 – cycles, the problem of identifying the connected domination number is equal to an art gallery problem, which is identified to be NP – hard. In [5], By Joseph Battle, Frank Harary and Yukihiro Kodama have proved that every planar graph with nine vertices has a non – planar complement. In [6], Jin Akiyama and Frank Harary have characterized all graphs for which G and its complement are outer planar. In [7], Enciso and Dutton have classified planar graph based on $\overline{\mathbf{G}}$ and also they have proved the following result.

R1. If G is a planar graph, then $\gamma(\overline{\mathbf{G}}) \leq 4$.

R2. If u is an up vertex for a graph in G, then u must be included in every possible γ – set [8].

2. Terminology

We consider only simple connected undirected graphs G = (V, E) with n vertices and m edges. The open neighbourhood of $v \in V(G)$ is defined by $N(v) = \{ u \in V(G) \mid uv \in E(G) \}$, while its closed neighborhood is $N[v] = N(v) \cup \{ v \}$. H is a sub graph of G, if $V(H) \subseteq V(G)$ and $uv \in E(H)$ implies $uv \in E(G)$. If H satisfies the added property that for every $uv \in E(H)$ if and only if $uv \in E(G)$, then H is said to be an induced sub graph of G and is denoted by $\langle H_i \rangle$. Two graphs are homeomorphic if one can be obtained from the other by the creation of edges in series or by the merging the edges in series. In graph theory, K_5 and $K_{3,3}$ are called Kuratowski′s graph. A path is a trail in which all vertices (except perhaps the first and last ones) are distinct, P_n denotes the path with n vertices. A cycle is a circuit in which no vertex except the first (which is also the last) appears more than once. C_n is a cycle with n vertices. K_n is a complete graph with n vertices. A star S_n is the complete bipartite graph $K_{1,n}$: a tree with one internal node and n leaves (but, no internal nodes and n + 1 leaves when n ≤ 1). The complement of a graph G is a graph $\overline{\mathbf{G}}$ on the same vertices ∍ two distinct vertices of $\overline{\mathbf{G}}$are adjacent if and only if they are not adjacent in G.

An elementary contraction of a graph G is obtained by merging two adjacent vertices u and v, i.e., by the removal of two vertices u and v and the addition of a new vertex x adjacent to those vertices to which u or v was adjacent. A graph G is contractible if it can be generated from G by a chain of elementary contractions [9]. For properties related to graph theory we refer to Harary[9].

A set of vertices D in G is said to be a dominating set if for every vertex of V – D is ⊥ to some vertex of D. The smallest possible cardinality of any dominating set D of G is called a minimum dominating set – abbreviated MDS. The cardinality of any MDS for G is called the domination number of G and it is denoted by γ (G).The private neighbourhood of v ∈ D is defined by pn [v, D] = N (v) – N (D – { v}). A vertex v is said to be selfish in the MDS D, if v is required only to dominate itself. A vertex of degree one is called pendant vertex and its neighbor is called a support vertex. If there is a γ – set of G containing v, the v is said to be good. If v does not belongs to any of the γ – set of G, then v is said to be a bad vertex. A vertex v is known to be a down vertex if γ(G - u) <γ (G). A vertex v is known to be a level vertex if γ(G - u) = γ (G). A vertex v is said to be a up vertex if γ(G - u) >γ (G). For properties related to domination we refer Haynes et al [8].

A subdivision of a graph G is a graph obtained from the subdivision of edges in G. The subdivision of some edge e with end vertices { u , v } generate a graph with one new vertex w, and with an edge set replacing e by two new edges, { u, w } and { w, v } and it is denoted by G_{sd}uv. Let w be the vertex introduced by subdividing uv. We shall denote this by G_{sd}uv = w. If G is any graph and D is a γ – set for G, then D ∪ { w } is a γ – set for G_{sd}uv implies γ (G_{sd}uv) ≥γ (G), ∀ u, v ∈V (G), u ⊥ v. A graph G is defined as DSS, if γ (G_{sd}uv) = γ (G), ∀ u, v ∈V (G), u ⊥ v [10]. In [10], the following result is proved.

R3. A graph G is domination subdivision stable if and only if ∀ u, v ∈V(G), either ∃ a γ – set containing u and v or ∃γ – set D such that

1. pn (u, D) = { v } or
2. v is 2 – dominated.

In this paper we consider graphs for which γ(G_{sd}uv) = γ (G) + 1.

3. Results and Discussions

In this section, we provide a necessary and sufficient condition for a graph to be NDSS and characterize the planarity and outer – planarity of NDSS graph.

3.1 Non - domination subdivision stable graph

Theorem 1

A graph G is NDSS if and only if

- everyγ – set of G is independent.
- G has no two dominated vertices.

Proof

Assume that G is NDSS

- If G has a γ – set D ∍ u, v ∈ D, u ⊥ v, then D itself is a γ – set for G_{sd}uv, a contradiction as G is NDSS, implies every γ – set of G is independent.
- If u ∈ D ∍ v is 2 – dominated, u adjacent to v, then D itself is a γ– set for G_{sd}uv, a contradiction as G is NDSS, implies G has no 2 – dominated vertices.

Conversely, assume that the conditions of the theorem are satisfied. If G is not NDSS, then by R3 ∃ a γ – set D ∍

- either u, v ∈ D.
- pn (u, D) = { v }
- v is 2 – dominated.

a contradiction to our assumption, implies G is NDSS.

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Observations

O1. If G is a NDSS graph, then any v ∈V(G) is not selfish.

Proof

If possible, assume that ∃ one v ∈V(G) ∍ v is selfish. Let D be any γ – set for G. D′ = D – { v } ∪ { w } is a γ – set for G_{sd}uv, implies G is not NDSS, a contradiction.

O2. If G is a NDSS graph, then G has no down vertices.

Proof

If possible, assume that ∃one v ∈V(G), v a down vertex. We know that if v is a down vertex, then ∃a γ – set D for G including v such that v is selfish, a contradiction, (by O1) implies G has no down vertices.

O3. If G is a NDSS graph, then a pendant vertex is always a level vertex.

Proof

Since pn(u, D) ≥ 2 for any NDSS graph, deg (v) = 1 , there exist no γ – set containing v . Also an up vertex is included in every γ – set, [R2] implies v is not an up vertex. By (O2) v is always a level vertex.

O4. If G is a NDSS graph, then ⟨pn(u, D) ⟩ is not complete for every u ∈ D, G ≠ K_n.

Proof

If possible assume that there exist one u ∈ D, such that ⟨pn(u, D) ⟩ is a clique. Let pn(u, D) = {u_1, u_2, …, u_k}. Since ⟨pn (u, D) ⟩ is a clique, D – { u } ∪ { u_i}, i = 1, 2,…, k is a γ – set for G ∍ for any v ∈ N (u_i) is 2 – dominated, a contradiction as G is NDSS.

O5. If G is a NDSS graph, then ∃no v_i∈ N (u, D) adjacent to every v_j∈ N (u, D) , i ≠ j, deg (v_j) ≥ 2.

Proof

Let u ∈ D. Let N (u) = { u_1, u_2, …, u_k}. If ∃one v_i∍v_i adjacent to every v_j, i ≠ j, j = 1, 2, … k then D – { u } ∪ { v_j} is a γ – set for G, ∍every w_i∈ N (u_j) is 2 – dominated, a contradiction.

O6. If G is a NDSS graph, then pn(u , D) ≥ 2.

Proof

If pn (u, D) = v for some u ∈ D, then D′ = D – { u } ∪ { w } is a γ – set for G ∍| D′ | = | D |, a contradiction as G is NDSS, implies pn (u, D) ≥ 2.

3. 2. Planarity

We recollect the following theorems on planar graphs.

R4. A graph is planar if and only if it does not contain eitherK_5 or $K_{3,3}$or any graph homeomorphic to either of them.

R5. A graph G is planar if and only if it does not have a subgraph contractible to Kuratowski′s graph[9].

R6. A necessary and sufficient condition for a graph G to be outer planar if it has no subgraphhomeomorphic to K_4 or $K_{2,3}$ except K_4 – x [9].

We shall prove that a NDSS graph is planar, non – planar, or non – outer planar using R4, R5 and R6.

If γ(G) = 1, then complement of G is disconnected and hence complement of G is not a NDSS graph. Also by R1, if G is a planar graph, then γ ($\overline{\mathbf{G}}$) ≤ 4. So in the remaining part of this section we limit our discussion to cases where 1 <γ(G) ≤ 4, 1 <γ ($\overline{\mathbf{G}}$) ≤ 4. In all graphs, in the remaining part of the discussion,

i. ----------- represents the newly added edges in the current discussion.

ii. When we apply edge contraction, a vertex receives a label of the contracted vertices. For example y: $bb_1x_1x_2$ means that the contracted edges are bb_1, b_1x_1, x_1x_2 and is assigned the new label as y.

Theorem 2

If G is a NDSS graph, then ⟨ V – D ⟩ is not complete, where D is a γ – set for G.

Proof

Let D = { u_1, u_2,…, u_k } be a γ - set for G. Let N (u_1) = { a_1, a_2,…, $\mathbf{a_{m_1}}$}, N (u_2) = { b_1, b2,…,$\mathbf{b_{m_2}}$},…, N (u_k) = { k_1, k_2,…, $\mathbf{k_{m_k}}$}. Since G is NDSS | m_i | ≥ 2, for all i = 1, 2, …, k. If 〈 V – D 〉 is complete, then D′ = { a_1, b_1, c_1, d_1 } is a γ - set for G, (a_1 dominates N (u_1) , N (u_2) ,…, N (u_k), b_1 dominates u_2, c_1 dominates u_3, d_1 dominates u_4) such that 〈 D′〉 is complete, a contradiction as G is NDSS.

Theorem 3

Let G be a NDSS graph. Let γ(G) = 3. Let D = { u_1, u_2, u_3 } be a γ – set for G. Let X_1 = pn (u_1, D) = { a_1, a_2,…, $\mathbf{a_{k_1}}$ },X_2 = pn (u_2, D) = { b_1, b_2,…, $\mathbf{b_{k_2}}$ }, X_3 = pn (u_3, D) = { c_1, c_2,…, $\mathbf{c_{k_3}}$ }. Then the following statements are true together

1. X_1 is collectively not adjacent to at least k_1 vertices in X_2, X_3
2. X_2 is collectively not adjacent to at least k_2 vertices in X_1, X_3.
3. X_3 is collectively not adjacent to at least k_3 vertices in X_1, X_2.

Proof

Since G is NDSS, | k_i | ≥ 2, i = 1, 2, 3. If k_1 vertices in X_1 are collectively not adjacent to at least k_1 vertices in X_2, then we can find k_1 non adjacent pairs (a_i, b_j) , i = 1 to k_1, j = 1 to k_2. If this is not true then there exist at least one a_i adjacent to every b_j. Similarly if k_1 vertices in X_1 are not collectively not adjacent to at least k_1 vertices in X_3, then there exist at least one a_i that is adjacent to every c_k, that is there exist one $\mathbf{a_{i_1}}$adjacent to every bj and some $\mathbf{a_{i_2}}$ adjacent to every c_k ($\mathbf{a_{i_1}}$may be equal to $\mathbf{a_{i_2}}$). Similarly there exist one $\mathbf{b_{j_1}}$adjacent to every a_i and some $\mathbf{b_{j_2}}$ adjacent to every c_k ($\mathbf{b_{j_1}}$may be equal to $\mathbf{b_{j_2}}$).

Similarly there exist one $\mathbf{c_{k_1}}$ adjacent to every a_i and some $\mathbf{c_{k_2}}$adjacent to every b_j ($\mathbf{c_{k_1}}$may be equal to $\mathbf{c_{k_2}}$). Then { $\mathbf{a_{i_1}}$, $\mathbf{b_{j_2}}$, $\mathbf{c_{k_1}}$ }is a γ - set for G such that { $\mathbf{a_{i_1}}$, $\mathbf{b_{j_2}}$, $\mathbf{c_{k_1}}$ } is not independent, a contradiction as G is NDSS.

Remark

1. Generalizing Theorem 3 if D = { u_1, u_2, …., u_m }, X_1 = pn (u_1 , D) = { a_1, a_2,…, $\mathbf{a_{k_1}}$}, X_2 = pn (u_2, D) = { b_1, b_2,…, $\mathbf{b_{k_2}}$ },..., X_m = pn (u_m, D) = { m_1, m_2,…, $\mathbf{m_{k_m}}$ }, then the following statements are true together

X_1 is collectively not adjacent to at least k_1 vertices in X_2 , X_3,…, X_m.

X_2 is collectively not adjacent to at least k_2 vertices in X_1, X_3,….,X_m.

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X_m is collectively not adjacent to at least k_m vertices in X_1, X_2,…,.X_{m-1}.

2. For every i_1 = 1 to k_1 , i_2 = 1 to k_2 ,...., i_m= 1 to k_m, there exist at least one pair ($\mathbf{a_{i_1}}$, $\mathbf{b_{i_2}}$) , ($\mathbf{a_{i_1}}$,$\mathbf{c_{i_3}}$) , …, ($\mathbf{a_{i_1}}$,$\mathbf{m_{i_m}}$) of vertices that are not adjacent. This means that every $\mathbf{a_{i_1}}$ not adjacent to at least one $\mathbf{b_{i_2}}$, $\mathbf{c_{i_3}}$,…, $\mathbf{m_{i_m}}$.

Theorem 4

If G is a NDSS graph such that γ(G) = 4, then $\overline{\mathbf{G}}$ is non planar.

Proof

Let D = { u_1, u_2, u_3, u_4 } be a γ - set for G. Let pn (u_1, D) ={ a_1, a_2,…, $\mathbf{a_{k_1}}$ }, pn (u_2, D) = { b_1, b_2,…, $\mathbf{b_{k_2}}$ }, pn (u_3, D) = { c_1, c_2,…, $\mathbf{c_{k_3}}$ },..., pn (u_4, D) ={ d_1, d_2,…, $\mathbf{d_{k_4}}$ }. Since G is NDSS, | k_i | ≥ 2 for all i = 1, 2, 3, 4. Since 〈 D 〉 is independent in G, 〈 D 〉 is complete in $\overline{\mathbf{G}}$.

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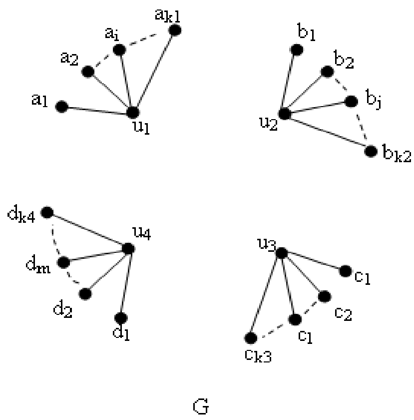

Figure 1.

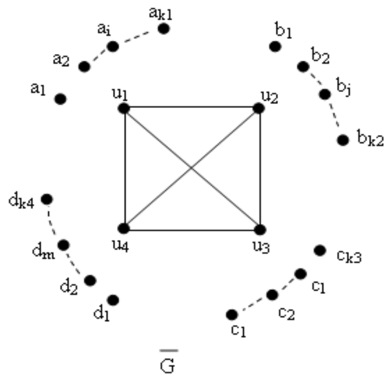

Figure 2.

By Theorem 3, we know that there exist at least 2 vertices in V – D which are not adjacent. Arbitrarily let us assume that some a_i, i = 1, 2, …, k_1 not adjacent to some b_j, j = 1, 2,…, k_2. Also a_i is adjacent to { u_2, u_3, u_4 }.

Since in G a_i not adjacent to b_j and b_j not adjacent to u_1, in $\overline{\mathbf{G}}$there exist an edge from a_i to b_j and b_j to u_1. Contracting edge a_ib_j, a_ib_j is adjacent to u_1. $\langle$ u_1, u_2, u_3, u_4, u_5: $a_ib_j\rangle$ is K_5, implies $\overline{\mathbf{G}}$ is non planar.

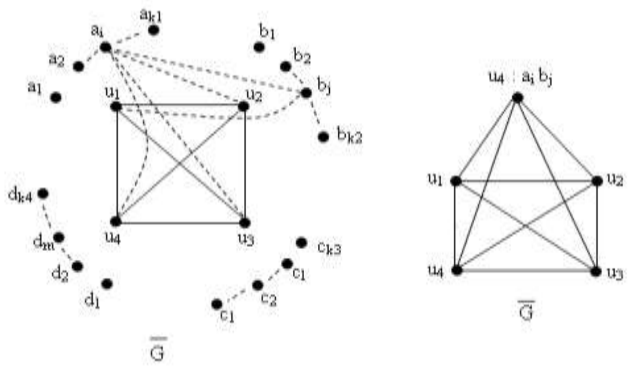

Figure 3.

Theorem 5

Let G be a NDSS graph such that γ(G) = 3 , then $\overline{\mathbf{G}}$ is non – planar.

Proof

Let D = { u_1, u_2, u_3 } be a γ - set for G. Let pn(u_1, D) = { a_1, a_2,…, $\mathbf{a_{k_1}}$ } , pn (u_2, D) = { b_1, b_2,…, $\mathbf{b_{k_2}}$}, pn (u_3,D) ={ c_1, c_2,…, $\mathbf{c_{k_3}}$}. Since G is NDSS, | k_i | ≥ 2 for all i = 1, 2, 3. Since $\langle$ D $\rangle$ is independent in G, $\langle$ D $\rangle$ is complete in $\overline{\mathbf{G}}$.

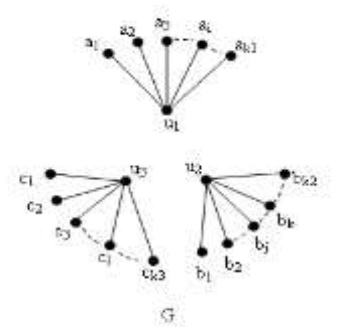

Figure4.

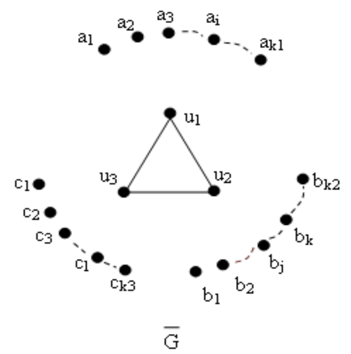

Figure 5.

Since G is NDSS by Theorem 2, $\langle$ V – D $\rangle$ is not complete, implies ∃ at least two vertices in V – D which are not adjacent. Arbitrarily let us assume that some a_i not adjacent to some b_j.

Since in G, a_i not adjacent to b_j they are ⊥in $\overline{\mathbf{G}}$. Also $b_j \perp u_1$.We know that a_i is adjacent to u_2 and b_j is adjacent to u_3. Contracting edge a_ib_j, a_i adjacent to u_1.$\langle u_1, u_2, u_3, u_4 : a_ib_j \rangle$ is K_4.

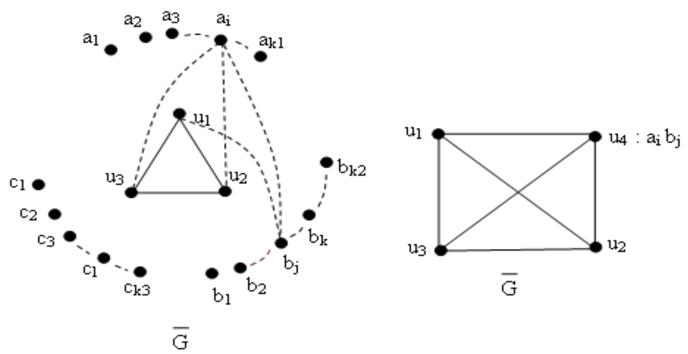

Figure 6.

By O6 and remark 2 of Theorem 3, there exist one b_k not adjacent to b_j, $k \neq j$ and some c_l not adjacent to b_k in G. In $\overline{\mathbf{G}}$, c_l adjacent to b_k, c_l adjacent to u_2. We know that b_k adjacent to u_4, u_1, u_3 and c_l adjacent to u_1. Contracting edges b_kc_l, c_lu_2, $\langle u_1, u_2, u_3, u_4, u_5 : b_kc_l \rangle$ is K_5, implies $\overline{\mathbf{G}}$ is non planar.

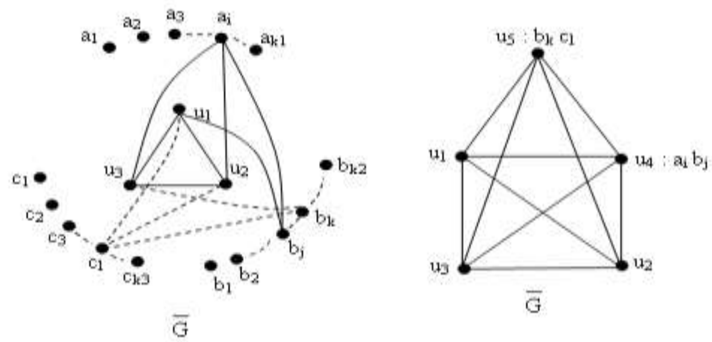

Figure 7.

If Fig. 8 (a) γ(G) = 2, $\overline{\mathbf{G}}$ non planar. For the graph in Fig. 8 (b) γ(G) = 2 , $\overline{\mathbf{G}}$ planar. So when γ (G) = 2, $\overline{\mathbf{G}}$ may or may not be planar. Contracting edges 63, 52 we see that ⟨ 1, 4, 63, 52 ⟩ is K_4, implies $\overline{\mathbf{G}}$ is non outer – planar. We generalize this result in Theorem 6.

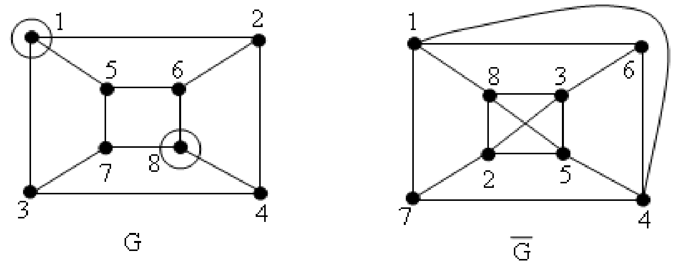

Figure 8 (a).

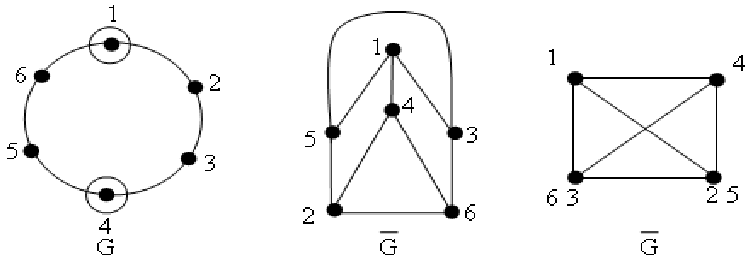

Figure 8 (b).

Theorem 6

Let G be a NDSS graph such that γ(G) = 2, then $\overline{\mathbf{G}}$ is non – outer planar.

Proof

Let D = { u_1, u_2 } be a γ – set for G. Let pn (u_1, D) ={ a_1, a_2,…, $\mathbf{a_{k_1}}$}, pn (u_2, D) = { b_1, b_2,…, $\mathbf{b_{k_2}}$}. Since G is NDSS, | k_i | ≥ 2 for all i = 1, 2. Since ⟨ D ⟩ is independent in G, ⟨ D ⟩ is complete in $\overline{\mathbf{G}}$.

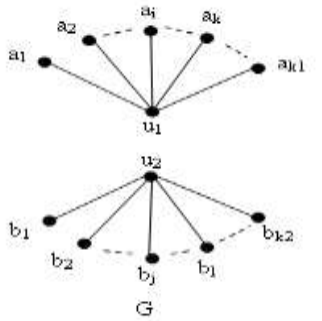

Figure 9.

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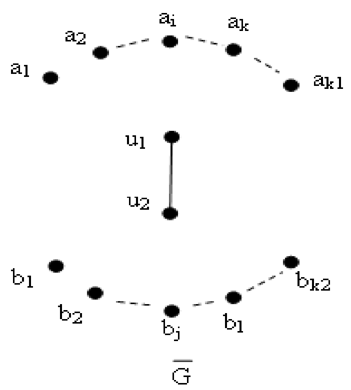

Figure 10.

Since G is NDSS by Theorem 2, $\langle V - D \rangle$ is not complete, implies ∃at least two vertices in V – D which are not adjacent. Arbitrarily, let us assume that some $a_i$not adjacent to some b_j .
Since in G a_i not adjacent to b_j they are ⊥in$\overline{\mathbf{G}}$.Also b_j adjacent to u_1. Contracting edge a_ib_j, a_i adjacent to u_1 implies $\langle u_1, u_2, u_3: a_ib_j \rangle$ is K_3.

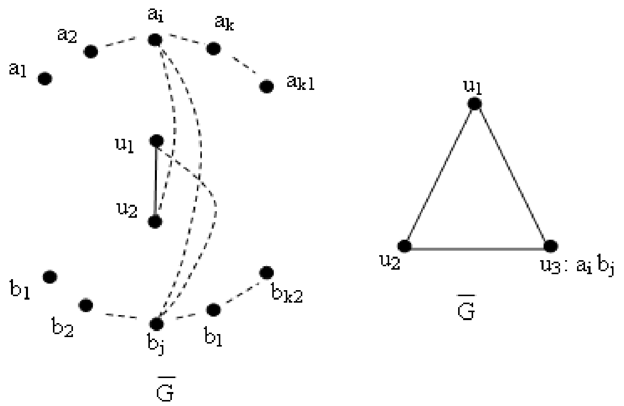

Figure 11.

By O6 and remark 2 of Theorem 3, there exist one b_k not adjacent to b_j , $k \neq j$ and some a_l, $l \neq i$ not adjacent to b_k in G. In $\overline{\mathbf{G}}$, b_k adjacent to b_j, a_l not adjacent to u_2, b_k not adjacent to u_1. Contracting a_lb_k, $\langle u_1, u_2, u_3, u_4 : a_lb_k \rangle$ is K_4, implies $\overline{\mathbf{G}}$ is non – outer planar.

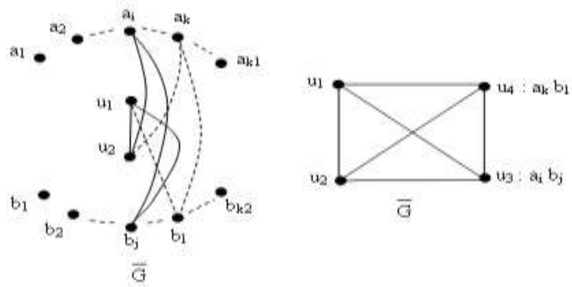

Figure 12.

4. Conclusion

This paper contributes to planarity characterization of NDSS graphs. We conclude that if G is an NDSS graph then,

- $\overline{\mathbf{G}}$is non – planar if $2 < \gamma (G) \leq 4$.
- $\overline{\mathbf{G}}$is non – outer planar if $\gamma (G) = 2$.

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