**Research Article**

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# Positive solutions of nonlinear fractional differential equations in non-zero self-distance spaces

https://doi.org/10.1515/gmj-2017-0040
Received July 31, 2016; accepted December 22, 2016

**Abstract:** This paper is concerned with the existence of positive solutions of three classes of nonlinear fractional differential equations using fixed point results in non-zero self-distance spaces. We introduce new concepts of generalized $\alpha$-weakly $(\psi, \varphi)_s$-contractive mappings involving rational terms and then develop fixed point results for weakly $\alpha$-admissible mappings. Some new examples and counterexamples are given to illustrate the applicability and effectiveness of these results over existing ones. In that way, we extend some previous results. For applications to fractional $q$-difference boundary value problems, the use of a $p$-Laplacian operator is suggested.

**Keywords:** Fixed point, $b$-metric-like space, altering distance function, weakly $\alpha$-admissible map, fractional differential equation

**MSC 2010:** 47H10, 34A08

## 1 Introduction

In recent years differential equations with fractional order have attracted many researchers because of their applications in many areas of science and engineering. The need for fractional order differential equations stems in part from the fact that many phenomena cannot be modelled by differential equations with integer derivatives. Analytical and numerical techniques have been implemented to study such equations.

The "Bible" of Fractional Calculus are the book [17] and the survey paper [12]. Some of the applications of fixed point theory in metric and ordered metric spaces to fractional differential and integral equations are discussed in [1, 5, 6, 11]. Baleanu, Rezapour and Mohammadi [6] used fixed-point methods of the form given in [16] to find the existence and uniqueness of a solution for the nonlinear fractional differential equation $D^\alpha u(t) = f(t, u(t))$ involving the Riemann–Liouville fractional derivative, with various boundary-value conditions. They considered three different classes of nonlinear fractional differential equations in their work.

The Banach contraction principle (BCP) is the most famous, simplest and one of the most versatile elementary results in fixed point theory in metric space structure. A huge amount of literature witnesses applications, generalizations and extensions of this principle carried out by several authors in different directions, e.g., by weakening the hypothesis, using different setups, considering different mappings and a generalized form of metric spaces.

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In this context, Matthews [13] is the one who introduced a generalization of metric spaces, called partial metric spaces. He showed that the BCP can be generalized to the partial metric context for applications in program verification. In partial metric spaces, the self-distance of an arbitrary point need not be equal to zero. This concept was further generalized by Hitzler and Seda [8] and, independently, by Amini-Harandi [3] under the name of dislocated metric spaces [8] resp. metric-like spaces [3]. This was further extended to partial $b$-metric spaces by Shukla [18] with the inclusion of properties of $b$-metric spaces (due to Bakhtin [4] and Czerwik [7]). Finally, Alghamdi, Hussain and Salimi [2] introduced $b$-metric-like spaces combining properties of metric-like spaces and $b$-metric spaces.

Contraction-type mappings have also been generalized in many forms. In the series of generalizations, Samet, Calogero Vetro and Pasquale Vetro [16] introduced the concept of $\alpha$-admissible maps and gave the concept of $\alpha$-$\psi$-contractive mappings, and generalized the Banach contraction theorem. Recently, Sintunavarat [19] introduced the notion of weakly $\alpha$-admissible maps and discussed respective fixed point results in metric space. In the subsequent work [20], he further derived fixed point results in $b$-metric spaces using weakly $\alpha$-admissible maps and gave application to the existence of a solution for nonlinear integral equations.

With the above discussion in mind, we prove the existence of a positive solution for three different classes of nonlinear fractional differential equations through generalized BCP in a space where self-distance is nonzero, that is, in a $b$-metric-like space. In order to do this, we first introduce a new concept of generalized $\alpha$-weakly $(\psi, \varphi)_s$-contractive mappings involving rational terms, and then develop fixed point results for weakly $\alpha$-admissible mappings. Further, we give some examples and counterexamples to illustrate the applicability and effectiveness of the results over existing results in metric and metric-like spaces. Finally, we use the constructed fixed point results to prove the existence of positive solutions of the aforementioned boundary-value problems for nonlinear fractional differential equations. For further applications to fractional $q$-difference boundary value problems, the use of a $p$-Laplacian operator is suggested.

Our improvements in this paper are five-fold:

1. The use of generalized metric spaces, i.e., $b$-metric-like spaces.
2. The use of the generalized weakly contraction condition with generalized distance.
3. The use of weakly $\alpha$-admissible mappings as opposed to $\alpha$-admissible mappings.
4. The contraction condition used by earlier authors is also sharpened.
5. Application to boundary-value problems for nonlinear fractional differential equations.

Moreover, from our fixed point results, we can derive the following types of fixed point results related to weakly $\alpha$-admissible mappings in $b$-metric-like spaces:

1. Fixed point results when the underlying spaces are endowed with a partial order.
2. Fixed point results when the underlying spaces are endowed with an arbitrary binary relation.
3. Fixed point results when the underlying spaces are endowed with a graph.
4. Fixed point results for cyclic mappings.

## 2 Preliminaries

Let us first recall some basic concepts and notations.

**Definition 2.1** ([4, 7]). Let $\mathcal{X}$ be a nonempty set and $s \geq 1$ a real number. A function $d_b : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ is called a $b$-metric if the following conditions hold for all $u, v, z \in \mathcal{X}$:
(i) $d_b(u, v) = 0$ if and only if $u = v$;
(ii) $d_b(u, v) = d_b(v, u)$;
(iii) $d_b(u, v) \leq s[d_b(u, z) + d_b(z, v)]$.
Then $(\mathcal{X}, d_b)$ is said to be a $b$-metric space.

**Definition 2.2** ([3]). Let $\mathcal{X}$ be a nonempty set. Let the mapping $\sigma : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$ satisfy for all $u, v, z \in \mathcal{X}$ the following conditions:
(i) $\sigma(u, v) = 0$ implies $u = v$;
(ii) $\sigma(u, v) = \sigma(v, u)$;
(iii) $\sigma(u, v) \leq \sigma(u, z) + \sigma(z, v)$.
Then $(\mathcal{X}, \sigma)$ is said to be a metric-like space.

Every partial metric space [13] is a metric-like space. Some examples of metric-like spaces are as follows.

**Example 2.3.** Let $\mathcal{X} = \mathbb{R}$. Then the mappings $\sigma_i : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$ $(i \in \{1, 2\})$ defined by

$$\sigma_1(u, v) = |u| + |v| + a, \quad \sigma_2(u, v) = |u - b| + |v - b|$$

are metric-like on $\mathcal{X}$, where $a \geq 0$ and $b \in \mathbb{R}$.

**Definition 2.4** ([2]). Let $\mathcal{X}$ be a nonempty set and $s \geq 1$ a real number. A function $\sigma_b : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$ is $b$-metric-like if the following conditions hold for all $u, v, z \in \mathcal{X}$:
(i) $\sigma_b(u, v) = 0$ implies $u = v$;
(ii) $\sigma_b(u, v) = \sigma_b(v, u)$;
(iii) $\sigma_b(u, v) \leq s[\sigma_b(u, z) + \sigma_b(z, v)]$.
Then the pair $(\mathcal{X}, \sigma_b)$ is called a $b$-metric-like space and the number $s$ is called the coefficient of $(\mathcal{X}, \sigma_b)$.

In a $b$-metric-like space $(\mathcal{X}, \sigma_b)$, the converse of condition (i) of Definition 2.4 may not be true and $\sigma_b(u, u)$ may be positive for some $u \in \mathcal{X}$. Every $b$-metric-like $\sigma_b$ on $\mathcal{X}$ generates a topology $\tau_{\sigma_b}$ on $\mathcal{X}$ whose base is the family of all open $\sigma_b$-balls $\{B_{\sigma_b}(u, \delta) : u \in \mathcal{X}, \delta > 0\}$, where $B_{\sigma_b}(u, \delta) = \{v \in \mathcal{X} : |\sigma_b(u, v) - \sigma_b(u, u)| < \delta\}$ for all $u \in \mathcal{X}$ and $\delta > 0$.

Clearly, every $b$-metric and partial $b$-metric [18] is $b$-metric-like with the same coefficient $s$. However, the converses of these facts need not hold [2].

**Example 2.5.** Let $\mathcal{X} = [0, \infty)$. Define functions $\sigma_{bi} : \mathcal{X}^2 \to [0, \infty)$ $(i \in \{1, 2\})$ by

$$\sigma_{b1}(x, y) = (x + y)^2, \quad \sigma_{b2}(x, y) = (\max\{x, y\})^2.$$

Then $(\mathcal{X}, \sigma_{bi})$ are $b$-metric-like spaces with constant $s = 2$. Clearly, $(\mathcal{X}, \sigma_{bi})$ are not $b$-metric or metric-like spaces.

**Example 2.6.** Let $\mathcal{X} = \mathbb{R}^+$, let $p > 1$ be a constant, and let $\sigma_b : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+$ be a function defined by

$$\sigma_b(x, y) = (x + y)^p \quad \text{for all } x, y \in \mathcal{X}.$$

Then $(\mathcal{X}, \sigma_b)$ is a $b$-metric-like space with coefficient $s = 2^{p-1}$, but it is not a partial $b$-metric space.

**Proposition 2.7** ([9]). *Let $(\mathcal{X}, \sigma)$ be a metric-like space and $\sigma_b(x, y) = [\sigma(x, y)]^p$, where $p > 1$ is a real number. Then $\sigma_b$ is $b$-metric-like with coefficient $s = 2^{p-1}$.*

**Example 2.8** ([9]). Let $\mathcal{X} = [0, 1]$ and let $p > 1$ be a real number. Then the mapping $\sigma_{b1} : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+$ defined by

$$\sigma_{b1} = (x + y - xy)^p$$

is $b$-metric-like on $\mathcal{X}$ with coefficient $s = 2^{p-1}$.

**Example 2.9** ([9]). Let $\mathcal{X} = \mathbb{R}$. Then the mappings $\sigma_{b_i} : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+$ $(i \in \{2, 3, 4\})$ defined by

$$\sigma_{b2}(x, y) = (|x| + |y| + a)^p, \quad \sigma_{b3}(x, y) = (|x - b| + |y - b|)^p, \quad \sigma_{b4}(x, y) = (x^2 + y^2)^p$$

are $b$-metric-like on $\mathcal{X}$, where $p > 1$, $a \geq 0$ and $b \in \mathbb{R}$.

Now, we define the concepts of Cauchy sequence and convergent sequence, as well as continuous mapping in a $b$-metric-like space.

**Definition 2.10** ([2, 18, 19]). Let $(\mathcal{X}, \sigma_b)$ be a $b$-metric-like space with coefficient $s \geq 1$, let $\{u_n\}$ be a sequence in $\mathcal{X}$ and $u \in \mathcal{X}$.
(i) The sequence $\{u_n\}$ is said to be convergent to $u$ with respect to $\tau_{\sigma_b}$ if $\lim_{n\to\infty} \sigma_b(u_n, u) = \sigma_b(u, u)$.
(ii) The sequence $\{u_n\}$ is called a Cauchy sequence in $(\mathcal{X}, \sigma_b)$ if $\lim_{n,m\to\infty} \sigma_b(u_n, u_m)$ exists and is finite.
(iii) $(\mathcal{X}, \sigma_b)$ is said to be a complete $b$-metric-like space if for every Cauchy sequence $\{u_n\}$ in $\mathcal{X}$ there exists $u \in \mathcal{X}$ such that

$$\lim_{n,m\to\infty} \sigma_b(u_n, u_m) = \lim_{n\to\infty} \sigma_b(u_n, u) = \sigma_b(u, u).$$

(iv) A mapping $\mathcal{J} : \mathcal{X} \to \mathcal{X}$ is said to be $\sigma_b$-continuous if

$$\lim_{n\to\infty} \sigma_b(x_n, x) = \sigma_b(x, x) \quad \text{implies} \quad \lim_{n\to\infty} \sigma_b(\mathcal{J}x_n, \mathcal{J}x) = \sigma_b(\mathcal{J}x, \mathcal{J}x).$$

It is clear that the limit of a sequence is usually not unique in a $b$-metric-like space (already partial metric spaces share this property).

**Definition 2.11.** For a nonempty set $\mathcal{X}$, let $\alpha : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ and $\mathcal{J}: \mathcal{X} \to \mathcal{X}$ be mappings. Then $\mathcal{J}$ is said to be
(i) $\alpha$-admissible if $x, y \in \mathcal{X}$ with $\alpha(x, y) \geq 1$ implies $\alpha(\mathcal{J}x, \mathcal{J}y) \geq 1$ (see [16]);
(ii) weakly $\alpha$-admissible if $x \in \mathcal{X}$ with $\alpha(x, \mathcal{J}x) \geq 1$ implies $\alpha(\mathcal{J}x, \mathcal{J}\mathcal{J}x) \geq 1$ (see [19]).

The following examples illustrate that a mapping can be weakly $\alpha$-admissible but not $\alpha$-admissible.

**Example 2.12.** Let $\mathcal{X} = [0, \infty)$. Define mappings $\alpha : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ and $\mathcal{J} : \mathcal{X} \to \mathcal{X}$ by

$$\alpha(x, y) = \begin{cases} e^{x+y} & \text{if } x, y \in [0, 1], \\ \ln(2x + y) & \text{otherwise,} \end{cases} \qquad \mathcal{J}(x) = \begin{cases} 2\tanh(\frac{3x}{2}) & \text{if } x \in [0, 1], \\ \ln 3x & \text{otherwise.} \end{cases}$$

It is easy to see that $\mathcal{J}$ is not an $\alpha$-admissible mapping. Indeed, for $x = 0$, $y = 1$, we have

$$\alpha(x, y) = \alpha(0, 1) = e > 1$$

but

$$\alpha(\mathcal{J}x, \mathcal{J}y) = \alpha(\mathcal{J}0, \mathcal{J}1) = \alpha(0, 2\tanh 1.5) < 1.$$

However, $\mathcal{J}$ is weakly $\alpha$-admissible. Indeed, suppose that $x \in \mathcal{X}$ such that $\alpha(x, \mathcal{J}x) \geq 1$. Then $x = 1$ and we obtain

$$\alpha(\mathcal{J}x, \mathcal{J}\mathcal{J}x) = \alpha(\mathcal{J}1, \mathcal{J}\mathcal{J}1) > 1.$$

**Example 2.13.** Let $\mathcal{X} = [0, \infty)$. Define mappings $\alpha : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ and $\mathcal{J} : \mathcal{X} \to \mathcal{X}$ by

$$\alpha(u, v) = \begin{cases} \cosh^{-1}(\frac{3u}{2} + 2v) & \text{if } u, v \in [0, 4], \\ \ln(3u + 2v) & \text{otherwise,} \end{cases} \qquad \mathcal{J}(u) = \begin{cases} \frac{u}{\sqrt{4+u^2}} & \text{if } u \in [0, 4], \\ 5u - 3 & \text{otherwise.} \end{cases}$$

It is easy to see that $\mathcal{T}$ is not an $\alpha$-admissible mapping. Indeed, for $u = 2$, $v = 0$, we have

$$\alpha(u, v) = \alpha(2, 0) = \cosh^{-1}(3) > 1$$

but

$$\alpha(\mathcal{J}u, \mathcal{J}v) = \alpha(\mathcal{J}2, \mathcal{J}0) = \alpha\Big(\frac{2}{\sqrt{8}}, 0\Big) = \cosh^{-1}(1.06) < 1.$$

However, $\mathcal{J}$ is weakly $\alpha$-admissible. Indeed, suppose that $u \in \mathcal{X}$ such that $\alpha(u, \mathcal{J}u) \geq 1$. Then $u = 2$ and we obtain

$$\alpha(\mathcal{J}u, \mathcal{J}\mathcal{J}u) = \alpha(\mathcal{J}2, \mathcal{J}\mathcal{J}2) = \alpha\Big(\frac{2}{\sqrt{8}}, \frac{1}{3}\Big) = \cosh^{-1}\Big(\frac{3}{\sqrt{2}} + \frac{2}{3}\Big) > 1.$$

From now on, we use the following terminology from paper [20]: For a nonempty set $\mathcal{X}$ and a mapping $\alpha : \mathcal{X} \times \mathcal{X} \to [0, \infty)$, we denote by $\mathcal{A}(\mathcal{X}, \alpha)$ and $\mathcal{WA}(\mathcal{X}, \alpha)$ the collection of all $\alpha$-admissible mappings on $\mathcal{X}$

and the collection of all weakly $\alpha$-admissible mappings on $\mathcal{X}$, respectively, that is,

$$\mathcal{A}(\mathcal{X},\alpha) := \{\mathcal{J} : \mathcal{X} \to \mathcal{X} \mid \mathcal{J} \text{ is an } \alpha\text{-admissible mapping}\},$$
$$\mathcal{WA}(\mathcal{X},\alpha) := \{\mathcal{J} : \mathcal{X} \to \mathcal{X} \mid \mathcal{J} \text{ is a weak } \alpha\text{-admissible mapping}\}.$$

Obviously,

$$\mathcal{A}(\mathcal{X},\alpha) \subset \mathcal{WA}(\mathcal{X},\alpha),$$

and, by Examples 2.12 and 2.13, the inclusion can be strict.

**Definition 2.14** (Altering distance function [10]). A function $\varphi : [0,+\infty) \to [0,+\infty)$ is called an altering distance function if the following properties are satisfied:
(i) $\varphi$ is continuous and non-decreasing;
(ii) $\varphi(t) = 0 \Leftrightarrow t = 0$.

**Definition 2.15** (Weakly contractive mapping [15]). Let $\mathcal{X}$ be a metric space. A mapping $\mathcal{J} : \mathcal{X} \to \mathcal{X}$ is called weakly contractive if

$$d(\mathcal{J}x, \mathcal{J}y) \le d(x,y) - \varphi(d(x,y)) \quad \text{for all } x, y \in \mathcal{X},$$

where $\varphi$ is an altering distance function.

# 3 Main results

We first introduce the notion of a generalized $\alpha$-weakly $(\psi,\varphi)_s$-contractive mapping in a $b$-metric-like space.

**Definition 3.1.** Let $(\mathcal{X},\sigma_b)$ be a $b$-metric-like space with the coefficient $s \ge 1$. A mapping $\mathcal{J}: \mathcal{X} \to \mathcal{X}$ is said to be a generalized $\alpha$-weakly $(\psi,\varphi)_s$-contractive mapping if there exist altering distance functions $\varphi$, $\psi$ and $\alpha: \mathcal{X} \times \mathcal{X} \to [0,\infty)$ such that

$$u, v \in \mathcal{X} \text{ with } \alpha(u,v) \ge 1 \;\Rightarrow\; \psi(s\sigma(\mathcal{J}u, \mathcal{J}v)) \le \psi(\Theta(u,v)) - \varphi(\Theta(u,v)), \tag{3.1}$$

where

$$\begin{aligned}\Theta(u,v) = \max\Big\{&\sigma_b(u,v), \sigma_b(v,\mathcal{J}v), \sigma_b(u,\mathcal{J}u), \frac{\sigma_b(u,\mathcal{J}v) + \sigma_b(v,\mathcal{J}u)}{4s},\\ &\frac{\sigma_b(u,\mathcal{J}u)\sigma_b(u,\mathcal{J}v) + \sigma_b(v,\mathcal{J}v)\sigma_b(v,\mathcal{J}u)}{1 + s[\sigma_b(u,\mathcal{J}u) + \sigma_b(v,\mathcal{J}v)]},\\ &\frac{\sigma_b(u,\mathcal{J}u)\sigma_b(u,\mathcal{J}v) + \sigma_b(v,\mathcal{J}v)\sigma_b(v,\mathcal{J}u)}{1 + \sigma_b(u,\mathcal{J}v) + \sigma_b(v,\mathcal{J}u)}\Big\}.\end{aligned} \tag{3.2}$$

We denote by $\Lambda_s(\mathcal{X},\alpha,\psi,\varphi)$ the collection of all generalized $\alpha$-weakly $(\psi,\varphi)_s$-contractive mappings on $(\mathcal{X},\sigma_b)$.

Now we are in a position to derive our first result of this section.

**Theorem 3.2.** *Let $(\mathcal{X},\sigma_b)$ be a $b$-complete $b$-metric-like space with coefficient $s \ge 1$, let $\alpha : \mathcal{X} \times \mathcal{X} \to [0,\infty)$ and $\mathcal{J} : \mathcal{X} \to \mathcal{X}$ be given mappings. Suppose that the following conditions hold:*
(A1) $\mathcal{J} \in \Lambda_s(\mathcal{X},\alpha,\psi,\varphi) \cap \mathcal{WA}(\mathcal{X},\alpha)$;
(A2) *there exists $u_0 \in \mathcal{X}$ such that $\alpha(u_0, \mathcal{J}u_0) \ge 1$;*
(A3) *$\alpha$ has a transitive property, that is, for $u, v, w \in \mathcal{X}$, $\alpha(u,v) \ge 1$ and $\alpha(v,w) \ge 1$ imply $\alpha(u,w) \ge 1$;*
(A4) *$\mathcal{J}$ is $\sigma_b$-continuous.*
*Then* $\mathrm{Fix}(\mathcal{J}) \ne \emptyset$.

*Proof.* By the given condition (A2), there exists $u_0 \in \mathcal{X}$ such that $\alpha(u_0, \mathcal{J}u_0) \ge 1$. Define a sequence $\{u_n\} \in \mathcal{X}$ by $u_{n+1} = \mathcal{J}u_n$ for $n = 0, 1, \ldots$. If there exists $n_0 \in \mathbb{N}$ such that $u_{n_0} = u_{n_0+1}$, then $u_{n_0} \in \mathrm{Fix}(\mathcal{J})$ and hence the proof is complete. Hence, we will assume that $u_n \neq u_{n+1}$ for all $n \in \mathbb{N}$. It follows that

$$\sigma_b(u_n, u_{n+1}) > 0 \quad \text{for all } n \in \mathbb{N}.$$

Hence, we have

$$\frac{1}{2s}\sigma_b(u_n, \mathcal{J}u_n) < \sigma_b(u_n, \mathcal{J}u_n) \quad \text{for all } n \in \mathbb{N}.$$

Step I. First we need to prove that

$$\lim_{n\to\infty} \sigma_b(u_n, u_{n+1}) = 0. \tag{3.3}$$

Using that $\mathcal{J} \in \mathcal{WA}(\mathcal{X}, \alpha)$ and $\alpha(u_0, \mathcal{J}u_0) \geq 1$, we have

$$\alpha(u_1, u_2) = \alpha(\mathcal{J}u_0, \mathcal{J}\mathcal{J}u_0) \geq 1.$$

Repeating this process, we obtain

$$\alpha(u_{n+1}, u_{n+2}) \geq 1. \tag{3.4}$$

It follows from $\mathcal{J} \in \Lambda_s(\mathcal{X}, \alpha, \psi, \varphi)$ that (owing to (3.4))

$$\begin{aligned}
\psi(\sigma_b(u_{n+1}, u_{n+2})) &= \psi(\sigma_b(\mathcal{J}u_n, \mathcal{J}u_{n+1}))\\
&\leq \psi(s\sigma_b(\mathcal{J}u_n, \mathcal{J}u_{n+1}))\\
&\leq \psi(\Theta(u_n, u_{n+1})) - \varphi(\Theta(u_n, u_{n+1}))
\end{aligned} \tag{3.5}$$

for all $n \in \mathbb{N}$, where

$$\begin{aligned}
\Theta(u_n, u_{n+1}) = \max\Bigg\{&\sigma_b(u_n, \mathcal{J}u_n), \sigma_b(u_n, \mathcal{J}u_n), \sigma_b(\mathcal{J}u_n, \mathcal{J}^2u_n), \frac{\sigma_b(u_n, \mathcal{J}^2u_n) + \sigma_b(\mathcal{J}u_n, \mathcal{J}u_n)}{4s},\\
&\frac{\sigma_b(u_n, \mathcal{J}u_n)\sigma_b(u_n, \mathcal{J}^2u_n) + \sigma_b(u_n, \mathcal{J}^2u_n)\sigma_b(\mathcal{J}u_n, \mathcal{J}u_n)}{1 + s[\sigma_b(u_n, \mathcal{J}u_n) + \sigma_b(\mathcal{J}u_n, \mathcal{J}^2u_n)]},\\
&\frac{\sigma_b(u_n, \mathcal{J}u_n)\sigma_b(u_n, \mathcal{J}^2u_n) + \sigma_b(\mathcal{J}u_n, \mathcal{J}^2u_n)\sigma_b(\mathcal{J}u_n, \mathcal{J}u_n)}{1 + \sigma_b(u_n, \mathcal{J}^2u_n) + \sigma_b(\mathcal{J}u_n, \mathcal{J}u_n)}\Bigg\}\\
= \max\Bigg\{&\sigma_b(u_n, \mathcal{J}u_n), \sigma_b(\mathcal{J}u_n, \mathcal{J}^2u_n),\\
&\frac{\sigma_b(u_n, \mathcal{J}u_n) + \sigma_b(\mathcal{J}u_n, \mathcal{J}^2u_n) + \sigma_b(\mathcal{J}u_n, u_n) + \sigma_b(u_n, \mathcal{J}u_n)}{4s},\\
&\frac{\sigma_b(u_n, \mathcal{J}u_n)\sigma_b(u_n, \mathcal{J}^2u_n) + \sigma_b(u_n, \mathcal{J}^2u_n)[\sigma_b(\mathcal{J}u_n, u_n) + \sigma_b(u_n, \mathcal{J}u_n)]}{1 + s[\sigma_b(u_n, \mathcal{J}u_n) + \sigma_b(\mathcal{J}u_n, \mathcal{J}^2u_n)]},\\
&\frac{\sigma_b(u_n, \mathcal{J}u_n)[\sigma_b(u_n, \mathcal{J}u_n) + \sigma_b(\mathcal{J}u_n, \mathcal{J}^2u_n)] + \sigma_b(\mathcal{J}u_n, \mathcal{J}^2u_n)[\sigma_b(\mathcal{J}u_n, u_n) + \sigma_b(u_n, \mathcal{J}u_n)]}{1 + \sigma_b(u_n, \mathcal{J}u_n) + \sigma_b(\mathcal{J}u_n, \mathcal{J}^2u_n) + \sigma_b(\mathcal{J}u_n, u_n) + \sigma_b(u_n, \mathcal{J}u_n)}\Bigg\}\\
= \max\Bigg\{&\sigma_b(u_n, \mathcal{J}u_n), \sigma_b(\mathcal{J}u_n, \mathcal{J}^2u_n), \frac{3\sigma_b(u_n, \mathcal{J}u_n) + \sigma_b(\mathcal{J}u_n, \mathcal{J}^2u_n)}{4s},\\
&\frac{3\sigma_b(u_n, \mathcal{J}u_n)\sigma_b(u_n, \mathcal{J}^2u_n)}{1 + s[\sigma_b(u_n, \mathcal{J}u_n) + \sigma_b(\mathcal{J}u_n, \mathcal{J}^2u_n)]},\\
&\frac{3\sigma_b(u_n, \mathcal{J}u_n)\sigma_b(\mathcal{J}u_n, \mathcal{J}^2u_n) + \sigma_b(u_n, \mathcal{J}u_n)\sigma_b(u_n, \mathcal{J}u_n)}{1 + 3\sigma_b(u_n, \mathcal{J}u_n) + \sigma_b(\mathcal{J}u_n, \mathcal{J}^2u_n)}\Bigg\}\\
= \max\{&\sigma_b(u_n, \mathcal{J}u_n), \sigma_b(\mathcal{J}u_n, \mathcal{J}^2u_n)\}.
\end{aligned}$$

If $\Theta(u_n, \mathcal{J}u_n) = \sigma_b(\mathcal{J}u_n, \mathcal{J}^2u_n)$ for some $n \in \mathbb{N}$, then inequality (3.5) implies that

$$\begin{aligned}
\psi(\sigma_b(\mathcal{J}u_n, \mathcal{J}^2u_n)) &\leq \psi(s\sigma_b(\mathcal{J}u_n, \mathcal{J}^2u_n))\\
&\leq \psi(\Theta(u_n, \mathcal{J}u_n)) - \varphi(\Theta(u_n, \mathcal{J}u_n))\\
&\leq \psi(\sigma_b(\mathcal{J}u_n, \mathcal{J}^2u_n)) - \varphi(\sigma_b(\mathcal{J}u_n, \mathcal{J}^2u_n))\\
&< \psi(\sigma_b(\mathcal{J}u_n, \mathcal{J}^2u_n)),
\end{aligned}$$

a contradiction. Therefore, $\Theta(u_n, \mathcal{J}u_n) = \sigma_b(u_n, \mathcal{J}u_n)$ for all $n \in \mathbb{N}$.

From (3.5), we have

$$\begin{aligned}\psi(\sigma_b(\mathcal{J}u_n, \mathcal{J}^2u_n)) &\le \psi(s\sigma_b(\mathcal{J}u_n, \mathcal{J}^2u_n))\\ &\le \psi(\Theta(u_n, \mathcal{J}u_n)) - \varphi(\Theta(u_n, \mathcal{J}u_n))\\ &< \psi(\sigma_b(u_n, \mathcal{J}u_n)),\end{aligned} \tag{3.6}$$

for all $n \in \mathbb{N}$. Since $\psi$ is a non-decreasing mapping, $\{\sigma_b(u_n, u_{n+1})\}$ is a decreasing sequence in $\mathbb{R}$ and then there exists $\rho \ge 0$ such that

$$\lim_{n\to\infty} \sigma_b(u_n, u_{n+1}) = \rho.$$

Passing to the limit as $n \to \infty$ in (3.6), we get

$$\psi(\rho) \le \psi(\rho) - \varphi(\rho) \le \psi(\rho)$$

and thus $\varphi(\rho) = 0$. This implies that $\rho = 0$, that is,

$$\lim_{n\to\infty} \sigma_b(u_n, \mathcal{J}u_n) = \lim_{n\to\infty} \sigma_b(u_n, u_{n+1}) = 0. \tag{3.7}$$

This proves that (3.3) holds.

Step II. Next, we prove that $\{u_n\}$ is a $b$-Cauchy sequence in $\mathcal{X}$.

Suppose, on the contrary, that there exist $\epsilon_0 > 0$ and subsequences $\{u_{m(k)}\}$ and $\{u_{n(k)}\}$ of $\{u_n\}$ such that $m(k) > n(k) \ge k$ and

$$\sigma_b(u_{m(k)}, u_{n(k)}) \ge \epsilon_0 \tag{3.8}$$

and $n(k)$ is the smallest number such that (3.8) holds, so that we have

$$\sigma_b(u_{m(k)}, u_{n(k)-1}) < \epsilon_0. \tag{3.9}$$

By Definition 2.4 (iii), (3.8) and (3.9), we get

$$\epsilon_0 \le \sigma_b(u_{m(k)}, u_{n(k)}) \le s\sigma_b(u_{m(k)}, u_{n(k)-1}) + s\sigma_b(u_{n(k)-1}, u_{n(k)}) < s\epsilon_0 + s\sigma_b(u_{n(k)-1}, u_{n(k)}). \tag{3.10}$$

Owing to (3.7), there exists $N_1 \in \mathbb{N}$ such that

$$\sigma_b(u_{m(k)-1}, \mathcal{J}u_{m(k)-1}) < \epsilon_0, \quad \sigma_b(u_{n(k)}, \mathcal{J}u_{n(k)}) < \epsilon_0, \quad \sigma_b(u_{m(k)}, \mathcal{J}u_{m(k)}) < \epsilon_0 \quad \text{for all } k > N_1, \tag{3.11}$$

which, together with (3.10), shows $\sigma_b(u_{m(k)}, u_{n(k)}) < 2s\epsilon_0$ for all $k > N_1$. Hence

$$\psi(\sigma_b(u_{m(k)}, u_{n(k)})) < \psi(2s\epsilon_0) \quad \text{for all } k > N_1. \tag{3.12}$$

From (3.7), (3.8) and (3.11), we get

$$\frac{1}{2s}\sigma_b(u_{m(k)}, \mathcal{J}u_{m(k)}) < \frac{\epsilon_0}{2s} < \sigma_b(u_{m(k)}, u_{n(k)}) \quad \text{for all } k > N_1.$$

Using the triangular inequality we deduce

$$\sigma_b(u_{m(k)}, u_{n(k)}) \le s\sigma_b(u_{m(k)}, u_{m(k)+1}) + s[\sigma_b(u_{m(k)+1}, u_{n(k)+1}) + \sigma_b(u_{n(k)+1}, u_{n(k)})]. \tag{3.13}$$

Passing to the limit as $k \to \infty$ in (3.13), by (3.6) we obtain

$$\frac{\epsilon_0}{s} \le \limsup_{k\to\infty} \sigma_b(u_{m(k)+1}, u_{n(k)+1}).$$

Therefore there exists $N_2 \in \mathbb{N}$ such that $\sigma_b(u_{m(k)+1}, u_{n(k)+1}) > 0$ for $k > N_2$, i.e. $\sigma_b(\mathcal{J}u_{m(k)}, \mathcal{J}u_{n(k)}) > 0$. Using the transitivity property of $\alpha$, we get $\alpha(u_{m(k)}, u_{n(k)}) \ge 1$. Therefore $\mathcal{J} \in \Lambda_s(\mathcal{X}, \alpha, \psi, \varphi)$ implies that

$$\begin{aligned}\psi(\sigma_b(u_{m(k)+1}, u_{n(k)+1})) &\le \psi(s\sigma_b(\mathcal{J}u_{m(k)}, \mathcal{J}u_{n(k)}))\\ &\le \psi(\Theta(u_{m(k)}, u_{n(k)})) - \varphi(\Theta(u_{m(k)}, u_{n(k)})).\end{aligned} \tag{3.14}$$

Using (3.2), (3.11)–(3.13), we have

$$\begin{aligned}
\Theta(u_{m(k)}, u_{n(k)}) = \max\Bigg\{ & \sigma_b(u_{m(k)}, u_{n(k)}), \sigma_b(u_{m(k)}, \mathcal{J}u_{m(k)}), \sigma_b(u_{n(k)}, \mathcal{J}u_{n(k)}), \\
& \frac{\sigma_b(u_{m(k)}, \mathcal{J}u_{n(k)}) + \sigma_b(u_{n(k)}, \mathcal{J}u_{m(k)})}{4s}, \\
& \frac{\sigma_b(u_{m(k)}, \mathcal{J}u_{m(k)})\sigma_b(u_{m(k)}, \mathcal{J}u_{n(k)}) + \sigma_b(u_{n(k)}, \mathcal{J}u_{n(k)})\sigma_b(u_{n(k)}, \mathcal{J}u_{m(k)})}{1 + s[\sigma_b(u_{m(k)}, \mathcal{J}u_{m(k)}) + \sigma_b(u_{n(k)}, \mathcal{J}u_{n(k)})]}, \\
& \frac{\sigma_b(u_{m(k)}, \mathcal{J}u_{m(k)})\sigma_b(u_{m(k)}, \mathcal{J}u_{n(k)}) + \sigma_b(u_{n(k)}, \mathcal{J}u_{n(k)})\sigma_b(u_{n(k)}, \mathcal{J}u_{m(k)})}{1 + \sigma_b(u_{m(k)}, \mathcal{J}u_{n(k)}) + \sigma_b(u_{n(k)}, \mathcal{J}u_{m(k)})}\Bigg\} \\
\le \max\Bigg\{ & \sigma_b(u_{m(k)}, u_{n(k)}), \sigma_b(u_{m(k)}, \mathcal{J}u_{m(k)}), \sigma_b(u_{n(k)}, \mathcal{J}u_{n(k)}), \\
& \frac{\sigma_b(u_{m(k)}, u_{n(k)}) + \sigma_b(u_{n(k)}, \mathcal{J}u_{n(k)}) + \sigma_b(u_{n(k)}, u_{m(k)}) + \sigma_b(u_{m(k)}, \mathcal{J}u_{m(k)})}{4s}, \\
& \frac{\sigma_b(u_{m(k)}, \mathcal{J}u_{m(k)})[\sigma_b(u_{m(k)}, u_{n(k)}) + \sigma_b(u_{n(k)}, \mathcal{J}u_{n(k)})]}{1 + s[\sigma_b(u_{m(k)}, \mathcal{J}u_{m(k)}) + \sigma_b(u_{n(k)}, \mathcal{J}u_{n(k)})]} \\
& \quad + \frac{\sigma_b(u_{n(k)}, \mathcal{J}u_{n(k)})[\sigma_b(u_{n(k)}, u_{m(k)}) + \sigma_b(u_{m(k)}, \mathcal{J}u_{m(k)})]}{1 + s[\sigma_b(u_{m(k)}, \mathcal{J}u_{m(k)}) + \sigma_b(u_{n(k)}, \mathcal{J}u_{n(k)})]}, \\
& \frac{\sigma_b(u_{m(k)}, \mathcal{J}u_{m(k)})[\sigma_b(u_{m(k)}, u_{n(k)}) + \sigma_b(u_{n(k)}, \mathcal{J}u_{n(k)})]}{1 + \sigma_b(u_{m(k)}, u_{n(k)}) + \sigma_b(u_{n(k)}, \mathcal{J}u_{n(k)}) + \sigma_b(u_{n(k)}, u_{m(k)}) + \sigma_b(u_{m(k)}, \mathcal{J}u_{m(k)})} \\
& \quad + \frac{\sigma_b(u_{n(k)}, \mathcal{J}u_{n(k)})[\sigma_b(u_{n(k)}, u_{m(k)}) + \sigma_b(u_{m(k)}, \mathcal{J}u_{m(k)})]}{1 + \sigma_b(u_{m(k)}, u_{n(k)}) + \sigma_b(u_{n(k)}, \mathcal{J}u_{n(k)}) + \sigma_b(u_{n(k)}, u_{m(k)}) + \sigma_b(u_{m(k)}, \mathcal{J}u_{m(k)})}\Bigg\} \\
\le \max\Bigg\{ & 2s\epsilon_0, \sigma_b(u_{m(k)}, \mathcal{J}u_{m(k)}), \sigma_b(u_{n(k)}, \mathcal{J}u_{n(k)}), \frac{2s\epsilon_0 + \epsilon_0 + 2s\epsilon_0 + \epsilon_0}{4s}, \\
& \frac{\sigma_b(u_{m(k)}, \mathcal{J}u_{m(k)})[2s\epsilon_0 + \epsilon_0] + \sigma_b(u_{n(k)}, \mathcal{J}u_{n(k)})[2s\epsilon_0 + \epsilon_0]}{1 + s[\epsilon_0 + \epsilon_0]}, \\
& \frac{[2s\epsilon_0 + \epsilon_0] + \epsilon_0[2s\epsilon_0 + \epsilon_0]}{1 + 2s\epsilon_0 + \sigma_b(u_{n(k)}, \mathcal{J}u_{n(k)}) + 2s\epsilon_0 + \epsilon_0}\Bigg\}
\end{aligned} \tag{3.15}$$

for $k > \max\{N_1, N_2\}$. Passing to the limit as $k \to \infty$ in (3.14) and using (3.15), we obtain

$$\psi(\epsilon_0) \le \psi(\epsilon_0) - \varphi(\epsilon_0).$$

This implies that $\varphi(\epsilon_0) = 0$ and hence $\epsilon_0 = 0$, a contradiction. Therefore $\{u_n\}$ is a $b$-Cauchy sequence in $\mathcal{X}$.

Step III. As $(\mathcal{X}, \sigma_b)$ is a $b$-complete $b$-metric-like space, there exists $u^* \in \mathcal{X}$ such that

$$\sigma_b(u^*, u^*) = \lim_{n\to\infty} \sigma_b(u_n, u^*) = \lim_{n,m\to\infty} \sigma_b(u_n, u_m) = 0.$$

We will show that $u^*$ is a fixed point for $\mathcal{J}$. Owing to condition (A4), we get

$$\lim_{n\to\infty} \sigma_b(\mathcal{J}u_n, \mathcal{J}u^*) = 0.$$

From the triangle inequality, we have

$$\sigma_b(u^*, \mathcal{J}u^*) \le s[\sigma_b(u^*, \mathcal{J}u_n) + \sigma_b(\mathcal{J}u_n, \mathcal{J}u^*)] \quad \text{for all } n \in \mathbb{N}.$$

Passing to the limit as $n \to \infty$ in the above inequality, we obtain

$$\sigma_b(u^*, \mathcal{J}u^*) = 0$$

and then $\mathcal{J}u^* = u^*$. This shows that $\mathrm{Fix}(\mathcal{J}) \neq \emptyset$, which completes the proof. □

We note that the previous result can still be valid for $\mathcal{J}$ not necessarily $b$-continuous. We have the following result.

**Theorem 3.3.** *Let $(\mathcal{X}, \sigma_b)$ be a b-complete b-metric-like space with coefficient $s \geq 1$, let $\alpha : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ and $\mathcal{J} : \mathcal{X} \to \mathcal{X}$ be given mappings. Suppose that the following conditions hold:*

(A1) $\mathcal{J} \in \Lambda_s(\mathcal{X}, \alpha, \psi, \varphi) \cap \mathcal{WA}(\mathcal{X}, \alpha)$;

(A2) *there exists $u_0 \in \mathcal{X}$ such that $\alpha(u_0, \mathcal{J}u_0) \geq 1$;*

(A3) *$\alpha$ has a transitive property;*

(A4*) *$\mathcal{X}$ is $\alpha$-regular, i.e., if $\{u_n\}$ is a sequence in $\mathcal{X}$ with $\alpha(u_n, u_{n+1}) \geq 1$ for $n \in \mathbb{N}$ and $u_n \to u^*$ as $n \to \infty$, then $\alpha(u_n, u^*) \geq 1$ for $n \in \mathbb{N}$.*

*Then* $\mathrm{Fix}(\mathcal{J}) \neq \emptyset$.

*Proof.* Following the proof of Theorem 3.2, we obtain a $\sigma_b$-Cauchy sequence $\{u_n\}$ in the $\sigma_b$-complete $b$-metric-like space $(\mathcal{X}, \sigma_b)$. Hence, there exists $u^* \in \mathcal{X}$ such that

$$\lim_{n\to\infty} \sigma_b(u_n, u^*) = 0,$$

that is, $u_n \to u^*$ as $n \to \infty$. By $\alpha$-regularity of $\mathcal{X}$, we have $\alpha(u_n, u^*) \geq 1$ for all $n \in \mathbb{N}$. It follows from (A1) that

$$\psi(s\sigma_b(\mathcal{J}u_n, \mathcal{J}u^*)) \leq \psi(\Theta(u_n, u^*)) - \varphi(\Theta(u_n, u^*)), \tag{3.16}$$

where

$$\begin{aligned}\Theta(u_n, u^*) = \max\Big\{&\sigma_b(u_n, u^*), \sigma_b(u^*, \mathcal{J}u^*), \sigma_b(u_n, \mathcal{J}u_n), \frac{\sigma_b(u_n, \mathcal{J}u^*) + \sigma_b(u^*, \mathcal{J}u_n)}{4s},\\ &\frac{\sigma_b(u_n, \mathcal{J}u_n)\sigma_b(u_n, \mathcal{J}u^*) + \sigma_b(u^*, \mathcal{J}u^*)\sigma_b(u^*, \mathcal{J}u_n)}{1 + s[\sigma_b(u_n, \mathcal{J}u_n) + \sigma_b(u^*, \mathcal{J}u^*)]},\\ &\frac{\sigma_b(u_n, \mathcal{J}u_n)\sigma_b(u_n, \mathcal{J}u^*) + \sigma_b(u^*, \mathcal{J}u^*)\sigma_b(u^*, \mathcal{J}u_n)}{1 + \sigma_b(u_n, \mathcal{J}u^*) + \sigma_b(u^*, \mathcal{J}u_n)}\Big\}.\end{aligned} \tag{3.17}$$

Applying the limit as $n \to \infty$ to (3.17) and using [9, Lemma 16], we get

$$\begin{aligned}\frac{\sigma_b(u^*, \mathcal{J}u^*)}{4s^2} &= \min\Big\{\sigma_b(u^*, \mathcal{J}u^*), \frac{\frac{\sigma_b(u^*, \mathcal{J}u^*)}{s}}{4s}\Big\}\\ &\leq \liminf_{n\to\infty} \Theta(u_n, u^*) \leq \limsup_{n\to\infty} \Theta(u_n, u^*)\\ &\leq \max\Big\{\sigma_b(u^*, \mathcal{J}u^*), \frac{s\sigma_b(u^*, \mathcal{J}u^*)}{4s}\Big\} = \sigma_b(u^*, \mathcal{J}u^*).\end{aligned} \tag{3.18}$$

Again, by using (3.16)–(3.18) and passing to the upper limit as $n \to \infty$ and using [9, Lemma 16], we get

$$\begin{aligned}\psi(\sigma_b(u^*, \mathcal{J}u^*)) &= \psi\Big(s\frac{1}{s}\sigma_b(u_{n+1}, \mathcal{J}u^*)\Big)\\ &\leq \psi(s \limsup_{n\to\infty} \sigma_b(u_{n+1}, \mathcal{J}u^*))\\ &\leq \psi(\limsup_{n\to\infty} \Theta(u_n, u^*)) - \varphi(\liminf_{n\to\infty} \Theta(u_n, u^*))\\ &\leq \psi(\sigma_b(u^*, \mathcal{J}u^*)) - \varphi(\sigma_b(u^*, \mathcal{J}u^*)),\end{aligned}$$

a contradiction, and hence $\sigma_b(u^*, \mathcal{J}u^*) = 0$. Therefore $u^* = \mathcal{J}u^*$. Hence $\mathrm{Fix}(\mathcal{J}) \neq \emptyset$. □

## 3.1 Consequences

We can derive several results from our main results. For example:

**Corollary 3.4.** *Let the conditions of Theorem 3.2 or Theorem 3.3 be satisfied, apart from condition* (A1)*, i.e., we have:*
(A1*) $\mathcal{J} \in \Lambda_s(\mathcal{X}, \alpha, \psi, \varphi) \cap \mathcal{A}(\mathcal{X}, \alpha)$;
(A2) *there exists* $u_0 \in \mathcal{X}$ *such that* $\alpha(u_0, \mathcal{J}u_0) \geq 1$;
(A3) $\alpha$ *has a transitive property;*
(A4) $\mathcal{J}$ *is* $\sigma_b$*-continuous, or*
(A4*) $\mathcal{X}$ *is* $\alpha$*-regular.*
*Then* $\operatorname{Fix}(\mathcal{J}) \neq \emptyset$.

**Corollary 3.5.** *Let* $(\mathcal{X}, d_b)$ *be a complete b-metric space with coefficient* $s \geq 1$*, let* $\psi, \varphi : [0, \infty) \to [0, \infty)$ *be altering distance functions, let* $\alpha : \mathcal{X} \times \mathcal{X} \to [0, \infty)$*, and let* $\mathcal{J} : \mathcal{X} \to \mathcal{X}$ *be a contractive mapping of the type* $\Lambda_s'(\mathcal{X}, \alpha, \psi, \varphi)$*, that is,*

$$u, v \in \mathcal{X} \text{ with } \alpha(u, v) \geq 1 \;\Rightarrow\; \psi(s d_b(\mathcal{J}u, \mathcal{J}v)) \leq \psi(\Theta_4(u, v)) - \varphi(\Theta_4(u, v)),$$

*where*

$$\Theta_4(u, v) = \max\bigg\{ d_b(u, v), d_b(v, \mathcal{J}v), d_b(u, \mathcal{J}u), \frac{d_b(u, \mathcal{J}v) + d_b(v, \mathcal{J}u)}{4s}, \\ \frac{d_b(u, \mathcal{J}u) d_b(u, \mathcal{J}v) + d_b(v, \mathcal{J}v) d_b(v, \mathcal{J}u)}{1 + s[d_b(u, \mathcal{J}u) + d_b(v, \mathcal{J}v)]}, \frac{d_b(u, \mathcal{J}u) d_b(u, \mathcal{J}v) + d_b(v, \mathcal{J}v) d_b(v, \mathcal{J}u)}{1 + d_b(u, \mathcal{J}v) + d_b(v, \mathcal{J}u)} \bigg\}.$$

*Suppose that the following conditions hold:*
(A1) $\mathcal{J} \in \Lambda_s'(\mathcal{X}, \alpha, \psi, \varphi) \cap \mathcal{A}(\mathcal{X}, \alpha)$;
(A2) *there exists* $u_0 \in \mathcal{X}$ *such that* $\alpha(u_0, \mathcal{J}u_0) \geq 1$;
(A3) $\alpha$ *has a transitive property;*
(A4) $\mathcal{J}$ *is b-continuous.*
*Then* $\operatorname{Fix}(\mathcal{J}) \neq \emptyset$.

## 4 Examples

The following example verifies the conditions of Theorem 3.2 and involvement of rational terms.

**Example 4.1.** Let $\mathcal{X} = \{0, 1, 2\}$ and let $\sigma_b : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ be defined by

$$\begin{aligned} &\sigma_b(0,0) = 0, && \sigma_b(1,1) = \frac{2}{3}, && \sigma_b(2,2) = \frac{5}{6}, \\ &\sigma_b(0,1) = \sigma_b(1,0) = \frac{1}{3}, && \sigma_b(0,2) = \sigma_b(2,0) = \frac{8}{3}, && \sigma_b(1,2) = \sigma_b(2,1) = 4. \end{aligned}$$

It is clear that $(\mathcal{X}, \sigma_b)$ is a $b$-complete $b$-metric like space with constant $s = \frac{14}{6}$, which is neither metric, nor metric-like space. Define mappings $\mathcal{J} : \mathcal{X} \to \mathcal{X}$ and $\alpha : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ by $\mathcal{J}0 = 0$, $\mathcal{J}1 = 0$, $\mathcal{J}2 = 1$, and

$$\alpha(u, v) = \begin{cases} \frac{1}{5} + \tanh(9u + v) & \text{if } u \geq v, \\ 0 & \text{otherwise.} \end{cases}$$

Under these assumption, we will show that all the conditions of Theorem 3.2 are satisfied.

*Proof.* Suppose that $u, v \in \mathcal{X}$, so that $\alpha(u, v) \geq 1$. Consider the functions $\psi(t) = 2t$ and $\varphi(t) = \frac{t}{2}$. Now $\sigma_b(\mathcal{J}0, \mathcal{J}0) = \sigma_b(\mathcal{J}0, \mathcal{J}1) = 0$, so the following four cases can be distinguished:

Case I: For $u = 0$ and $v = 2$ (or $u = 1$ and $v = 2$),

$$\psi(s\sigma_b(\mathcal{J}u, \mathcal{J}v)) = \psi\Big(\frac{14}{6}\sigma_b(\mathcal{J}0, \mathcal{J}2)\Big) = \psi\Big(\Big(\frac{14}{6}\Big)\sigma_b(0, 1)\Big) = \psi\Big(\frac{14}{18}\Big) = \frac{14}{9} = 1.55 \tag{4.1}$$

and

$$\begin{aligned}\Theta(0,2) = \max\Big\{&\sigma_b(0,2), \sigma_b(2,\mathcal{J}2), \sigma_b(0,\mathcal{J}0), \frac{\sigma_b(0,\mathcal{J}2)+\sigma_b(2,\mathcal{J}0)}{4\frac{14}{6}},\\ &\frac{\sigma_b(0,\mathcal{J}0)\sigma_b(0,\mathcal{J}2)+\sigma_b(2,\mathcal{J}2)\sigma_b(2,\mathcal{J}0)}{1+\frac{14}{6}[\sigma_b(0,\mathcal{J}0)+\sigma_b(2,\mathcal{J}2)]}, \frac{\sigma_b(0,\mathcal{J}0)\sigma_b(0,\mathcal{J}2)+\sigma_b(2,\mathcal{J}2)\sigma_b(2,\mathcal{J}0)}{1+\sigma_b(0,\mathcal{J}2)+\sigma_b(2,\mathcal{J}0)}\Big\}\\ = \max\Big\{&\sigma_b(0,2), \sigma_b(2,1), \sigma_b(0,0), \frac{\sigma_b(0,1)+\sigma_b(2,0)}{\frac{28}{3}},\\ &\frac{\sigma_b(0,0)\sigma_b(0,1)+\sigma_b(2,1)\sigma_b(2,0)}{1+\frac{7}{3}[\sigma_b(0,0)+\sigma_b(2,1)]}, \frac{\sigma_b(0,0)\sigma_b(0,1)+\sigma_b(2,1)\sigma_b(2,0)}{1+\sigma_b(0,1)+\sigma_b(2,0)}\Big\}\\ = \max\Big\{&\frac{8}{3}, 4, 0, \frac{9}{8}, 1.03, 5.64\Big\}\\ = 5.64 \quad &\text{(corresponding to rational term).}\end{aligned} \tag{4.2}$$

From (4.1) and (4.2) it is clear that inequality (3.1) will be in the form

$$\psi(s\sigma_b(\mathcal{J}u,\mathcal{J}v)) \le \psi(\Theta(u,v)) - \varphi(\Theta(u,v)),$$

that is, $1.55 \le \psi(5.64) - \varphi(5.64) \le 8.46$, which is true.

Case II: For $u = 2$ and $v = 2$,

$$\psi(s\sigma_b(\mathcal{J}u,\mathcal{J}v)) = \psi\Big(\frac{14}{6}\sigma_b(\mathcal{J}2,\mathcal{J}2)\Big) = \psi\Big(\Big(\frac{14}{6}\Big)\sigma_b(1,1)\Big) = \psi\Big(\frac{28}{18}\Big) = \frac{28}{9} = 3.11, \tag{4.3}$$

and

$$\begin{aligned}\Theta(2,2) = \max\Big\{&\sigma_b(2,2), \sigma_b(2,\mathcal{J}2), \sigma_b(2,\mathcal{J}2), \frac{\sigma_b(2,\mathcal{J}2)+\sigma_b(2,\mathcal{J}2)}{4\frac{14}{6}},\\ &\frac{\sigma_b(2,\mathcal{J}2)\sigma_b(2,\mathcal{J}2)+\sigma_b(2,\mathcal{J}2)\sigma_b(2,\mathcal{J}2)}{1+\frac{14}{6}[\sigma_b(2,\mathcal{J}2)+\sigma_b(2,\mathcal{J}2)]}, \frac{\sigma_b(2,\mathcal{J}2)\sigma_b(2,\mathcal{J}2)+\sigma_b(2,\mathcal{J}2)\sigma_b(2,\mathcal{J}2)}{1+\sigma_b(2,\mathcal{J}2)+\sigma_b(2,\mathcal{J}2)}\Big\}\\ = \max\Big\{&\sigma_b(2,2), \sigma_b(2,1), \sigma_b(2,2), \frac{\sigma_b(2,1)+\sigma_b(2,1)}{\frac{28}{3}},\\ &\frac{\sigma_b(2,1)\sigma_b(2,1)+\sigma_b(2,1)\sigma_b(2,1)}{1+\frac{7}{3}[\sigma_b(2,1)+\sigma_b(2,1)]}, \frac{\sigma_b(2,1)\sigma_b(2,1)+\sigma_b(2,1)\sigma_b(2,1)}{1+\sigma_b(2,1)+\sigma_b(2,1)}\Big\}\\ = \max\Big\{&\frac{5}{6}, 4, \frac{5}{6}, 0.85, 1.6, 3.55\Big\} = 4.\end{aligned} \tag{4.4}$$

Therefore, by (4.3) and (4.4), inequality (3.1) reduces to

$$\psi(s\sigma_b(\mathcal{J}u,\mathcal{J}v)) \le \psi(\Theta(u,v)) - \varphi(\Theta(u,v)),$$

that is, $3.11 \le \psi(4) - \varphi(4) \le 6$, which is true.

Case III: For $u = 2$ and $v = 1$,

$$\psi(s\sigma_b(\mathcal{J}u,\mathcal{J}v)) = \psi\Big(\frac{14}{6}\sigma_b(\mathcal{J}0,\mathcal{J}2)\Big)\psi\Big(\Big(\frac{14}{6}\Big)\sigma_b(0,1)\Big) = \psi\Big(\frac{14}{18}\Big) = \frac{14}{9} = 1.55, \tag{4.5}$$

and

$$\begin{aligned}\Theta(2,1) = \max\Big\{&\sigma_b(2,1), \sigma_b(1,\mathcal{J}2), \sigma_b(2,\mathcal{J}2), \frac{\sigma_b(2,\mathcal{J}1)+\sigma_b(1,\mathcal{J}2)}{4\frac{14}{6}},\\ &\frac{\sigma_b(2,\mathcal{J}2)\sigma_b(2,\mathcal{J}1)+\sigma_b(1,\mathcal{J}1)\sigma_b(1,\mathcal{J}2)}{1+\frac{14}{6}[\sigma_b(2,\mathcal{J}2)+\sigma_b(1,\mathcal{J}1)]}, \frac{\sigma_b(2,\mathcal{J}2)\sigma_b(2,\mathcal{J}1)+\sigma_b(1,\mathcal{J}1)\sigma_b(1,\mathcal{J}2)}{1+\sigma_b(2,\mathcal{J}1)+\sigma_b(1,\mathcal{J}2)}\Big\}\end{aligned}$$

$$= \max\Big\{\sigma_b(2,1), \sigma_b(1,1), \sigma_b(2,1), \frac{\sigma_b(2,0)+\sigma_b(1,1)}{\frac{28}{3}},$$
$$\frac{\sigma_b(2,1)\sigma_b(2,0)+\sigma_b(1,0)\sigma_b(1,1)}{1+\frac{7}{3}[\sigma_b(0,0)+\sigma_b(2,1)]}, \frac{\sigma_b(2,1)\sigma_b(2,0)+\sigma_b(1,0)\sigma_b(1,1)}{1+\sigma_b(2,0)+\sigma_b(1,1)}\Big\}$$
$$= \max\Big\{4, \frac{2}{3}, 4, \frac{4}{21}, 2.27, 2.3\Big\} = 4. \tag{4.6}$$

Therefore, by (4.5) and (4.6), inequality (3.1) is satisfied, since

$$\psi(s\sigma_b(\mathcal{J}u, \mathcal{J}v)) \le \psi(\Theta(u,v)) - \varphi(\Theta(u,v))$$

reduces to $1.55 \le \psi(4) - \varphi(4) \le 6$.

Case IV: For $u = 2$ and $v = 0$, it clearly follows as in Case I.

This implies that (3.1) holds for all the cases, thus $\mathcal{J} \in \Lambda_s(\mathcal{X}, \alpha, \psi, \varphi)$. It is easy to see that the mapping $\mathcal{J}$ is not $\alpha$-admissible but weakly $\alpha$-admissible.

First, we show that $\mathcal{J}$ is not an $\alpha$-admissible mapping. Indeed, for $u = 1$, $v = 2$, we see that

$$\alpha(u,v) = \alpha(1,2) = \frac{1}{5} + \tanh(11) > 1$$

but

$$\alpha(\mathcal{J}u, \mathcal{J}v) = \alpha(\mathcal{J}1, \mathcal{J}2) = \alpha(0,1) = 0 < 1.$$

Next, we show that $\mathcal{J}$ is weakly $\alpha$-admissible. Suppose that $u \in X$ such that $\alpha(u, \mathcal{J}u) \ge 1$. Then $u, \mathcal{J}u \in [0,2]$ and

$$\alpha(u, \mathcal{J}u) = \alpha(2,1) = \frac{1}{5} + \tanh(19) > 1.$$

This implies that $\mathcal{J}\mathcal{J}u \in [0,2]$ and so $x = 2$. Now we obtain

$$\alpha(\mathcal{J}u, \mathcal{J}\mathcal{J}u) = \alpha(\mathcal{J}2, \mathcal{J}\mathcal{J}2) = \alpha(1,0) = \frac{1}{5} + \tanh(9) \ge 1.$$

Also, we can see that $\mathcal{J}$ is continuous and there is $u_0 = 1$ such that

$$\alpha(x_0, \mathcal{J}x_0) = \alpha(1, \mathcal{J}1) = \alpha(1,0) \ge 1.$$

From the definition of $\alpha$, it is clear that $\alpha$ has a transitive property.

Therefore, all the conditions of Theorem 3.2 are satisfied. Thus we can conclude that $\mathrm{Fix}(\mathcal{J}) \neq \emptyset$. In this example, it is easy to see that $0 \in \mathrm{Fix}(\mathcal{J})$.

It can be observed that inequality (3.1) is satisfied neither in metric $d(x,y) = |x-y|$, nor in metric-like $\sigma(x,y) = \max\{x,y\}$. □

The following example shows that Theorem 3.2 is not true in metric spaces and metric-like spaces. We show that the contraction condition (2.1) of Sintunavarat [20] is not suitable in $b$-metric like space $(X, \sigma_b)$ in this example.

**Example 4.2.** Let $\mathcal{X} = [0,\infty)$ and let $\sigma_b : \mathcal{X} \times \mathcal{X} \to [0,\infty)$ be defined by

$$\sigma_b(u,v) = \max\{u^2, v^2\} \quad \text{for all } u, v \in \mathcal{X}.$$

Clearly $(\mathcal{X}, \sigma_b)$ is a complete $b$-metric like space with constant $s = 4$. Let a mapping $\mathcal{J} : \mathcal{X} \to \mathcal{X}$ be given by

$$\mathcal{J}u = \begin{cases} \frac{3u}{8} & \text{if } u \in [0,4], \\ \ln(5u-2) & \text{if } u > 4. \end{cases}$$

Let $\alpha : \mathcal{X} \times \mathcal{X} \to [0,\infty)$ be given by

$$\alpha(u,v) = \begin{cases} \sinh^{-1}(3u+2v) - \frac{2}{3} & \text{if } u, v \in [0,3], \\ \ln(\frac{u+5v}{2}) & \text{otherwise.} \end{cases}$$

Now, using control functions $\psi(t) = t$ and $\varphi(t) = \frac{5}{16}t$, we have to prove that

(A1) $\mathcal{J} \in \Lambda_s(\mathcal{X}, \alpha, \psi, \varphi) \cap \mathcal{WA}(\mathcal{X}, \alpha)$;
(A2) there exists $x_0 \in \mathcal{X}$ such that $\alpha(x_0, \mathcal{J}x_0) \geq 1$;
(A3) $\alpha$ has the transitive property;
(A4) $\mathcal{J}$ is $\sigma_b$-continuous.

*Proof.* In order to prove that $\mathcal{J}: \mathcal{X} \to \mathcal{X}$ is a contractive mapping, without loss of generality suppose that $0 \leq v \leq u \leq 4$. Then

$$\psi(s\sigma_b(\mathcal{J}u, \mathcal{J}v)) \leq \psi\Big(4\max\Big\{\Big(\frac{3u}{8}\Big)^2, \Big(\frac{3v}{8}\Big)^2\Big\}\Big) \leq \psi\Big(4\frac{9u^2}{64}\Big) = \psi\Big(\frac{9u^2}{16}\Big) = \frac{9u^2}{16}$$

and

$$\Theta(u, v) = \max\Big\{u^2, u^2, v^2, \frac{u^2 + v^2}{8}, \frac{(u^2)(u^2) + (v^2)(v^2)}{1 + 2[u^2 + v^2]}, \frac{(u^2)(u^2) + (v^2)(v^2)}{1 + u^2 + v^2}\Big\} = u^2.$$

Hence,

$$\psi(s\sigma_b(\mathcal{J}u, \mathcal{J}v)) = \frac{9u^2}{16} \leq u^2 - \frac{5}{16}u^2 \leq \psi(u^2) - \varphi(u^2) = \psi(\Theta(u, v)) - \varphi(\Theta(u, v))$$

and inequality (3.1) is satisfied. Thus $\mathcal{J} \in \Lambda_s(X, \alpha, \psi, \varphi)$.

We will show now that $\mathcal{J} \in \mathcal{WA}(X, \alpha)$ and $\mathcal{J} \notin \mathcal{A}(X, \alpha)$. First we show that $\mathcal{J}$ is not an $\alpha$-admissible mapping. Indeed, for $u = 0$, $v = 2$, we have that

$$\alpha(u, v) = \alpha(0, 3) = \sinh^{-1}(6) - \frac{2}{3} > 1,$$

but

$$\alpha(\mathcal{J}u, \mathcal{J}v) = \alpha(\mathcal{J}0, \mathcal{J}3) = \alpha\Big(0, \frac{9}{8}\Big) = \sinh^{-1}\Big(\frac{18}{8}\Big) - \frac{2}{3} < 1.$$

Suppose now that $u \in \mathcal{X}$ is such that $\alpha(u, \mathcal{J}u) \geq 1$. Then $u, \mathcal{J}u \in [0, 4]$. This implies that $\mathcal{J}\mathcal{J}u \in [0, 4)$ and so $u = 3$. Now we obtain

$$\alpha(\mathcal{J}u, \mathcal{J}\mathcal{J}u) = \alpha(\mathcal{J}3, \mathcal{J}\mathcal{J}3) = \alpha\Big(\frac{9}{8}, \frac{27}{64}\Big) = \sinh^{-1}\Big(\frac{135}{32}\Big) - \frac{2}{3} > 1.$$

Hence, $\mathcal{J}$ is weakly $\alpha$-admissible.

Also, we can see that $\mathcal{J}$ is continuous and there is $u_0 = 1$ such that

$$\alpha(u_0, \mathcal{J}u_0) = \alpha(1, \mathcal{J}1) = \alpha\Big(1, \frac{3}{8}\Big) = \sinh^{-1}\Big(\frac{15}{4}\Big) - \frac{2}{3} \geq 1.$$

We can also see that $\alpha$ has a transitive property, for all $u, v, w \in [0, 4]$.

Therefore, all the conditions of Theorem 3.2 are satisfied. We conclude that $\mathrm{Fix}(\mathcal{J}) \neq \emptyset$. In this example, it is easy to see that $0 \in \mathrm{Fix}(\mathcal{J})$.

Finally, we show that in this case the contraction condition (2.1) of Sintunavarat [20] is not true in the $b$-metric like space $(X, \sigma_b)$. Indeed, in this space the mentioned condition takes the form

$$x, y \in \mathcal{X} \text{ with } \alpha(x, y) \geq 1 \;\Rightarrow\; \psi(s^3\sigma_b(\mathcal{J}x, \mathcal{J}y)) \leq \psi(M_s(x, y)) - \varphi(M_s(x, y)) + L\psi(N(x, y)), \tag{4.7}$$

where

$$M_s(x, y) = \Big\{\sigma_b(x, y), \sigma_b(x, \mathcal{J}x), \sigma_b(y, \mathcal{J}y), \frac{\sigma_b(x, \mathcal{J}y) + \sigma_b(y, \mathcal{J}x)}{2s}\Big\}$$

and

$$N(x, y) = \min\{\sigma_b(x, \mathcal{J}x), \sigma_b(y, \mathcal{J}x)\}.$$

Denote by $\mathcal{L}$ and $\mathcal{R}$, respectively, the left-hand and right-hand sides of condition (4.7). Take $0 \leq y < x \leq 4$, and $s = 4$, $L = 0$. Then

$$\mathcal{L} = \psi\Big(4^3\sigma_b\Big(\frac{3x}{8}, \frac{3y}{8}\Big)\Big) = 64\frac{9x^2}{64} = 9x^2$$

and

$$\mathcal{R} = \psi(M_s(x,y)) - \varphi(M_s(x,y)) + L\psi(N(x,y)),$$
$$M_s(x,y) = \max\left\{x^2, x^2, y^2, \frac{x^2+y^2}{4}\right\} = x^2,$$
$$N(x,y) = \max\{x^2, y^2\} = y^2.$$

Then we get

$$\mathcal{R} = \psi(x^2) - \varphi(x^2) + L\psi(y^2) \le x^2 - \frac{5}{16}x^2 \le \frac{11}{16}x^2.$$

Consequently, we have $\mathcal{L} \nleq \mathcal{R}$ and hence the contraction condition (4.7) is not true in the $b$-metric-like space $(X, \sigma_b)$.

It can again be observed that inequality (3.1) is satisfied neither in metric $d(x,y) = |x-y|$, nor in metric-like $\sigma(x,y) = \max\{x,y\}$. □

## 5 Application to fractional differential equations

This section is devoted to the existence of solutions for a nonlinear fractional differential equation as an application of Theorem 3.3. It is inspired by the paper [6].

Recall that the Caputo derivative of fractional order $\beta$ is defined by

$$^{c}D^{\beta}(g(t)) = \frac{1}{\Gamma(n-\beta)} \int_0^t (t-s)^{n-\beta-1} g^{(n)}(s)\,ds \quad (n-1 < \beta < n,\ n = [\beta]+1),$$

where $g : [0,\infty) \to \mathbb{R}$ is a continuous function, $[\beta]$ denotes the integer part of the positive real number $\beta$ and $\Gamma$ is the gamma function.

In addition, the Riemann–Liouville fractional derivative of order $\beta$ for a continuous function $g(t)$ is defined by

$$D^{\beta}(g(t)) = \frac{1}{\Gamma(n-\beta)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{g(s)}{(t-s)^{\beta-n-1}}\,ds \quad (n-1 < \beta < n,\ n = [\beta]+1),$$

provided the right-hand side is point-wise defined on $(0,\infty)$.

In what follows, we consider three different classes of nonlinear fractional differential equations and prove the existence of their positive solutions through assumptions using three forms of distance functions via Theorem 3.2.

**Class 1.** First we consider the nonlinear fractional differential equation of the form

$$^{c}D^{\beta}(x(t)) = f(t, x(t)) \quad (0 < t < 1,\ 1 < \beta \le 2) \tag{5.1}$$

with the integral boundary conditions

$$x(0) = 0, \quad x(1) = \int_0^{\eta} x(s)\,ds \quad (0 < \eta < 1),$$

where $x \in C([0,1], \mathbb{R})$, and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

To get the result, we consider $\mathcal{X} = C([0,1], \mathbb{R})$ endowed with the $b$-metric-like

$$\sigma_b(u,v) = \max_{t\in[0,1]} \left[|u(t)| + |v(t)|\right]^2,$$

with the constant $s = 2$.

**Theorem 5.1.** *Let $\mathcal{J}\colon \mathcal{X} \to \mathcal{X}$ be the operator defined by*

$$\mathcal{J}u(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{(\beta-1)} f(s, u(s))\, ds - \frac{2t}{(2-\eta^2)\Gamma(\beta)} \int_0^t (1-s)^{(\beta-1)} f(s, u(s))\, ds$$
$$+ \frac{2t}{(2-\eta^2)\Gamma(\beta)} \int_0^\eta \Big( \int_0^s (s-m)^{(\beta-1)} f(m, u(m))\, dm \Big)\, ds, \tag{5.2}$$

*where $t \in [0,1]$. Suppose the following assertions hold:*

(F1) *$f : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function, non-decreasing in the second variable;*
(F2) *there exists $x_0 \in \mathcal{X}$ such that $x_0(c) \le \mathcal{J}x_0(c)$ for all $c \in [0,1]$;*
(F3) *for each $t \in [0,1]$ and $x, y \in \mathcal{X}$ with $x(w) \le y(w)$ for all $w \in [0,1]$, we have*

$$[|f(t, x(t))| + |f(t, y(t))|]^2 \le \frac{1}{10\sqrt{2}} \Gamma(\beta+1) \Delta_1(x,y)(t),$$

*where*

$$\Delta_1(x,y)(t) = \max\Bigg\{ (|x(t)| + |y(t)|)^2, (|x(t)| + |\mathcal{J}x(t)|)^2, (|y(t)| + |\mathcal{J}y(t)|)^2,$$
$$\frac{(|x(t)| + |\mathcal{J}y(t)|)^2 + (|y(t)| + |\mathcal{J}x(t)|)^2}{8},$$
$$\frac{(|x(t)| + |\mathcal{J}x(t)|)^2 (|x(t)| + |\mathcal{J}y(t)|)^2 + (|y(t)| + |\mathcal{J}y(t)|)^2 (|y(t)| + |\mathcal{J}x(t)|)^2}{1 + 2[(|x(t)| + |\mathcal{J}x(t)|)^2 + (|y(t)| + |\mathcal{J}y(t)|)^2]},$$
$$\frac{(|x(t)| + |\mathcal{J}x(t)|)^2 (|x(t)| + |\mathcal{J}y(t)|)^2 + (|y(t)| + |\mathcal{J}y(t)|)^2 (|y(t)| + |\mathcal{J}x(t)|)^2}{1 + (|x(t)| + |\mathcal{J}y(t)|)^2 + (|x(t)| + |\mathcal{J}y(t)|)^2} \Bigg\}^{\frac{1}{2}}.$$

*Then equation (5.1) has at least one solution $u^* \in \mathcal{X}$.*

*Proof.* Define a function $\alpha : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ by

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x(c) \le y(c) \text{ for all } c \in I, \\ \eta & \text{otherwise,} \end{cases} \tag{5.3}$$

where $\eta \in (0,1)$. It is easy to see that $\alpha$ has a transitive property. Indeed, for all $x, y, z \in \mathcal{X}$:

$$\begin{aligned} \alpha(x,y) \ge 1,\ \alpha(y,z) \ge 1 &\Rightarrow x(c) \le y(c),\ y(c) \le z(c) \text{ for all } c \in I \\ &\Rightarrow x(c) \le z(c) \text{ for all } c \in I \\ &\Rightarrow \alpha(x,y) \ge 1. \end{aligned}$$

Since $\mathcal{J}$ is non-decreasing in the second variable, it follows that $\mathcal{J} \in \mathcal{A}(\mathcal{X}, \alpha)$. From (F2) and (5.3), we get $\alpha(x_0, \mathcal{J}x_0) \ge 1$. To prove condition (A4*) of Theorem 3.3, let $\{x_n\}$ be an increasing sequence in $\mathcal{X}$. Then by definition (5.3) of $\alpha$, we have $\alpha(x_n, x_{n+1}) \ge 1$ for all $n \in \mathbb{N}$. If $x_n \to x \in \mathcal{X}$ as $n \to \infty$, then as in the paper [14], we get $x_n(c) \le x(c)$ for any $c \in I$. Therefore by (5.3), $\alpha(x_n, x) \ge 1$ for all $n \in \mathbb{N}$. Thus condition (A4*) holds.

Now we have to check that $\mathcal{J} \in \Lambda_s(\mathcal{X}, \alpha, \psi, \varphi)$. For this, let $u, v \in \mathcal{X}$ be such that $\alpha(u,v) \ge 1$, that is, $u(t) \le v(t)$ for all $t \in I$. For all $t \in I$, by the conditions (F3) and (5.2), we have

$$|\mathcal{J}u(t)| + |\mathcal{J}v(t)| = \Bigg| \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{(\beta-1)} f(s, u(s))\, ds - \frac{2t}{(2-\eta^2)\Gamma(\beta)} \int_0^1 (1-s)^{(\beta-1)} f(s, u(s))\, ds$$
$$+ \frac{2t}{(2-\eta^2)\Gamma(\beta)} \int_0^\eta \Big( \int_0^s (s-m)^{(\beta-1)} f(m, u(m))\, dm \Big)\, ds \Bigg|$$

$$
\begin{aligned}
&+\left|\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{(\beta-1)}f(s,v(s))\,ds+\frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^1 (1-s)^{(\beta-1)}f(s,v(s))\,ds\right.\\
&\left.-\frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^\eta \bigg(\int_0^s (s-m)^{(\beta-1)}f(m,v(m))\,dm\bigg)\,ds\right|\\
\le{}& \frac{1}{\Gamma(\beta)}\int_0^t |t-s|^{2(\beta-1)}[|f(s,u(s))|+|f(s,v(s))|]\,ds\\
&+\frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^1 (1-s)^{2(\beta-1)}[|f(s,u(s))|+|f(s,v(s))|]\,ds\\
&+\frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^\eta\int_0^s (s-m)^{2(\beta-1)}\big||f(m,u(m))|+|f(m,v(m))|\,dm\big|\,ds\\
\le{}& \frac{1}{\Gamma(\beta)}\int_0^t |t-s|^{(\beta-1)}\frac{\Gamma(\beta+1)}{10}\Delta_1(u,v)(s)\,ds\\
&+\frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^1 (1-s)^{(\beta-1)}\frac{\Gamma(\beta+1)}{10}\Delta_1(u,v)(s)\,ds\\
&+\frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^\eta \bigg(\int_0^s |s-m|^{(\beta-1)}\frac{\Gamma(\beta+1)}{10}\Delta_1(m)\,dm\bigg)\,ds,\\
\le{}& \frac{1}{10}\Gamma(\beta+1)(\Theta(u,v))^{\frac12}\times\sup_{t\in(0,1)}\bigg(\frac{1}{\Gamma(\beta)}\int_0^t |t-s|^{(\beta-1)}\,ds\\
&+\frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^1 (1-s)^{(\beta-1)}\,ds+\frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^\eta\int_0^s |s-m|^{(\beta-1)}\,dm\,ds\bigg)\\
\le{}& \frac{1}{2\sqrt2}(\Theta(u,v))^{\frac12}
\end{aligned}
$$

which implies that

$$\sigma_b(\mathcal{J}(u),\mathcal{J}(v))\le\frac18\Theta(u,v),$$

where $\Theta(u,v)$ is given in (3.2).

Now, considering the control functions $\psi,\varphi:[0,+\infty)\to[0,+\infty)$ given by

$$\psi(t)=t,\quad \varphi(t)=\frac{t}{2}\quad\text{for } t\ge0,$$

we get

$$\psi(s\sigma(\mathcal{J}u,\mathcal{J}v))\le\psi(\Theta(u,v))-\varphi(\Theta(u,v)).$$

Thus $\mathcal{J}\in\Lambda_s(\mathcal{X},\alpha,\psi,\varphi)$. Therefore, by Theorem 3.3 we conclude that there is a fixed point $u^*\in\mathcal{X}$ of the operator $\mathcal{J}$ and $u^*$ is also a solution to the integral equation (5.2) and the fractional differential equation (5.1). □

**Class 2.** Secondly, we consider the nonlinear fractional differential equation of the form

$$D^\beta(x(t))+f(t,x(t))=0\quad(0\le t\le1,\ 1<\beta)\tag{5.4}$$

with the two-point boundary conditions

$$x(0)=0,\quad x(1)=0,$$

where $f:I=[0,1]\times\mathbb{R}\to\mathbb{R}$ is a continuous function.

This problem is equivalent to the integral equation

$$u(t) = \int_0^1 G(t,\zeta) f(\zeta, u(\zeta))\, d\zeta \quad \text{for all } t \in I,$$

where the corresponding Green function is given by

$$G(t,\zeta) = \begin{cases} (t(1-\zeta))^{\beta-1} - (t-\zeta)^{\beta-1} & \text{if } 0 \le \zeta < t \le 1, \\ \frac{(t(1-\zeta))^{\alpha-1}}{\Gamma(\beta)} & \text{if } 0 \le t \le \zeta \le 1. \end{cases}$$

Here we consider $\mathcal{X} = C([0,1], \mathbb{R})$ endowed with a different $b$-metric-like,

$$\sigma_b(u,v) = \max_{t\in I} [|u(t)| + |v(t)|]^p,$$

making $(\mathcal{X}, \sigma_b)$ a $b$-metric-like space with the constant $2^{p-1}$.

**Theorem 5.2.** *Let $\mathcal{J}\colon \mathcal{X} \to \mathcal{X}$ be the operator defined by*

$$\mathcal{J}u(t) = \int_0^1 G(t,\zeta) f(\zeta, u(\zeta))\, d\zeta, \tag{5.5}$$

*where $t \in [0,1]$. Suppose the following assertions hold:*

(F1) *$f : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function, non-decreasing in the second variable;*

(F2) *there exists $x_0 \in \mathcal{X}$ such that $x_0(c) \le \int_0^1 G(c,\zeta) f(s, x_0(\zeta))\, d\zeta$ for all $c \in I$;*

(F3) *there exists $p > 1$ such that the following condition holds: for all $t \in I$ and $x, y \in \mathcal{X}$ with $x(w) \le y(w)$ for all $w \in I$,*

$$|f(t,x(t))| + |f(t,y(t))| \le \Big[\frac{2}{\Gamma(\beta)}\Big(1 + \Big(1 - \frac{1}{\Gamma(\beta)}\Big)\frac{1}{4^\beta}\Big)\Big]^{-1} \Delta_2(x,y)(t),$$

*where*

$$\begin{aligned}\Delta_2(x,y)(t) = \Bigg(\frac{1}{2}\max\Bigg\{&(|x(t)| + |y(t)|)^p, (|x(t)| + |\mathcal{J}x(t)|)^p, (|y(t)| + |\mathcal{J}y(t)|)^p, \\ &\frac{(|x(t)| + |\mathcal{J}y(t)|)^p + (|y(t)| + |\mathcal{J}x(t)|)^p}{2^{p+1}}, \\ &\frac{(|x(t)| + |\mathcal{J}x(t)|)^p (|x(t)| + |\mathcal{J}y(t)|)^p + (|y(t)| + |\mathcal{J}y(t)|)^p (|y(t)| + |\mathcal{J}x(t)|)^p}{1 + 2^{p-1}[(|x(t)| + |\mathcal{J}x(t)|)^p + (|y(t)| + |\mathcal{J}y(t)|)^p]}, \\ &\frac{(|x(t)| + |\mathcal{J}x(t)|)^p (|x(t)| + |\mathcal{J}y(t)|)^p + (|y(t)| + |\mathcal{J}y(t)|)^p (|y(t)| + |\mathcal{J}x(t)|)^p}{1 + (|x(t)| + |\mathcal{J}y(t)|)^p + (|x(t)| + |\mathcal{J}y(t)|)^p}\Bigg\}\Bigg)^{\frac{1}{p}}.\end{aligned}$$

*Then there exists a fixed point $u^* \in \mathcal{X}$ of $\mathcal{J}$, that is, equation* (5.4) *has at least one solution $u^* \in \mathcal{X}$.*

*Proof.* We can define $\alpha$ on $\mathcal{X}$ and prove conditions (A1)–(A4*) as in Theorem 5.1 (see (5.3)).

Here we have only to check that $\mathcal{J} \in \Lambda_s(\mathcal{X}, \alpha, \psi, \varphi)$. For this, let $x, y \in \mathcal{X}$ be such that $\alpha(x,y) \ge 1$, that is, $x(c) \le y(c)$ for all $c \in I$. For all $c \in I$, by the conditions (F3) and (5.5), we have

$$\begin{aligned}|(\mathcal{J}x)(c)| + |(\mathcal{J}y)(c)| &= \Big|\int_0^1 G(c,\zeta) f(\zeta, x(\zeta))\, d\zeta\Big| + \Big|\int_0^1 G(c,\zeta) f(\zeta, y(\zeta))\, d\zeta\Big| \\ &\le \Big(\int_0^1 |G(c,\zeta)|\, d\zeta\Big)\Big(\int_0^1 [|f(\zeta, x(\zeta))| + |f(\zeta, y(\zeta))|]\, d\zeta\Big) \\ &\le \Big(\int_0^1 G(c,\zeta)\, d\zeta\Big)\Big(\int_0^1 \Big[\frac{2}{\Gamma(\beta)}\Big(1 + \Big(1 - \frac{1}{\Gamma(\beta)}\Big)\frac{1}{4^\beta}\Big)\Big]^{-1} \Delta_2(x,y)(\zeta)\, d\zeta\Big)\end{aligned}$$

$$
\begin{aligned}
&= \left( \int_0^c [(c(1-\zeta))^{\beta-1} - (c-\zeta)^{\beta-1}]\, d\zeta + \int_c^1 \frac{(c(1-\zeta))^{\alpha-1}}{\Gamma(\beta)}\, d\zeta \right) \\
&\quad \times \left[ \frac{1}{\Gamma(\beta)} \left( 1 + \left( 1 - \frac{1}{\Gamma(\beta)} \right) \frac{1}{4^\beta} \right) \right]^{-1} \frac{1}{2^{(p+1)/p}} \\
&\quad \times \max \Bigg\{ (|x(\zeta)| + |y(\zeta)|)^p, (|x(\zeta)| + |\mathcal{J}x(\zeta)|)^p, (|y(\zeta)| + |\mathcal{J}y(\zeta)|)^p, \\
&\qquad \frac{(|x(\zeta)| + |\mathcal{J}y(\zeta)|)^p + (|y(\zeta)| + |\mathcal{J}x(\zeta)|)^p}{2^{p+1}}, \\
&\qquad \frac{(|x(\zeta)| + |\mathcal{J}x(\zeta)|)^p (|x(\zeta)| + |\mathcal{J}y(\zeta)|)^p + (|y(\zeta)| + |\mathcal{J}y(\zeta)|)^p (|y(\zeta)| + |\mathcal{J}x(\zeta)|)^p}{1 + 2^{p-1}[(|x(\zeta)| + |\mathcal{J}x(\zeta)|)^p + (|y(\zeta)| + |\mathcal{J}y(\zeta)|)^p]}, \\
&\qquad \frac{(|x(\zeta)| + |\mathcal{J}x(\zeta)|)^p (|x(\zeta)| + |\mathcal{J}y(\zeta)|)^p + (|y(\zeta)| + |\mathcal{J}y(\zeta)|)^p (|y(\zeta)| + |\mathcal{J}x(\zeta)|)^p}{1 + (|x(\zeta)| + |\mathcal{J}y(\zeta)|)^p + (|x(\zeta)| + |\mathcal{J}y(\zeta)|)^p} \Bigg\}^{\frac{1}{p}} \\
&= \frac{1}{\Gamma(\beta)} \left( c^\beta + \frac{1}{\Gamma(\beta)} [c(1-c)]^\beta \right) \times \left[ \frac{1}{\Gamma(\beta)} \left( 1 + \left( 1 - \frac{1}{\Gamma(\beta)} \right) \frac{1}{4^\beta} \right) \right]^{-1} \frac{1}{2^{p+1}} \\
&\quad \times \max \Bigg\{ (|x(\zeta)| + |y(\zeta)|)^p, (|x(\zeta)| + |\mathcal{J}x(\zeta)|)^p, (|y(\zeta)| + |\mathcal{J}y(\zeta)|)^p, \\
&\qquad \frac{(|x(\zeta)| + |\mathcal{J}y(\zeta)|)^p + (|y(\zeta)| + |\mathcal{J}x(\zeta)|)^p}{2^{p+1}}, \\
&\qquad \frac{(|x(\zeta)| + |\mathcal{J}x(\zeta)|)^p (|x(\zeta)| + |\mathcal{J}y(\zeta)|)^p + (|y(\zeta)| + |\mathcal{J}y(\zeta)|)^p (|y(\zeta)| + |\mathcal{J}x(\zeta)|)^p}{1 + 2^{p-1}[(|x(\zeta)| + |\mathcal{J}x(\zeta)|)^p + (|y(\zeta)| + |\mathcal{J}y(\zeta)|)^p]}, \\
&\qquad \frac{(|x(\zeta)| + |\mathcal{J}x(\zeta)|)^p (|x(\zeta)| + |\mathcal{J}y(\zeta)|)^p + (|y(\zeta)| + |\mathcal{J}y(\zeta)|)^p (|y(\zeta)| + |\mathcal{J}x(\zeta)|)^p}{1 + (|x(\zeta)| + |\mathcal{J}y(\zeta)|)^p + (|x(\zeta)| + |\mathcal{J}y(\zeta)|)^p} \Bigg\}^{\frac{1}{p}}.
\end{aligned}
$$

This implies that

$$\sigma_b(\mathcal{J}(x), \mathcal{J}(y)) = \sup_{c \in I} (|(\mathcal{J}x)(c)| + |(\mathcal{J}y)(c)|)^p \leq \frac{1}{2^{p+1}} \Theta(x, y),$$

where $\Theta(x, y)$ is given in (3.2).

Now, considering the control functions $\psi, \varphi : [0, +\infty) \to [0, +\infty)$ given by

$$\psi(t) = t, \quad \varphi(t) = \frac{3t}{4} \quad \text{for } t \geq 0,$$

we get

$$\psi(2^{p-1} \sigma(\mathcal{J}x, \mathcal{J}y)) \leq \psi(\Theta(x, y)) - \varphi(\Theta(x, y)).$$

Thus $\mathcal{J} \in \Lambda_s(\mathcal{X}, \alpha, \psi, \varphi)$. Therefore, by Theorem 3.3 we conclude that there is a fixed point $u^* \in \mathcal{X}$ of the operator $\mathcal{J}$ and $u^*$ is also a solution to the integral equation (5.5) and the fractional differential equation (5.4). □

**Class 3.** Finally, we consider the nonlinear fractional differential equation of the form

$$D^\alpha(x(t)) + D^\beta(x(t)) = f(t, x(t)) \quad (0 \leq t \leq 1,\ 0 < \beta < \alpha < 1) \tag{5.6}$$

with the two-point boundary conditions

$$x(0) = 0, \quad x(1) = 0,$$

where $f : I = [0, 1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

This problem is equivalent to the integral equation

$$u(t) = \int_0^1 G(t - \zeta) f(\zeta, u(\zeta))\, d\zeta \quad \text{for all } t \in I,$$

where the corresponding Green function is given by

$$G(t,\zeta) = t^{\alpha-1}E_{\alpha-\beta}(-t^{\alpha-\beta}),$$

where $E_{\alpha-\beta}$ is the generalized Mittag-Leffler function.

Here we consider $\mathcal{X} = C([0,1],\mathbb{R})$ endowed with a different $b$-metric-like,

$$\sigma_b(u,v) = \max_{t\in I}[|u(t)| + |v(t)| + a]^p,$$

making $(\mathcal{X},\sigma_b)$ a $b$-metric-like space with the constant $2^{p-1}$ ($a \geq 0$ is fixed).

**Theorem 5.3.** *Let $\mathcal{J}\colon \mathcal{X}\to\mathcal{X}$ be the operator defined by*

$$\mathcal{J}u(t) = \int_0^1 G(t-\zeta)f(\zeta,u(\zeta))\,d\zeta, \tag{5.7}$$

*where $t\in[0,1]$. Suppose the following assertions hold:*

(F1) *$f: I\times\mathbb{R}\to\mathbb{R}$ is a continuous function, non-decreasing in the second variable;*

(F2) *there exists $x_0\in\mathcal{X}$ such that $x_0(c) \leq \int_0^1 G(c-\zeta)f(s,x_0(\zeta))\,d\zeta$ for all $c\in I$;*

(F3) *there exist $p>1$ and $a\geq 0$ satisfying the following condition: for all $t\in I$ and $x,y\in\mathcal{X}$ with $x(w)\leq y(w)$ for all $w\in I$, we have*

$$|f(t,x(t))| + |f(t,y(t))| \leq \alpha\Big[\frac{1}{2}\Delta_3(x,y)(t) - a\Big],$$

*where*

$$\begin{aligned}\Delta_3(x,y)(t) = \bigg(\frac{1}{2}\max\bigg\{&(|x(t)|+|y(t)|+a)^p, (|x(t)|+|\mathcal{J}x(t)|+a)^p, (|y(t)|+|\mathcal{J}y(t)|+a)^p,\\ &\frac{(|x(t)|+|\mathcal{J}y(t)|+a)^p + (|y(t)|+|\mathcal{J}x(t)|+a)^p}{2^{p+1}},\\ &\frac{(|x(t)|+|\mathcal{J}x(t)|+a)^p(|x(t)|+|\mathcal{J}y(t)|+a)^p + (|y(t)|+|\mathcal{J}y(t)|+a)^p(|y(t)|+|\mathcal{J}x(t)|+a)^p}{1+2^{p-1}[(|x(t)|+|\mathcal{J}x(t)|+a)^p + (|y(t)|+|\mathcal{J}y(t)|+a)^p]},\\ &\frac{(|x(t)|+|\mathcal{J}x(t)|+a)^p(|x(t)|+|\mathcal{J}y(t)|+a)^p + (|y(t)|+|\mathcal{J}y(t)|+a)^p(|y(t)|+|\mathcal{J}x(t)|+a)^p}{1+(|x(t)|+|\mathcal{J}y(t)|+a)^p + (|x(t)|+|\mathcal{J}y(t)|+a)^p}\bigg\}\bigg)^{\frac{1}{p}}.\end{aligned}$$

*Then there exists a point $u^*\in\mathcal{X}$ which satisfies* (5.7)*, that is, equation* (5.6) *has at least one solution $u^*\in\mathcal{X}$.*

*Proof.* We can again define $\alpha$ on $\mathcal{X}$ and prove conditions (A1)–(A4*) as in Theorem 5.1 (see (5.3)). We have again only to check that $\mathcal{J}\in\Lambda_s(\mathcal{X},\alpha,\psi,\varphi)$. For this, let $x,y\in\mathcal{X}$ be such that $\alpha(x,y)\geq 1$, that is, $x(c)\leq y(c)$ for all $c\in I$. For all $c\in I$, by the conditions (F3) and (5.7), we have

$$\begin{aligned}|(\mathcal{J}x)(c)| + |(\mathcal{J}y)(c)| &= \bigg|\int_0^1 G(c,\zeta)f(\zeta,x(\zeta))\,d\zeta\bigg| + \bigg|\int_0^1 G(c,\zeta)f(\zeta,y(\zeta))\,d\zeta\bigg|\\ &\leq \bigg(\int_0^1 |G(c,\zeta)|\,d\zeta\bigg)\bigg(\int_0^1 [|f(\zeta,x(\zeta))| + |f(\zeta,y(\zeta))|]\,d\zeta\bigg)\\ &\leq \bigg(\int_0^1 G(c,\zeta)\,d\zeta\bigg)\bigg(\int_0^1 \alpha\Big[\frac{1}{2}\Delta_3(x,y)(\zeta) - a\Big]\,d\zeta\bigg)\\ &\leq \sup_{c\in I}\bigg(\int_0^1 G(c,\zeta)\,d\zeta\bigg)\bigg(\int_0^1 \alpha\Big[\frac{1}{2}\Delta(x,y)(\zeta) - a\Big]\,d\zeta\bigg).\end{aligned}$$

Here it should be noted that

$$G(c) = c^{\alpha-1}E_{\alpha-\beta}(-c^{\alpha-\beta}) \leq c^{\alpha-1}\frac{1}{1} + |-c^{\alpha-\beta}| \leq c^{\alpha-1} \quad \text{for all } t\in I.$$

Thus

$$\sup_{c\in I}\int_0^1 |G(c,\zeta)|\,d\zeta \le \frac{1}{\alpha}.$$

Therefore

$$\sigma_b(\mathcal{J}(x),\mathcal{J}(y)) = \sup_{c\in I}\big(|(\mathcal{J}x)(c)| + |(\mathcal{J}y)(c)| + a\big)^p \le \Big[\frac{1}{\alpha}\times\alpha\times\frac{1}{2}\Theta(x,y)\Big]^p \le \frac{1}{2^{p+1}}\Theta(x,y),$$

where $\Theta(x,y)$ is given in (3.2).

Now, considering the control functions $\psi,\varphi:[0,+\infty)\to[0,+\infty)$ given by

$$\psi(t)=t,\quad \varphi(t)=\frac{3t}{4}\quad \text{for } t\ge 0,$$

we get

$$\psi(2^{p-1}\sigma(\mathcal{J}x,\mathcal{J}y)) \le \psi(\Theta(x,y)) - \varphi(\Theta(x,y)).$$

Thus, $\mathcal{J}\in\Lambda_s(\mathcal{X},\alpha,\psi,\varphi)$. Therefore, by Theorem 3.3 we conclude that there is a fixed point $u^*\in\mathcal{X}$ of the operator $\mathcal{J}$ and $u^*$ is also a solution to the integral equation (5.7) and the fractional differential equation (5.6). □

# 6 Some suggestions for further work

On the lines of our work, the following fractional $q$-difference boundary-value problems with $p$-Laplacian operator can also be discussed:

(i) $D_q^\gamma(\phi_p(D_q^\alpha u(t))) + f(t,u(t)) = 0$, $0<t<1$, $2<\alpha<3$,

$$u(0)=(D_q u)(0)=0,\quad (D_q u)(1)=\beta(D_q u)(\eta),$$

where $0<\gamma<1$, $2<\alpha<3$, $0<\beta\eta^{\alpha-2}<1$, $D_{0+}^\alpha$ is the Riemann–Liouville fractional derivative, and $\phi_p(s)=|s|^{p-2}s$, $p>1$.

(ii) $D_q^\gamma(\phi_p(D_q^\delta y(t))) + f(t,y(t)) = 0$, $0<t<1$, $0<\gamma<1$, $3<\delta<4$,

$$y(0)=(D_q y)(0)=(D_q^2 y)(0)=0,\quad a_1(D_q y)(1)+a_2(D_q^2 y)(1)=0,\quad a_1+|a_2|\ne 0,\quad D_{0^+}^\gamma y(t)|_{t=0}=0.$$

**Funding:** The first author is thankful to the United State-India Education Foundation, New Delhi, India and IIE/CIES, Washington, DC, USA for Fulbright-Nehru PDF Award (no. 2052/FNPDR/2015).

# References

[1] R. P. Agarwal, M. Benchohra and S. Hamani, A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions, *Acta Appl. Math.* **109** (2010), no. 3, 973–1033.
[2] M. A. Alghamdi, N. Hussain and P. Salimi, Fixed point and coupled fixed point theorems on $b$-metric-like spaces, *J. Inequal. Appl.* **2013** (2013), Paper No. 402.
[3] A. Amini-Harandi, Metric-like spaces, partial metric spaces and fixed points, *Fixed Point Theory Appl.* **2012** (2012), Paper No. 204.
[4] I. A. Bakhtin, The contraction mapping principle in almost metric space (in Russian), *Func. Anal. Gos. Ped. Inst. Unianowsk* **30** (1989), 26–37.
[5] D. Baleanu, R. P. Agarwal, H. Mohammadi and S. Rezapour, Some existence results for a nonlinear fractional differential equation on partially ordered Banach spaces, *Bound. Value Probl.* **2013** (2013), Paper No. 112.
[6] D. Baleanu, S. Rezapour and H. Mohammadi, Some existence results on nonlinear fractional differential equations, *Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **371** (2013), no. 1990, Article ID 20120144.
[7] S. Czerwik, Nonlinear set-valued contraction mappings in $b$-metric spaces, *Atti Sem. Mat. Fis. Univ. Modena* **46** (1998), no. 2, 263–276.

[8] P. Hitzler and A. Seda, *Mathematical Aspects of Logic Programming Semantics*, Chapman & Hall/CRC Stud. Inform. Ser., CRC Press, Boca Raton, 2011.
[9] N. Hussain, J. R. Roshan, V. Parvaneh and Z. Kadelburg, Fixed points of contractive mappings in $b$-metric-like spaces, *Sci. World J.* **2014** (2014), Article ID 471827.
[10] M. S. Khan, M. Swaleh and S. Sessa, Fixed point theorems by altering distances between the points, *Bull. Aust. Math. Soc.* **30** (1984), no. 1, 1–9.
[11] S. Liang and Y. Song, Existence and uniqueness of positive solutions to nonlinear fractional differential equation with integral boundary conditions, *Lith. Math. J.* **52** (2012), no. 1, 62–76.
[12] J. T. Machado, V. Kiryakova and F. Mainardi, Recent history of fractional calculus, *Commun. Nonlinear Sci. Numer. Simul.* **16** (2011), no. 3, 1140–1153.
[13] S. G. Matthews, Partial metric topology, in: *Papers on General Topology and Applications* (Flushing 1992), Ann. New York Acad. Sci. 728, New York Academy of Sciences, New York (1994), 183–197.
[14] J. J. Nieto and R. Rodríguez-López, Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations, *Acta Math. Sin. (Engl. Ser.)* **23** (2007), no. 12, 2205–2212.
[15] B. E. Rhoades, Some theorems on weakly contractive maps, *Nonlinear Anal.* **47** (2001), no. 4, 2683–2693.
[16] B. Samet, C. Vetro and P. Vetro, Fixed point theorems for $\alpha$-$\psi$-contractive type mappings, *Nonlinear Anal.* **75** (2012), no. 4, 2154–2165.
[17] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Integrals and Derivatives*, Gordon & Breach Science, Yverdon, 1993,
[18] S. Shukla, Partial b-metric spaces and fixed point theorems, *Mediterr. J. Math.* **11** (2014), no. 2, 703–711.
[19] W. Sintunavarat, A new approach to $\alpha$-$\psi$-contractive mappings and generalized Ulam–Hyers stability, well-posedness and limit shadowing results, *Carpathian J. Math.* **31** (2015), no. 3, 395–401.
[20] W. Sintunavarat, Fixed point results in $b$-metric spaces approach to the existence of a solution for nonlinear integral equations, *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM* **110** (2016), no. 2, 585–600.