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Random fixed point theorems for a finite family of asymptotically quasi-nonexpansive in the intermediate sense random operators

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Abstract. In this paper, we establish some strong convergence theorems of modified general composite implicit random iteration process to a common random fixed point for a finite family of asymptotically quasi-nonexpansive in the intermediate sense random operators.

Keywords. Asymptotically quasi-nonexpansive in the intermediate sense random operator, modified general composite implicit random iteration process, common random fixed point, strong convergence, separable uniformly convex Banach space.

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1 Introduction

Random approximations and random fixed point theorems are stochastic generalizations of classical approximations and fixed point theorems. The study of random fixed point theorems was initiated by Prague school of probabilities in the 1950's. The interest in these problems was enhanced after the publication of the survey article of Bharucha-Reid [5]. Random fixed point theory and applications have been further developed rapidly in recent years (see [2, 3, 8, 9, 14, 18] and references therein). In 2001, Xu and Ori [19] introduced implicit iteration process to approximate common fixed of a finite family of nonexpansive mappings. This process proved to be the main tool to approximate common fixed point of various class of mappings in deterministic operator theory. Zhao and Chang [20] studied convergence of modified implicit iteration process to common fixed point of a finite family of asymptotically nonexpansive mappings. Sun [17] proved a necessary and sufficient condition for convergence of implicit iteration process to a common fixed point of asymptotically quasi-nonexpansive mappings.

In 2007, Plubteing, Kumam and Wangkeeree [11] studied the implicit random iteration process with errors which converges strongly to a common fixed point of a finite family of asymptotically quasi-nonexpansive random operators on an unbounded set in uniformly convex Banach spaces. They gave a necessary and sufficient condition of the said scheme and mappings and also they proved some strong convergence theorems.

Recently Su and Li [15] and Su and Qin [16] respectively introduced the composite implicit iteration process and the general iteration algorithm which properly include the implicit iteration process.

Very recently, Beg and Thakur [4] introduced the modified general composite implicit random iteration process and they proved a necessary and sufficient condition for strong convergence of this iteration process to a common random fixed point of a finite family of random asymptotically nonexpansive mappings in separable Banach spaces. They also proved some strong convergence theorems for the said iteration scheme and mappings in uniformly separable convex Banach spaces.

The purpose of this paper is to propose the modified general composite implicit random iteration process and to give a necessary and sufficient condition for strong convergence of this iteration process to a common random fixed point of a finite family of asymptotically quasi-nonexpansive in the intermediate sense random operators in separable Banach spaces. We also prove some strong convergence theorems for the said iteration scheme and operators in uniformly separable convex Banach spaces. The results presented in this paper extend and improve the recent ones announced by Plubtieng, Kumam and Wangkeeree [11], Beg and Thakur [4] and some others.

2 Preliminaries

Let (Ω, Σ) be a measurable space (Σ is sigma algebra) and let C be a nonempty subset of a Banach space X. A mapping $\xi\colon \Omega \to X$ is measurable if $\xi^{-1}(U) \in \Sigma$, for each open subset U of X. The mapping $T\colon \Omega \times C \to C$ is a random map if and only if for each fixed $x \in C$, the mapping $T(\cdot, x)\colon \Omega \to C$ is measurable and it is continuous if for each $\omega \in \Omega$, the mapping $T(\omega, \cdot)\colon C \to X$ is continuous. A measurable mapping $\xi\colon \Omega \to X$ is a random fixed point of a random map $T\colon \Omega \times C \to X$ if and only if $T(\omega, \xi(\omega)) = \xi(\omega)$, for each $\omega \in \Omega$. We denote the set of random fixed points of a random map T by $\mathrm{RF}(T)$.

Let $B(x_0, r)$ denote the spherical ball centered at x_0 with radius r, defined as the set $\{x \in X : \|x - x_0\| \le r\}$. We denote the nth iterate

$$T(\omega, T(\omega, T(\omega, \dots, T(\omega, x) \dots)))$$

of T by $T^n(\omega, x)$. The letter I denotes the random mapping

$$I\colon \Omega \times C \to C$$

defined by $I(\omega, x) = x$ and $T^0 = I$.

Let C be a closed and convex subset of a separable Banach space X and the sequence of functions $\{\xi_n\}$ is pointwise convergent, that is, $\xi_n(\omega) \to q := \xi(\omega)$. Then the closedness of C implies that ξ is a mapping from Ω to C. Since C is a subset of separable Banach space X, if T is a continuous random operator then, by [1, Lemma 8.2.3], the mapping $\omega \to T(\omega, f(\omega))$ is a measurable function for any measurable function f from Ω to C. Thus $\{\xi_n\}$ is a sequence of measurable functions. Hence $\xi\colon \Omega \to C$, being the limit of the sequence of measurable functions, is also measurable [2, Remark 2.3].

Definition 2.1. Let $T\colon \Omega \times C \to C$ be a random operator, where C is a nonempty convex subset of a separable Banach space X.

(1) T is said to be asymptotically nonexpansive random operator if there exists a sequence of measurable mappings $h_n\colon \Omega \to [1, \infty)$ with $\lim_{n\to\infty} h_n(\omega) = 1$, for each $\omega \in \Omega$, such that for $x, y \in C$, we have

$$\|T^n(\omega, x) - T^n(\omega, y)\| \le h_n(\omega)\|x - y\|, \quad \text{for each } \omega \in \Omega. \tag{2.1}$$

(2) T is said to be asymptotically quasi-nonexpansive random operator if for each $\omega \in \Omega$ we have

$$G(\omega) = \{x \in C : x = T(\omega, x)\} \neq \emptyset$$

and there exists a sequence of measurable mappings $h_n\colon \Omega \to [1, \infty)$ with $\lim_{n\to\infty} h_n(\omega) = 1$, for each $\omega \in \Omega$, such that for $x \in C$ and $y \in G(\omega)$, the following inequality holds:

$$\|T^n(\omega, x) - y\| \le h_n(\omega)\|x - y\|, \quad \text{for each } \omega \in \Omega. \tag{2.2}$$

(3) T is said to be asymptotically quasi-nonexpansive in the intermediate sense random operator provided that T is uniformly continuous and

$$\limsup_{n\to\infty} \sup_{\substack{x\in C\\ y\in G(\omega)}} \big(\|T^n(\omega, x) - y\| - \|x - y\|\big) \le 0, \quad \text{for each } \omega \in \Omega, \tag{2.3}$$

where $G(\omega) = \{x \in C : x = T(\omega, x)\} \neq \emptyset$.

Definition 2.2. The modified random Mann iteration scheme is a sequence of function $\{\xi_n\}$ defined by

$$\xi_{n+1}(\omega) = (1 - \alpha_n)\xi_n(\omega) + \alpha_n T^n(\omega, \xi_n(\omega)), \quad \text{for each } \omega \in \Omega, \tag{2.4}$$

where $0 \le \alpha_n \le 1$, $n = 1, 2, \ldots$ and $\xi_0\colon \Omega \to C$ is an arbitrary measurable mapping.

Since C is a convex set, it follows that for each n, ξ_n is a mapping from Ω to C.

Definition 2.3. The modified random Ishikawa iteration scheme is the sequences of function $\{\xi_n\}$ and $\{\eta_n\}$ defined by

$$\begin{aligned} \xi_{n+1}(\omega) &= (1-\alpha_n)\xi_n(\omega) + \alpha_n T^n(\omega, \eta_n(\omega)), \\ \eta_n(\omega) &= (1-\beta_n)\xi_n(\omega) + \beta_n T^n(\omega, \xi_n(\omega)), \quad \text{for each } \omega \in \Omega, \end{aligned} \tag{2.5}$$

where $0 \le \alpha_n, \beta_n \le 1$, $n = 1, 2, \dots$ and $\xi_0 \colon \Omega \to C$ is an arbitrary measurable mapping. Also $\{\xi_n\}$ and $\{\eta_n\}$ are sequences of functions from Ω to C.

Definition 2.4. Let $\{T_1, T_2, \dots, T_N\}$ be a family of asymptotically quasi-nonexpansive in the intermediate sense random operators from $\Omega \times C \to C$, where C is a closed, convex subset of a separable Banach space X. Let

$$F = \bigcap_{i=1}^{N} \mathrm{RF}(T_i) \neq \emptyset,$$

where $\mathrm{RF}(T_i)$ is the set of all random fixed points of a random operator T_i for each $i \in \{1, 2, \dots, N\}$. Let $\xi_0 \colon \Omega \to C$ be any fixed measurable map, and $\{\alpha_n\} \subset [0, 1]$, then the sequence of function $\{\xi_n\}$ defined by

$$\begin{aligned} \xi_1(\omega) &= \alpha_1\xi_0(\omega) + (1-\alpha_1)T_1(\omega, \xi_1(\omega)), \\ \xi_2(\omega) &= \alpha_2\xi_1(\omega) + (1-\alpha_2)T_2(\omega, \xi_2(\omega)), \\ &\vdots \\ \xi_N(\omega) &= \alpha_N\xi_{N-1}(\omega) + (1-\alpha_N)T_N(\omega, \xi_N(\omega)), \\ \xi_{N+1}(\omega) &= \alpha_{N+1}\xi_N(\omega) + (1-\alpha_{N+1})T_1^2(\omega, \xi_{N+1}(\omega)), \\ &\vdots \\ \xi_{2N}(\omega) &= \alpha_{2N}\xi_{2N-1}(\omega) + (1-\alpha_{2N})T_N^2(\omega, \xi_{2N}(\omega)), \\ \xi_{2N+1}(\omega) &= \alpha_{2N+1}\xi_{2N}(\omega) + (1-\alpha_{2N+1})T_1^3(\omega, \xi_{2N+1}(\omega)), \\ &\vdots \end{aligned} \tag{2.6}$$

is called the modified implicit random iteration process for a finite family of asymptotically quasi-nonexpansive in the intermediate sense random operators $\{T_1, T_2, \dots, T_N\}$.

Since each $n \ge 1$ can be written as

$$n = (k-1)N + i,$$

where $i = i(n) \in \{1, 2, \dots, N\}$, $k = k(n) \ge 1$ is a positive integer and $k(n) \to \infty$ as $n \to \infty$. Hence the above iteration process can be written in the following compact form:

$$\xi_n(\omega) = \alpha_n\xi_{n-1}(\omega) + (1-\alpha_n)T_{i(n)}^{k(n)}(\omega, \xi_n(\omega)), \quad \text{for all } n \ge 1. \tag{2.7}$$

Now, we define a modified general composite implicit random iteration process.

Definition 2.5. Let $\{T_1, T_2, \ldots, T_N\}$ be a family of asymptotically quasi-nonexpansive in the intermediate sense random operators from $\Omega \times C \to C$, where C is a closed, convex subset of a separable Banach space X with

$$F = \bigcap_{i=1}^{N} \mathrm{RF}(T_i) \neq \emptyset.$$

Let $\xi_0 \colon \Omega \to C$ be any fixed measurable map. Then the sequence of function $\{\xi_n\}$ defined by

$$\begin{aligned}\xi_n(\omega) &= \alpha_n \xi_{n-1}(\omega) + (1-\alpha_n) T_{i(n)}^{k(n)}(\omega, \eta_n(\omega)), \quad \text{for all } n \geq 1, \\ \eta_n(\omega) &= a_n \xi_n(\omega) + b_n \xi_{n-1}(\omega) + c_n T_{i(n)}^{k(n)}(\omega, \xi_n(\omega)) \\ &\qquad + d_n T_{i(n)}^{k(n)}(\omega, \xi_{n-1}(\omega)), \quad \text{for all } n \geq 1,\end{aligned} \tag{2.8}$$

where $\{\alpha_n\}, \{a_n\}, \{b_n\}, \{c_n\}, \{d_n\} \subset [0, 1]$ such that $a_n + b_n + c_n + d_n = 1$, and $n = (k-1)N + i$, where $i = i(n) \in \{1, 2, \ldots, N\}$, $k = k(n) \geq 1$ is a positive integer and $k(n) \to \infty$ as $n \to \infty$.

Remark 2.6. By proper selection of the sequences $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ and $\{d_n\}$ it can be seen that the modified Mann iteration scheme, modified Ishikawa iteration scheme, and modified implicit iteration process can easily be obtained from (2.8).

In the sequel we need the following lemmas to prove our main results:

Lemma 2.7 (see [10]). *Let $\{a_n\}$, $\{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1 + \delta_n) a_n + b_n, \quad n \geq 1.$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists. In particular, if $\{a_n\}$ has a subsequence converging to zero, then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.8 (Schu [12]). *Let E be a uniformly convex Banach space and $0 < a \leq t_n \leq b < 1$ for all $n \geq 1$. Suppose that $\{x_n\}$ and $\{y_n\}$ are sequences in E satisfying*

$$\limsup_{n\to\infty} \|x_n\| \leq r, \quad \limsup_{n\to\infty} \|y_n\| \leq r, \quad \lim_{n\to\infty} \|t_n x_n + (1-t_n) y_n\| = r$$

for some $r \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

3 Main results

Theorem 3.1. *Let E be a real separable uniformly convex Banach space and C be a nonempty closed convex subset of E. Let $\{T_i\}_{i=1}^N$ be N uniformly L-Lipschitzian asymptotically quasi-nonexpansive in the intermediate sense random operators from $\Omega \times C \to C$. Let $F = \bigcap_{i=1}^N \mathrm{RF}(T_i) \neq \emptyset$. Let $\{\xi_n(\omega)\}$ be the sequence defined by* (2.8)*. Put*

$$A_n(\omega) = \max\Big\{ \sup_{\substack{\xi(\omega)\in F \\ n\geq 1}} \big(\|T_i^n(\omega, \xi_n(\omega)) - \xi(\omega)\| - \|\xi_n(\omega) - \xi(\omega)\|\big) \vee 0 : i \in I \Big\}. \tag{3.1}$$

Assume that

$$\sum_{n=1}^{\infty} A_n(\omega) < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} (1 - \alpha_n) < \infty$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ with $0 < a \leq \alpha_n \leq b < 1$ for some $a, b \in (0, 1)$. Then:

(a) $\lim_{n\to\infty} \|\xi_n(\omega) - \xi(\omega)\|$ *exists for all* $\omega \in \Omega$.

(b) $\lim_{n\to\infty} d(\xi_n(\omega), F)$ *exists, with* $d(\xi_n(\omega), F) = \inf_{\xi(\omega)\in F} \|\xi_n(\omega) - \xi(\omega)\|$.

(c) $\lim_{n\to\infty} \|\xi_n(\omega) - T_l(\omega, \xi_n(\omega))\| = 0$, *for all* $1 \leq l \leq N$.

Proof. Let $\xi(\omega) \in F$. Using (2.8) and (3.1), we have

$$\begin{aligned}
\|\eta_n(\omega) - \xi(\omega)\| &= \|a_n\xi_n(\omega) + b_n\xi_{n-1}(\omega) + c_n T_{i(n)}^{k(n)}(\omega, \xi_n(\omega)) \\
&\qquad + d_n T_{i(n)}^{k(n)}(\omega, \xi_{n-1}(\omega)) - \xi(\omega)\| \\
&\leq a_n\|\xi_n(\omega) - \xi(\omega)\| + b_n\|\xi_{n-1}(\omega) - \xi(\omega)\| \\
&\qquad + c_n\|T_{i(n)}^{k(n)}(\omega, \xi_n(\omega)) - \xi(\omega)\| \\
&\qquad + d_n\|T_{i(n)}^{k(n)}(\omega, \xi_{n-1}(\omega)) - \xi(\omega)\| \\
&\leq a_n\|\xi_n(\omega) - \xi(\omega)\| + b_n\|\xi_{n-1}(\omega) - \xi(\omega)\| \\
&\qquad + c_n(\|\xi_n(\omega) - \xi(\omega)\| + A_n(\omega)) \\
&\qquad + d_n(\|\xi_{n-1}(\omega) - \xi(\omega)\| + A_n(\omega)) \\
&\leq (a_n + c_n)\|\xi_n(\omega) - \xi(\omega)\| + (b_n + d_n)\|\xi_{n-1}(\omega) - \xi(\omega)\| \\
&\qquad + (c_n + d_n)A_n(\omega) \\
&\leq \|\xi_n(\omega) - \xi(\omega)\| + \|\xi_{n-1}(\omega) - \xi(\omega)\| \\
&\qquad + (c_n + d_n)A_n(\omega).
\end{aligned} \tag{3.2}$$

Using (2.8), (3.1) and (3.2), we have

$$\begin{aligned}
\|\xi_n(\omega) - \xi(\omega)\| &= \|\alpha_n \xi_{n-1}(\omega) + (1-\alpha_n) T_{i(n)}^{k(n)}(\omega, \eta_n(\omega)) - \xi(\omega)\| \\
&\le \alpha_n \|\xi_{n-1}(\omega) - \xi(\omega)\| + (1-\alpha_n) \|T_{i(n)}^{k(n)}(\omega, \eta_n(\omega)) - \xi(\omega)\| \\
&\le \alpha_n \|\xi_{n-1}(\omega) - \xi(\omega)\| + (1-\alpha_n)(\|\eta_n(\omega) - \xi(\omega)\| + A_n(\omega)) \\
&\le \alpha_n \|\xi_{n-1}(\omega) - \xi(\omega)\| \\
&\qquad + (1-\alpha_n)\big(\|\xi_n(\omega) - \xi(\omega)\| + \|\xi_{n-1}(\omega) - \xi(\omega)\| \\
&\qquad\qquad + (c_n + d_n) A_n(\omega)\big) \\
&\qquad + (1-\alpha_n) A_n(\omega) \\
&\le \|\xi_{n-1}(\omega) - \xi(\omega)\| + (1-\alpha_n)\|\xi_n(\omega) - \xi(\omega)\| \\
&\qquad + (1-\alpha_n)(1 + c_n + d_n) A_n(\omega) \\
&\le \|\xi_{n-1}(\omega) - \xi(\omega)\| + (1-\alpha_n)\|\xi_n(\omega) - \xi(\omega)\| \\
&\qquad + 2(1-\alpha_n) A_n(\omega) \qquad (3.3)
\end{aligned}$$

which on simplifying, we have

$$\begin{aligned}
\|\xi_n(\omega) - \xi(\omega)\| &\le \frac{1}{\alpha_n} \|\xi_{n-1}(\omega) - \xi(\omega)\| + \frac{2(1-\alpha_n)}{\alpha_n} A_n(\omega) \\
&= \left(1 + \frac{1-\alpha_n}{\alpha_n}\right) \|\xi_{n-1}(\omega) - \xi(\omega)\| + \frac{2(1-\alpha_n)}{\alpha_n} A_n(\omega) \\
&\le \left(1 + \frac{1-\alpha_n}{a}\right) \|\xi_{n-1}(\omega) - \xi(\omega)\| + \frac{2(1-a)}{a} A_n(\omega) \\
&= (1 + B_n) \|\xi_{n-1}(\omega) - \xi(\omega)\| + \frac{2(1-a)}{a} A_n(\omega), \qquad (3.4)
\end{aligned}$$

where $B_n = (1-\alpha_n)/a$. Since by hypothesis

$$\sum_{n=1}^{\infty} (1-\alpha_n) < \infty,$$

it follows that

$$\sum_{n=1}^{\infty} B_n < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} A_n(\omega) < \infty.$$

Taking infimum over all $\xi(\omega) \in F$, we have

$$d(\xi_n(\omega), F) \le (1 + B_n) d(\xi_{n-1}(\omega), F) + \frac{2(1-a)}{a} A_n(\omega). \qquad (3.5)$$

Lemma 2.7implies that $\lim_{n\to\infty}\|\xi_n(\omega)-\xi(\omega)\|$ and $\lim_{n\to\infty} d(\xi_n(\omega), F)$ exist. Without loss of generality, we can assume that

$$\lim_{n\to\infty}\|\xi_n(\omega)-\xi(\omega)\| = d, \tag{3.6}$$

where $d \geq 0$ is some number. Since $\{\|\xi_n(\omega)-\xi(\omega)\|\}$ is a convergent sequence, so $\{\xi_n(\omega)\}$ is a bounded sequence in C.

It follows from (3.2) and (3.6) that

$$\lim_{n\to\infty}\|\eta_n(\omega)-\xi(\omega)\| \leq d,$$

which further gives that

$$\begin{aligned}\limsup_{n\to\infty}\|T_{i(n)}^{k(n)}(\omega,\eta_n(\omega))-\xi(\omega)\| &\leq \limsup_{n\to\infty}\big(\|\eta_n(\omega)-\xi(\omega)\|+A_n(\omega)\big)\\ &\leq d.\end{aligned} \tag{3.7}$$

Also from (2.8), we have

$$\begin{aligned}&\lim_{n\to\infty}\|\alpha_n(\xi_{n-1}(\omega)-\xi(\omega))+(1-\alpha_n)(T_{i(n)}^{k(n)}(\omega,\eta_n(\omega))-\xi(\omega))\|\\ &\quad=\lim_{n\to\infty}\|\xi_n(\omega)-\xi(\omega)\|=d.\end{aligned} \tag{3.8}$$

Lemma 2.8 and (3.6)–(3.8) imply that

$$\lim_{n\to\infty}\|T_{i(n)}^{k(n)}(\omega,\eta_n(\omega))-\xi_{n-1}(\omega)\|=0. \tag{3.9}$$

Again from (2.8) and (3.9), we have

$$\begin{aligned}\lim_{n\to\infty}\|\xi_n(\omega)-\xi_{n-1}(\omega)\| &\leq \lim_{n\to\infty}(1-\alpha_n)\|T_{i(n)}^{k(n)}(\omega,\eta_n(\omega))-\xi_{n-1}(\omega)\|\\ &=0,\end{aligned} \tag{3.10}$$

and thus

$$\lim_{n\to\infty}\|\xi_n(\omega)-\xi_{n+j}(\omega)\|=0, \quad \text{for all } j=1,2,\ldots,N. \tag{3.11}$$

On the other hand, from (3.9) and (3.10), we have

$$\begin{aligned}\lim_{n\to\infty}\|\xi_n(\omega)-T_{i(n)}^{k(n)}(\omega,\eta_n(\omega))\| &\leq \lim_{n\to\infty}\Big[\|\xi_n(\omega)-\xi_{n-1}(\omega)\|\\ &\qquad+\|\xi_{n-1}(\omega)-T_{i(n)}^{k(n)}(\omega,\eta_n(\omega))\|\Big]\\ &=0.\end{aligned} \tag{3.12}$$

Now,

$$
\begin{aligned}
&\|T_{i(n)}^{k(n)}(\omega, \xi_n(\omega)) - \xi_n(\omega)\| \\
&\quad \le \|\xi_n(\omega) - \xi_{n-1}(\omega)\| + \|T_{i(n)}^{k(n)}(\omega, \eta_n(\omega)) - \xi_{n-1}(\omega)\| \\
&\qquad + \|T_{i(n)}^{k(n)}(\omega, \eta_n(\omega)) - T_{i(n)}^{k(n)}(\omega, \xi_n(\omega))\| \\
&\quad \le \|\xi_n(\omega) - \xi_{n-1}(\omega)\| + \|T_{i(n)}^{k(n)}(\omega, \eta_n(\omega)) - \xi_{n-1}(\omega)\| \\
&\qquad + \big(\|\eta_n(\omega) - \xi_n(\omega)\| + A_n(\omega)\big) \\
&\quad \le \|\xi_n(\omega) - \xi_{n-1}(\omega)\| + \|T_{i(n)}^{k(n)}(\omega, \eta_n(\omega)) - \xi_{n-1}(\omega)\| \\
&\qquad + \big(\|\eta_n(\omega) - \xi_{n-1}(\omega)\| + \|\xi_n(\omega) - \xi_{n-1}(\omega)\|\big) + A_n(\omega) \\
&\quad = 2\|\xi_n(\omega) - \xi_{n-1}(\omega)\| + \|T_{i(n)}^{k(n)}(\omega, \eta_n(\omega)) - \xi_{n-1}(\omega)\| \\
&\qquad + \|\eta_n(\omega) - \xi_{n-1}(\omega)\| + A_n(\omega).
\end{aligned} \tag{3.13}
$$

Again, by using (2.8), we have

$$
\begin{aligned}
&\|\eta_n(\omega) - \xi_{n-1}(\omega)\| \\
&\quad = \|a_n\xi_n(\omega) + b_n\xi_{n-1}(\omega) + c_n T_{i(n)}^{k(n)}(\omega, \xi_n(\omega)) \\
&\qquad + d_n T_{i(n)}^{k(n)}(\omega, \xi_{n-1}(\omega) - \xi_{n-1}(\omega)\| \\
&\quad = \|c_n\big(T_{i(n)}^{k(n)}(\omega, \xi_n(\omega)) - \xi_n(\omega)\big) + d_n\big(T_{i(n)}^{k(n)}(\omega, \xi_{n-1}(\omega)) - \xi_n(\omega)\big) \\
&\qquad + (a_n + c_n + d_n)(\xi_n(\omega) - \xi_{n-1}(\omega))\| \\
&\quad \le c_n\|T_{i(n)}^{k(n)}(\omega, \xi_n(\omega)) - \xi_n(\omega)\| + d_n\|T_{i(n)}^{k(n)}(\omega, \xi_{n-1}(\omega)) - \xi_n(\omega)\| \\
&\qquad + (a_n + c_n + d_n)\|\xi_n(\omega) - \xi_{n-1}(\omega)\| \\
&\quad \le c_n\|T_{i(n)}^{k(n)}(\omega, \xi_n(\omega)) - \xi_n(\omega)\| \\
&\qquad + d_n\Big[\|T_{i(n)}^{k(n)}(\omega, \xi_{n-1}(\omega) - T_{i(n)}^{k(n)}(\omega, \xi_n(\omega))\| \\
&\qquad\qquad + \|T_{i(n)}^{k(n)}(\omega, \xi_n(\omega)) - \xi_n(\omega)\|\Big] \\
&\qquad + (a_n + c_n + d_n)\|\xi_n(\omega) - \xi_{n-1}(\omega)\| \\
&\quad \le (c_n + d_n)\|T_{i(n)}^{k(n)}(\omega, \xi_n(\omega)) - \xi_n(\omega)\| + d_n\big(\|\xi_{n-1}(\omega) - \xi_n(\omega)\| + A_n(\omega)\big) \\
&\qquad + (a_n + c_n + d_n)\|\xi_n(\omega) - \xi_{n-1}(\omega)\| \\
&\quad \le (c_n + d_n)\|T_{i(n)}^{k(n)}(\omega, \xi_n(\omega)) - \xi_n(\omega)\| + (1 - b_n + d_n)\|\xi_n(\omega) - \xi_{n-1}(\omega)\| \\
&\qquad + d_n A_n(\omega).
\end{aligned} \tag{3.14}
$$

Substituting (3.14) into (3.13), we get

$$
\begin{aligned}
&\|T_{i(n)}^{k(n)}(\omega, \xi_n(\omega) - \xi_n(\omega)\| \\
&\quad \le 2\|\xi_n(\omega) - \xi_{n-1}(\omega)\| + \|T_{i(n)}^{k(n)}(\omega, \eta_n(\omega)) - \xi_{n-1}(\omega)\| \\
&\qquad + (c_n + d_n)\|T_{i(n)}^{k(n)}(\omega, \xi_n(\omega)) - \xi_n(\omega)\| \\
&\qquad + (1 - b_n + d_n)\|\xi_n(\omega) - \xi_{n-1}(\omega)\| + A_n(\omega) + d_n A_n(\omega) \\
&\quad \le (3 - b_n + d_n)\|\xi_n(\omega) - \xi_{n-1}(\omega)\| + \|T_{i(n)}^{k(n)}(\omega, \eta_n(\omega)) - \xi_{n-1}(\omega)\| \\
&\qquad + (c_n + d_n)\|T_{i(n)}^{k(n)}(\omega, \xi_n(\omega)) - \xi_n(\omega)\| + (1 + d_n)A_n(\omega). \\
&\quad \le (3 + d_n)\|\xi_n(\omega) - \xi_{n-1}(\omega)\| + \|T_{i(n)}^{k(n)}(\omega, \eta_n(\omega)) - \xi_{n-1}(\omega)\| \\
&\qquad + (c_n + d_n)\|T_{i(n)}^{k(n)}(\omega, \xi_n(\omega)) - \xi_n(\omega)\| + (1 + d_n)A_n(\omega). \qquad (3.15)
\end{aligned}
$$

Since $c_n + d_n \le \tau < 1$, the above inequality becomes

$$
\begin{aligned}
(1 - \tau)\|T_{i(n)}^{k(n)}(\omega, \xi_n(\omega)) - \xi_n(\omega)\| \le {} & (3 + d_n)\|\xi_n(\omega) - \xi_{n-1}(\omega)\| \\
& + \|T_{i(n)}^{k(n)}(\omega, \eta_n(\omega)) - \xi_{n-1}(\omega)\| \\
& + (1 + d_n)A_n(\omega). \qquad (3.16)
\end{aligned}
$$

From (3.9), (3.10) and (3.16), we have

$$
\lim_{n\to\infty} \|T_{i(n)}^{k(n)}(\omega, \xi_n(\omega)) - \xi_n(\omega)\| = 0. \qquad (3.17)
$$

Since $\{T_l : 1 \le l \le N\}$ is uniformly L-Lipschitzian and since any positive integer $n > N$ can be written as $n = (k(n) - 1)N + i(n)$, $i(n) \in \{1, 2, \dots, N\}$, we have

$$
\begin{aligned}
\|\xi_{n-1}(\omega) - T_n(\omega, \xi_n(\omega))\| &\le \|\xi_{n-1}(\omega) - T_{i(n)}^{k(n)}(\omega, \eta_n(\omega))\| \\
&\quad + \|T_{i(n)}^{k(n)}(\omega, \eta_n(\omega)) - T_n(\omega, \xi_n(\omega))\| \\
&\le \sigma_n + L\|T_{i(n)}^{k(n)-1}(\omega, \eta_n(\omega)) - \xi_n(\omega)\| \\
&\le \sigma_n + L\Big[\|T_{i(n)}^{k(n)-1}(\omega, \eta_n(\omega)) - \xi_{n-1}(\omega)\| \\
&\qquad + \|\xi_{n-1}(\omega) - \xi_n(\omega)\|\Big] \qquad (3.18)
\end{aligned}
$$

where

$$
\sigma_n = \|\xi_{n-1}(\omega) - T_{i(n)}^{k(n)}(\omega, \eta_n(\omega))\|.
$$

From (3.9) we have $\sigma_n \to 0$ as $n \to \infty$. Again

$$
\begin{aligned}
&\|T_{i(n)}^{k(n)-1}(\omega, \eta_n(\omega)) - \xi_{n-1}(\omega)\| \\
&\quad \le \|T_{i(n)}^{k(n)-1}(\omega, \eta_n(\omega)) - T_{i(n-N)}^{k(n)-1}(\omega, \xi_{n-N}(\omega))\| \\
&\qquad + \|T_{i(n-N)}^{k(n)-1}(\omega, \xi_{n-N}(\omega)) - T_{i(n-N)}^{k(n)-1}(\omega, \eta_{n-N}(\omega))\| \\
&\qquad + \|T_{i(n-N)}^{k(n)-1}(\omega, \eta_{n-N}(\omega)) - \xi_{(n-N)-1}(\omega)\| \\
&\qquad + \|\xi_{(n-N)-1}(\omega) - \xi_n(\omega)\|. \qquad (3.19)
\end{aligned}
$$

Since for each $n > N$, $n = (n - N) \pmod N$, and $n = (k(n) - 1) + i(n)$, we have

$$
n - N = (k(n-1) - 1)N + i(n) = (k(n-N) - 1)N + i(n-N),
$$

that is, $k(n - N) = k(n) - 1$ and $i(n - N) = i(n)$. Therefore from (3.19), we have

$$
\begin{aligned}
&\|T_{i(n)}^{k(n)-1}(\omega, \eta_n(\omega)) - \xi_{n-1}(\omega)\| \\
&\quad \le \|T_{i(n)}^{k(n)-1}(\omega, \eta_n(\omega)) - T_{i(n)}^{k(n)-1}(\omega, \xi_{n-N}(\omega))\| \\
&\qquad + \|T_{i(n-N)}^{k(n)-1}(\omega, \xi_{n-N}(\omega)) - T_{i(n-N)}^{k(n)-1}(\omega, \eta_{n-N}(\omega))\| \\
&\qquad + \|T_{i(n-N)}^{k(n)-1}(\omega, \eta_{n-N}(\omega)) - \xi_{(n-N)-1}(\omega)\| \\
&\qquad + \|\xi_{(n-N)-1}(\omega) - \xi_n(\omega)\| \\
&\quad \le L\|\eta_n(\omega) - \xi_{n-N}(\omega)\| + L\|\xi_{n-N}(\omega) - \eta_{n-N}(\omega)\| \\
&\qquad + \sigma_{n-N} + \|\xi_{(n-N)-1}(\omega) - \xi_n(\omega)\|. \qquad (3.20)
\end{aligned}
$$

By (3.18), (3.19) and (3.20), we have

$$
\begin{aligned}
&\|\xi_{n-1}(\omega) - T_n(\omega, \xi_n(\omega))\| \\
&\quad \le \sigma_n + L^2\Big(\|\eta_n(\omega) - \xi_{n-N}(\omega)\| + \|\xi_{n-N}(\omega) - \eta_{n-N}(\omega)\|\Big) \\
&\qquad + L(\sigma_{n-N} + \|\xi_{(n-N)-1}(\omega) - \xi_n(\omega)\| + \|\xi_n(\omega) - \xi_{n-1}(\omega)\|) \\
&\quad \le \sigma_n + L^2\Big(\|\eta_n(\omega) - \xi_n(\omega)\| + \|\xi_n(\omega) - \xi_{n-N}(\omega)\| \\
&\qquad + \|\xi_{n-N}(\omega) - \eta_{n-N}(\omega)\|\Big) \\
&\qquad + L\Big(\sigma_{n-N} + \|\xi_{(n-N)-1}(\omega) - \xi_n(\omega)\| + \|\xi_n(\omega) - \xi_{n-1}(\omega)\|\Big) \\
&\quad \to 0 \quad \text{as } n \to \infty. \qquad (3.21)
\end{aligned}
$$

It follows from (3.11) and (3.21) that

$$\lim_{n\to\infty} \|\xi_n(\omega) - T_n(\omega, \xi_n(\omega))\| \\ \le \lim_{n\to\infty} [\|\xi_n(\omega) - \xi_{n-1}(\omega)\| + \|\xi_{n-1}(\omega) - T_n(\omega, \xi_n(\omega))\|] = 0. \quad (3.22)$$

Consequently, for any $l = 1, 2, \dots, N$, from (3.11) and (3.21), we have

$$\begin{aligned} \lim_{n\to\infty} &\|\xi_n(\omega) - T_{n+l}(\omega, \xi_n(\omega))\| \\ &\le \|\xi_n(\omega) - \xi_{n-1}(\omega)\| + \|\xi_{n+l}(\omega) - T_{n+l}(\omega, \xi_{n+l}(\omega))\| \\ &\qquad + \|T_{n+l}(\omega, \xi_{n+l}(\omega)) - T_{n+l}(\omega, \xi_n(\omega))\| \\ &\le (1+L)\|\xi_n(\omega) - \xi_{n+l}(\omega)\| + \|\xi_{n+l}(\omega) - T_{n+l}(\omega, \xi_{n+l}(\omega))\| \\ &\to 0 \quad \text{as } n \to \infty, \end{aligned} \quad (3.23)$$

which implies that

$$\lim_{n\to\infty} \|\xi_n(\omega) - T_{n+l}(\omega, \xi_n(\omega))\| = 0 \quad \text{for all } l = 1, 2, \dots, N.$$

Since for each $l = 1, 2, \dots, N$,

$$\left\{\|\xi_n(\omega) - T_l(\omega, \xi_n(\omega))\|\right\}_{n=1}^{\infty} \subset \bigcup_{j=1}^{N} \{\|\xi_n(\omega) - T_{n+j}(\omega, \xi_n(\omega))\|\}_{n=1}^{\infty},$$

we have

$$\lim_{n\to\infty} \|\xi_n(\omega) - T_l(\omega, \xi_n(\omega))\| = 0, \quad (3.24)$$

for all $l = 1, 2, \dots, N$. This completes the proof. □

Next, we prove necessary and sufficient conditions for the strong convergence of the general composite implicit random iteration process to a common random fixed point of a finite family of asymptotically quasi-nonexpansive in the intermediate sense random operators.

Theorem 3.2. *Let E be a real separable uniformly convex Banach space and C be a nonempty closed convex subset of E. Let $\{T_i\}_{i=1}^N$ be N asymptotically quasi-nonexpansive in the intermediate sense random operators from $\Omega \times C \to C$. Let*

$$F = \bigcap_{i=1}^{N} \mathrm{RF}(T_i) \neq \emptyset.$$

Let $\{\xi_n(\omega)\}$ be the sequence defined by (2.8). *Put*

$$A_n(\omega) = \max\Big\{ \sup_{\substack{\xi(\omega)\in F \\ n\ge 1}} \big(\|T_i^n(\omega, \xi_n(\omega)) - \xi(\omega)\| - \|\xi_n(\omega) - \xi(\omega)\|\big) \vee 0 : i \in I\Big\}.$$

Assume that

$$\sum_{n=1}^{\infty} A_n(\omega) < \infty \quad \textit{and} \quad \sum_{n=1}^{\infty}(1-\alpha_n) < \infty,$$

where $\{\alpha_n\}$ *is a sequence in* $(0, 1)$ *with* $0 < a \le \alpha_n \le b < 1$ *for some* $a, b \in (0, 1)$. *Then the sequence* $\{\xi_n(\omega)\}$ *converges to a common random fixed point of random operators* $\{T_i : i = 1, 2, \ldots, N\}$ *if and only if* $\liminf_{n\to\infty} d(\xi_n(\omega), F) = 0$.

Proof. If for some $\xi(\omega) \in F$, $\lim_{n\to\infty} \|\xi_n(\omega) - \xi(\omega)\| = 0$ for each $\omega \in \Omega$, then obviously

$$\liminf_{n\to\infty} d(\xi_n(\omega), F) = 0.$$

Conversely, suppose that $\liminf_{n\to\infty} d(\xi_n(\omega), F) = 0$. Then we have

$$\lim_{n\to\infty} d(\xi_n(\omega), F) = 0, \quad \text{for each } \omega \in \Omega.$$

Thus for any $\varepsilon > 0$ there exists a positive integer N_1 such that for $n \ge N_1$,

$$d(\xi_n(\omega), F) < \frac{\varepsilon}{6}, \quad \text{for each } \omega \in \Omega. \tag{3.25}$$

Since $\lim_{n\to\infty} \|\xi_n(\omega) - \xi(\omega)\|$ exists for all $\xi(\omega) \in F$, we have

$$\|\xi_n(\omega) - \xi(\omega)\| < K, \tag{3.26}$$

for all $n \ge 1$ and some positive number K.

Again since

$$\sum_{n=1}^{\infty} B_n < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} A_n(\omega) < \infty,$$

there exist positive integers N_2 and N_3 such that

$$\sum_{j=n}^{\infty} B_j < \frac{\varepsilon}{6K}, \quad \text{for all } n \ge N_2, \tag{3.27}$$

and

$$\sum_{j=n}^{\infty} A_j(\omega) < \frac{a\varepsilon}{12(1-a)}, \quad \text{for all } n \ge N_3. \tag{3.28}$$

Let $N = \max\{N_1, N_2, N_3\}$. It follows from (3.4), that

$$\|\xi_n(\omega) - \xi(\omega)\| \le \|\xi_{n-1}(\omega) - \xi(\omega)\| + KB_n + \frac{2(1-a)}{a} A_n(\omega). \tag{3.29}$$

Now, for each $m, n \geq N$ and each $\omega \in \Omega$, we have

$$\begin{aligned}
\|\xi_n(\omega) - \xi_m(\omega) &\leq \|\xi_n(\omega) - \xi(\omega)\| + \|\xi_m(\omega) - \xi(\omega)\| \\
&\leq \|\xi_N(\omega) - \xi(\omega)\| + K \sum_{j=N+1}^{n} B_j + \frac{2(1-a)}{a} \sum_{j=N+1}^{n} A_j(\omega) \\
&\quad + \|\xi_N(\omega) - \xi(\omega)\| + K \sum_{j=N+1}^{n} B_j \\
&\quad + \frac{2(1-a)}{a} \sum_{j=N+1}^{n} A_j(\omega) \\
&= 2\|\xi_N(\omega) - \xi(\omega)\| + 2K \sum_{j=N+1}^{n} B_j \\
&\quad + \frac{4(1-a)}{a} \sum_{j=N+1}^{n} A_j(\omega) \\
&< 2.\frac{\varepsilon}{6} + 2K.\frac{\varepsilon}{6K} + \frac{4(1-a)}{a}.\frac{a\varepsilon}{12(1-a)} < \varepsilon. \qquad (3.30)
\end{aligned}$$

This implies that $\{\xi_n(\omega)\}$ is a Cauchy sequence for each $\omega \in \Omega$. Therefore we get $\xi_n(\omega) \to p(\omega)$ for each $\omega \in \Omega$, and $p: \Omega \to C$, being the limit of the sequence of measurable function, is also measurable. Now, $\lim_{n\to\infty} d(\xi_n(\omega), F) = 0$ for each $\omega \in \Omega$, and the set F is closed, we have $p(\omega) \in F$, that is, $p(\omega)$ is a common random fixed point of the random operators $\{T_i : i = 1, 2, \ldots, N\}$. This completes the proof. □

Recall that the following: A mapping $T: C \to C$ where C is a subset of a Banach space E with $F(T) \neq \emptyset$ is said to satisfy *condition* (A) (see [13]) if there exists a nondecreasing function $f: [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$ such that for all $x \in C$,

$$\|x - Tx\| \geq f(d(x, F(T))),$$

where $d(x, F(T)) = \inf\{\|x - p\| : p \in F(T)\}$.

A family $\{T_i\}_{i=1}^{N}$ of N self-mappings of C with $\mathcal{F} = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$ is said to satisfy

(1) *condition* (B) (see [7]) on C if there is a nondecreasing function $f: [0, 1] \to [0, 1]$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$ and all $x \in C$ such that

$$\max_{1 \leq l \leq N} \{\|x - T_l x\|\} \geq f(d(x, \mathcal{F})).$$

(2) *condition* $(\overline{\mathrm{C}})$ (see [6]) on C if there is a nondecreasing function $f:[0,1] \to [0,1]$ with $f(0)=0$, $f(r)>0$ for all $r \in (0,\infty)$ and all $x \in C$ such that

$$\|x - T_l x\| \geq f(d(x,\mathcal{F}))$$

for at least one T_l, $l = 1,2,\ldots,N$; or in other words at least one of the mappings T_l satisfies condition (A).

Condition (B) reduces to condition (A) when all but one of the mappings T_i are identities. Also conditions (B) and $(\overline{\mathrm{C}})$ are equivalent (see [6]).

A random operator $T:\Omega\times C \to C$ is said to satisfy condition (A), condition (B) or condition $(\overline{\mathrm{C}})$ if the map $T(\omega,\cdot):C \to C$ does so, for each $\omega \in \Omega$.

Let $T:\Omega \times C \to C$ be a random map. Then T is said to be:

(i) a completely continuous random operator if for a sequence of measurable mappings ξ_n from $\Omega \to C$ such that $\{\xi_n(\omega)\}$ is bounded for each $\omega \in \Omega$, then $T(\omega,\xi_n(\omega))$ has convergent subsequence for each $\omega \in \Omega$.

(ii) a demicompact random operator if for a sequence of measurable mappings ξ_n from $\Omega \to C$ such that $\{\xi_n(\omega) - T(\omega,\xi_n(\omega))\}$ converges, there exists a subsequence say $\{\xi_{n_j}(\omega)\}$ of $\{\xi_n(\omega)\}$ that converges strongly to some $\xi(\omega)$ for each $\omega \in \Omega$, where ξ is a measurable mapping from Ω to C.

(iii) a semi-compact random operator if for a sequence of measurable mappings ξ_n from $\Omega \to C$ such that $\lim_{n\to\infty} \|\xi_n(\omega) - T(\omega,\xi_n(\omega))\| \to 0$, for each $\omega \in \Omega$, there exists a subsequence say $\{\xi_{n_j}(\omega)\}$ of $\{\xi_n(\omega)\}$ that converges strongly to some $\xi(\omega)$ for each $\omega \in \Omega$, where ξ is a measurable mapping from Ω to C.

Senter and Dotson [13] established a relation between condition (A) and demicompactness. They actually showed that condition (A) is weaker than demicompactness for a nonexpansive mapping.

Every compact operator is demicompact. As every completely continuous operator $T:C \to C$ is continuous and demicompact, it satisfies condition (A). Therefore to study strong convergence of the sequence $\{\xi_n(\omega)\}$ defined by (2.8) we use condition $(\overline{\mathrm{C}})$ instead of the complete continuity of the operators $T_1, T_2, \ldots, T_N$.

Theorem 3.3. *Let E be a real separable uniformly convex Banach space and C be a nonempty closed convex subset of E. Let $\{T_i\}_{i=1}^N$ be N uniformly L-Lipschitzian asymptotically quasi-nonexpansive in the intermediate sense random operators as in Theorem* 3.1 *and satisfying* condition $(\overline{\mathrm{C}})$. *Let $F = \bigcap_{i=1}^N \mathrm{RF}(T_i) \neq \emptyset$. Let $\{\xi_n(\omega)\}$ be the sequence defined by* (2.8). *Put*

$$A_n(\omega) = \max\Big\{ \sup_{\substack{\xi(\omega)\in F\\ n\geq 1}} \big(\|T_i^n(\omega,\xi_n(\omega)) - \xi(\omega)\| - \|\xi_n(\omega) - \xi(\omega)\|\big) \vee 0 : i \in I \Big\}.$$

Assume that

$$\sum_{n=1}^{\infty} A_n(\omega) < \infty \quad \textit{and} \quad \sum_{n=1}^{\infty}(1-\alpha_n) < \infty,$$

where $\{\alpha_n\}$ *is a sequence as in Theorem* 3.1. *Then the sequence* $\{\xi_n(\omega)\}$ *converges to a common random fixed point of the random operators* $\{T_i : i = 1, 2, \dots, N\}$.

Proof. By Theorem 3.1, $\lim_{n\to\infty} \|\xi_n(\omega) - \xi(\omega)\|$ and $\lim_{n\to\infty} d(\xi_n(\omega), F)$ exist. Let one of the mappings T_i, say T_s, $s \in \{1, 2, \dots, N\}$, satisfy condition (A). Also by Theorem 3.1, we have $\lim_{n\to\infty} \|\xi_n(\omega) - T_s\xi_n(\omega)\| = 0$, so we get that $\lim_{n\to\infty} f(d(\xi_n(\omega), F)) = 0$, for each $\omega \in \Omega$. By the property of f and the fact that the limit $\lim_{n\to\infty} d(\xi_n(\omega), F)$ exists, we have $\lim_{n\to\infty} d(\xi_n(\omega), F) = 0$, for each $\omega \in \Omega$. By Theorem 3.2, we obtain $\{\xi_n(\omega)\}$ converges strongly to a common random fixed point in F. This completes the proof. □

Theorem 3.4. *Let E be a real separable uniformly convex Banach space and C be a nonempty closed convex subset of E. Let $\{T_i\}_{i=1}^N$ be N uniformly L-Lipschitzian asymptotically quasi-nonexpansive in the intermediate sense random operators as in Theorem* 3.1 *such that one of the mappings in $\{T_1, T_2, \dots, T_N\}$ is semi-compact. Let $F = \bigcap_{i=1}^N \mathrm{RF}(T_i) \neq \emptyset$. Let $\{\xi_n(\omega)\}$ be the sequence defined by* (2.8). *Put*

$$A_n(\omega) = \max\Big\{ \sup_{\substack{\xi(\omega)\in F \\ n\geq 1}} \big(\|T_i^n(\omega, \xi_n(\omega)) - \xi(\omega)\| - \|\xi_n(\omega) - \xi(\omega)\|\big) \vee 0 : i \in I \Big\}.$$

Assume that

$$\sum_{n=1}^{\infty} A_n(\omega) < \infty \quad \textit{and} \quad \sum_{n=1}^{\infty}(1-\alpha_n) < \infty,$$

where $\{\alpha_n\}$ *is a sequence as in Theorem* 3.1. *Then the sequence* $\{\xi_n(\omega)\}$ *converges to a common random fixed point of the random operators* $\{T_i : i = 1, 2, \dots, N\}$.

Proof. Suppose that T_{i_0} is semi-compact for some $i_0 \in \{1, 2, \dots, N\}$. By Theorem 3.1, we have $\lim_{n\to\infty} \|\xi_n(\omega) - T_{i_0}\xi_n(\omega)\| = 0$. So there exists a subsequence $\{\xi_{n_j}(\omega)\}$ of $\{\xi_n(\omega)\}$ such that $\lim_{n_j\to\infty} \xi_{n_j}(\omega) \to \xi_0(\omega)$ for each $\omega \in \Omega$. Obviously ξ_0 is a measurable mapping from $\Omega \to C$. Now again by Theorem 3.1 we have

$$\lim_{n_j\to\infty} \|\xi_{n_j}(\omega) - T_l(\omega, \xi_{n_j}(\omega))\| = 0,$$

for each $\omega \in \Omega$ and all $l \in \{1, 2, \dots, N\}$. So $\|\xi_0(\omega) - T_l(\omega, \xi_0(\omega))\| = 0$ for all $l \in \{1, 2, \dots, N\}$. This implies that $\xi_0(\omega) \in F$, also $\liminf_{n\to\infty} d(\xi_n(\omega), F) = 0$. By Theorem 3.2, we obtain $\{\xi_n(\omega)\}$ converges strongly to a common random fixed point in F. This completes the proof. □

Remark 3.5. Our results extend and improve the corresponding results of Plubtieng, Kumam and Wangkeeree [11] to the case of the more general class of asymptotically quasi-nonexpansive random operators and general composite implicit random iteration process considered in this paper.

Remark 3.6. Our results also extend and improve the corresponding results of Beg and Thakur [4] to the case of the more general class of asymptotically quasi-nonexpansive random operators considered in this paper.

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Bibliography

[1] J. P. Aubin and H. Frankowska *Set-Valued Analysis*, Birkhäuser-Verlag, Boston, 1990.

[2] I. Beg and M. Abbas, Iterative procedures for solutions of random operator equations in Banach spaces, *J. Math. Anal. Appl.* **315** (2006), no. 1, 181–201.

[3] I. Beg and M. Abbas, Random fixed point theorems for a random operator on an unbounded subset of a Banach space, *Appl. Math. Lett.* **21** (2008), no. 10, 1001–1004.

[4] I. Beg and B. Thakur, Solution of random operator equations using general composite implicit iteration process, *Int. J. Mod. Math.* **4** (2009), no. 1, 19–34.

[5] A. T. Bharucha-Reid, Fixed point theorems in Probabilistic analysis, *Bull. Amer. Math. Soc.* **82** (1976), no. 5, 641–657.

[6] C. E. Chidume and B. Ali, Weak and strong convergence theorems for finite families of asymptotically nonexpansive mappings in Banach spaces, *J. Math. Anal. Appl.* **330** (2007), no. 1, 377–387.

[7] C. E. Chidume and N. Shahzad, Strong convergence of an implicit iteration process for a finite family of nonexpansive mappings, *Nonlinear Anal.* **62** (2005), 1149–1156.

[8] S. Itoh, Random fixed point theorems with an application to random differential equations in Banach spaces, *J. Math. Anal. Appl.* **67** (1979), no. 2, 261–273.

[9] T. C. Lin, Random approximations and random fixed point theorems for continuous 1-set contractive random maps, *Proc. Amer. Math. Soc.* **123** (1995), 1167–1176.

[10] M. O. Osilike, S. C. Aniagbosor and B. G. Akuchu, Fixed points of asymptotically demicontractive mappings in arbitrary Banach spaces, *Panamer. Math. J.* **12** (2002), 77–78.

[11] S. Plubtieng, P. Kumam and R. Wangkeeree, Approximation of a common random fixed point for a finite family of random operators, *Int. J. Math. Math. Sci.* **2007** (2007), Article ID 69626.

[12] J. Schu, Weak and strong convergence theorems to fixed points of asymptotically nonexpansive mappings, *Bull. Aust. Math. Soc.* **43** (1991), 153–159.

[13] H. F. Senter and W. G. Dotson Jr., Approximating fixed points of nonexpansive mappings, *Proc. Amer. Math. Soc.* **44** (1974), 375–380.

[14] N. Shahzad, Random fixed points of pseudo-contractive random operators, *J. Math. Anal. Appl.* **296** (2004), 302–308.

[15] Y. Su and S. Li, Composite implicit iteration process for common fixed points of a finite family of strictly pseudocontractive maps, *J. Math. Anal. Appl.* **320** (2006), 882–891.

[16] Y. Su and X. Qin, General iteration algorithm and convergence rate optimal model for common fixed points of nonexpansive mapping, *Appl. Math. Comput.* **186** (2007), 271–278.

[17] Z. Sun, Strong convergence of an implicit iteration process for a finite family of asymptotically quasi-nonexpansive mappings, *J. Math. Anal. Appl.* **286** (2003), 351–358.

[18] H. K. Xu, Some random fixed point theorems for condensing and nonexpansive operators, *Proc. Amer. Math. Soc.* **100** (1990), 103–123.

[19] H. K. Xu and R. G. Ori, An implicit iteration process for nonexpansive mappings, *Numer. Funct. Anal. Optim.* **22** (2001), 767–773.

[20] Y. Zhou and S. S. Chang, Convergence of implicit iteration process for a finite family of asymptotically nonexpansive mappings in Banach spaces, *Numer. Funct. Anal. Optim.* **23** (2002), 911–921.

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