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Results on the approximate controllability of fractional hemivariational inequalities of order $1 < r < 2$

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Abstract

In this paper, we investigate the approximate controllability of fractional evolution inclusions with hemivariational inequalities of order $1 < r < 2$. The main results of this paper are verified by using the fractional theories, multivalued analysis, cosine families, and fixed-point approach. At first, we discuss the existence of the mild solution for the class of fractional systems. After that, we establish the approximate controllability of linear and semilinear control systems. Finally, an application is presented to illustrate our theoretical results.

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1 Introduction

Fractional calculus, as a significant area of mathematics, was initiated in 1695. Currently, the concept of fractional calculation has been powerfully tested in many social, physical, signal and image processing, biological, control theory, and engineering problems, etc. For more specifics, refer to the book [32] and the research papers [1–6, 14]. The notion of exact controllability has an essential role in mathematical control theories and technology. In recent years, many authors have done fruitful achievements on exact and approximate controllability of different nonlinear dynamical problems; one can refer to the research articles [16, 30].

It is common knowledge that many problems from mechanics (elasticity theory, semipermeability, electrostatics, hydraulics, fluid flow), economics, and so on can be modeled by subdifferential inclusions or hemivariational inequalities, and we refer to [31] for more applications of hemivariational inequalities. Recently, the existence of solutions for hemivariational inequalities has been proved by many authors. Furthermore, the hemivariational inequalities were initiated by Panagiotopoulos in 1980. They are handled for the mechanical problems with nonconvex, nonsmooth superpotentials, and optimal control problems; several researchers investigated the existence and approximate controllability results by applying the various approaches and different fixed-point theorems. We suggest the articles [23–25] and the references therein. Recently, in [20], the authors discussed the existence results for quasilinear parabolic hemivariational inequalities by use of the theory of multivalued pseudomonotone operators, the notion of the generalized gradient of Clarke, and the property of the first eigenfunction. The authors built a Landesman–Lazer theory in the nonsmooth framework of quasilinear parabolic hemivariational inequalities. Inspired by the above-mentioned paper, in [21], the authors studied the approximate controllability for control systems described by a class of hemivariational inequalities. The authors introduced the concept of mild solutions for hemivariational inequalities and then the approximate controllability was formulated and proved by utilizing a fixed-point theorem of multivalued maps and properties of generalized Clarke subdifferential.

Currently, a growing number of researchers have made successful progress in existence and exact controllability results for fractional evolution systems of order $r \in (1,2)$. In particular, in [41], the authors introduced a new approach for finding mild solutions for the considered system. Additionally, the authors developed the controllability of fractional differential systems with order $r \in (1,2)$ by applying fractional calculus, cosine families, multivalued analysis, the Laplace transform, measure of noncompactness, Mainardi's Wright-type function, and a fixed-point theorem. Further, the authors established the nonlocal condition in fractional system with order $r \in (1,2)$ (see [12]). Moreover, there are some interesting and improved outcomes on the existence and exact controllability of the fractional system of order $r \in (1,2)$ with delay, without delay by referring to the theory of fractional calculus, cosine families, multivalued analysis, and the fixed-point approach. For more details, see the research articles [26–29].

The approximate controllability for fractional evolution inclusions of order $r \in (1,2)$ by using the ideas of hemivariational inequalities, Mainardi's Wright function and strongly continuous cosine families is still an untreated topic [18, 19]. The above facts inspired the present work. Therefore, we consider the fractional differential hemivariational inequalities with order $r \in (1,2)$ having the form

$$\begin{cases} {}^{C}D_t^r u(t) \in Au(t) + \mathrm{B}x(t) + \partial \mathcal{G}(t, u(t)), & t \in W := [0, c], \\ u(0) = u_0, \qquad u'(0) = u_1 \in U, \end{cases} \tag{1.1}$$

where ${}^{C}D_t^r$ denotes the Caputo fractional derivative of order $r \in (1,2)$, $A : D(A) \subseteq U \times U$ is the infinitesimal generator of a strongly continuous cosine family $\{C(t)\}_{t \geq 0}$ in a Hilbert space U. The control function $x(\cdot)$ takes values in $L^2(W, Y)$ and the admissible controls' set Y is a Hilbert space. Furthermore, B is a bounded linear operator from Y into U, and $\partial \mathcal{G}(t, \cdot)$ denotes the generalized Clarke's subdifferential of a locally Lipschitz function $\mathcal{G}(t, \cdot)$ mapping U into $\mathbb{R}$.

The main contributions of our manuscript are as follows: (i) A new set of sufficient conditions are formulated and used to prove the existence and approximate controllability results of fractional evolution inclusions with hemivariational inequalities of order $1 < r < 2$ under simple and fundamental assumptions on the system operators, in particular, that the corresponding linear system is approximately controllable. (ii) In this paper, we establish sufficient conditions for the approximate controllability results of fractional evolution inclusions with hemivariational inequalities for the linear system. (iii) Further, we extend the result to obtain the conditions for the solvability of approximate controllability results for fractional evolution inclusions with hemivariational inequalities for the semilinear case. (iv) We show that our result has no analog for the concept of complete controllability, and finally we give an example of the system which is not completely controllable, but approximately controllable. (v) More precisely, the controllability problem can be converted into a solvability problem of a functional operator equation in an appropriate Hilbert spaces and the fixed point approach is used to solve the problem.

The rest of this manuscript is organized as follows: Sect. 2 gives basic definitions and preliminary results to be used in this paper. In Sect. 3, we present the existence of solutions to (1.1) and investigate the approximate controllability of the linear system (4.1) in Sect. 4. In Sect. 5, we provide the approximate controllability of (1.1). Finally, an application is provided to illustrate the theory of the obtained results.

2 Preliminaries

Now, we introduce the well-known definitions, lemmas, notations, and facts about fractional calculus which will be used in the sequel [15, 32]. Let X be a Banach space with the norm $\|\cdot\|_X$; X^* denotes its dual and $(\cdot,\cdot)_X$ is the duality pairing between X^* and X; $\mathbb{C}(W,X)$ denotes the Banach space of all continuous functions from W into X with the $\|u\|_{\mathbb{C}(W,X)} = \sup_{t\in W}\|u(t)\|_X$. We set $P := \sup_{t\in[0,\infty)}\|C(t)\| < +\infty$.

Denote by $D(A)$ and $R(A)$ the domain and range of A, respectively. We denote the resolvent set of A by $\rho(A)$ and the resolvent of A by

$$R(\Lambda,A) = (\Lambda I - A)^{-1} \in L_c(X).$$

We now present some theories based on fractional calculus, which are discussed in [15].

Definition 2.1 The Riemann–Liouville fractional integral of order $\gamma \in \mathbb{R}^+$ with the lower limit zero for a function $g : [0,\infty) \to \mathbb{R}^+$ is defined by

$$I^\gamma g(t) = \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1} g(s)\,ds, \quad t > 0,$$

provided the right-hand side is pointwise defined on $[0,\infty)$, where $\Gamma(\cdot)$ is the gamma function.

Definition 2.2 The Riemann–Liouville fractional derivative of order $\gamma \in \mathbb{R}^+$ with the lower limit zero for a function $g : [0,\infty) \to \mathbb{R}^+$ is defined by

$${}^{\mathrm{L}}D^\gamma g(t) = \frac{1}{\Gamma(k-\gamma)}\frac{d^k}{dt^k}\int_0^t (t-s)^{k-\gamma-1} g^{(k)}(s)\,ds, \quad t > 0, k-1 < \gamma < k.$$

Definition 2.3 The Caputo fractional derivative of order $\gamma \in \mathbb{R}^+$ with the lower limit zero for a function $g : [0,\infty) \to \mathbb{R}^+$ is defined by

$${}^{\mathrm{C}}D^\gamma g(t) = {}^{\mathrm{L}}D^\gamma \left(g(t) - \sum_{i=1}^{k-1} \frac{g^{(i)}(0)}{i!} t^i \right), \quad t > 0, k-1 < \gamma < k.$$

Remark 2.4

(1) The Caputo derivative of a constant is equal to zero.

(2) If $g \in \mathbb{C}[0,\infty)$, then

$$ {}^{C}D^{\gamma}g(t) = \frac{1}{\Gamma(k-\gamma)}\int_0^t (t-s)^{k-\gamma-1}g^{(k)}(s)\,ds = I^{k-\gamma}g^{(k)}(t), $$

with $t > 0$, $k-1 < \gamma < k$.

(3) If g is an abstract function with values in U, then the integrals which appear in Definitions 2.2 and 2.3 are taken in Bochner's sense.

Let us recall the well-known definitions and results of multivalued analysis. For more details on multivalued analysis, see the books [10, 13].

(i) A multivalued map $E : X \to 2^X \setminus \{\emptyset\} := \psi(X)$ is convex (closed) valued, if $E(u)$ is convex (closed) for all $u \in X$.

(ii) A multivalued map E is said to be upper semicontinuous on X if for each $u \in X$, the set $E(u)$ is a nonempty, closed subset of X, and if for each open set $\mathbb{V}$ of X containing $E(u)$, there exists an open neighborhood Q of u such that

$$ E(Q) \subseteq \mathbb{V}. $$

(iii) E is said to be completely continuous if $E(\mathbb{V})$ is relatively compact, for every bounded subset $\mathbb{V} \subseteq X$.

(iv) Let (F,Ψ) and (X,d) be a measurable space and separable metric space, respectively. A multivalued mapping $E : W \to \psi(X)$ is said to be measurable, if for every closed set $\mathscr{F} \subseteq X$, we have

$$ E^{-1}(\mathscr{F}) = \{t \in W : E(t) \cap \mathscr{F} \neq \emptyset\} \in \Psi. $$

We present the definition of the Clarke's subdifferential for a locally Lipschitz function μ mapping X into $\mathbb{R}$ [13]. We denote by $\mu^0(m,n)$ the Clarke's generalized directional derivative of μ at m in the direction n, i.e.,

$$ \mu^0(m,n) := \lim_{\sigma \to 0^+} \sup_{\nu \to m} \frac{\mu(\nu + \sigma n) - \mu(\nu)}{\sigma}. $$

Recall also that the generalized Clarke's subdifferential of μ at m is the subset of X^* given by

$$ \partial\mu(m) := \{m^* \in X^* : \mu^0(m,n) \geq \langle m^*, n\rangle \text{ for all } n \in X\}. $$

Lemma 2.5 ([25]) *Let μ be locally Lipschiz of rank $\mathscr{F}$ near m, then*

(a) *$\partial\mu(m)$ is a nonempty, convex, weak*-compact subset of X^* and $\|m^*\|_{X^*} \leq \mathscr{F}$, for every $m^* \in \partial\mu(m)$;*

(b) *for every $n \in X$, one has $\mu^0(m,n) = \max\{\langle m^*, n\rangle : \text{for all } m^* \in \partial\mu(m)\}$.*

Definition 2.6 ([34]) A one-parameter family $\{C(t)\}_{t\in\mathbb{R}}$ of bounded linear operators mapping the Hilbert space U to itself is said to be a strongly continuous cosine family if and only if

(i) $C(0) = I$;
(ii) $C(s+t) + C(s-t) = 2C(s)C(t)$ for all $s, t \in \mathbb{R}$;
(iii) $C(t)u$ is strongly continuous in t on $\mathbb{R}$ for each fixed point $u \in U$.

Let $\{S(t)\}_{t\in\mathbb{R}}$ denote the strongly continuous sine family associated with the strongly continuous cosine family $\{C(t)\}_{t\in\mathbb{R}}$, where

$$S(t)u = \int_0^t C(s)u\,ds, \quad u \in U, t \in \mathbb{R}. \tag{2.1}$$

Additionally, A is called an infinitesimal generator of a cosine family $\{C(t)\}_{t\in\mathbb{R}}$ if

$$Au = \frac{d^2}{dt^2}C(t)u\bigg|_{t=0}, \quad \forall u \in D(A),$$

where $D(A)$ is determined by

$$D(A) = \{u \in U : C(t)u \in \mathbb{C}^2(\mathbb{R}, U)\}.$$

Denote a set

$$\mathcal{A} = \{u \in U : C(t)u \in \mathbb{C}^1(\mathbb{R}, U)\}.$$

The infinitesimal generator A is a closed, densely-defined operator in U. We set $b = \frac{r}{2}$ for $r \in (1,2)$, as in [12, 40].

Definition 2.7 Any $u \in \mathbb{C}(W, U)$ is said to be a mild solution of system (1.1) on W if

$$u(t) = C_b(t)u_0 + K_b(t)u_1 + \int_0^t (t-s)^{b-1} T_b(t-s)g(s)\,ds + \int_0^t (t-s)^{b-1} T_b(t-s)\mathrm{B}x(s)\,ds, \quad t \in W,$$

where

$$C_b(t) = \int_0^\infty S_b(\tau)C(t^b\tau)\,d\tau, \qquad K_b(t) = \int_0^t C_b(s)\,ds,$$
$$T_b(t) = \int_0^\infty b\tau S_b(\tau)S(t^b\tau)\,d\tau, \qquad S_b(\tau) = \frac{1}{b}\tau^{-1-\frac{1}{b}}\zeta_b(\tau^{-\frac{1}{b}}),$$
$$\zeta_b(\tau) = \frac{1}{\pi}\sum_{k=1}^\infty (-1)^{k-1}\tau^{-kb-1}\frac{\Gamma(kb+1)}{k!}\sin(k\pi b), \quad \tau \in (0,\infty),$$

and $S_b(\cdot)$ is the Mainardi's Wright-type function defined on $(0,\infty)$ such that

$$S_b(\tau) \geq 0 \quad \text{for } \tau \in (0,\infty) \quad \text{and} \quad \int_0^\infty S_b(\tau)\,d\tau = 1.$$

Let

$$N_c(\mathcal{G}) = \{u(c) \in U : u(\cdot) \text{ is a mild solution of system (1.1) corresponding to a control } x \in L^2(W, Y) \text{ with } u_0, u_1 \in U\},$$

which is said to be the reachable set of (1.1). If $\mathcal{G} = 0$, then this system is said to be the corresponding linear system of (1.1). Let $N_c(0)$ denote the reachable set of the linear system.

Definition 2.8 ([18]) The control system (1.1) is said to be approximately controllable on $W = [0, c]$, if $\overline{N_c(\mathcal{G})} = U$, where $\overline{N_c(\mathcal{G})}$ denotes the closure of $N_c(\mathcal{G})$. Clearly, the corresponding linear system is approximately controllable on W, if $\overline{N_c(0)} = U$.

Lemma 2.9 ([12]) *The operators $C_b(t)$, $K_b(t)$, and $T_b(t)$ have the following properties*:

(i) *For any $t \geq 0$, $C_b(t)$, $K_b(t)$, and $T_b(t)$ are linear and bounded operators such that for all $u \in U$,*

$$\|C_b(t)u\| \leq P\|u\|, \qquad \|K_b(t)u\| \leq P\|u\|t, \qquad \|T_b(t)u\| \leq \frac{P}{\Gamma(2b)}\|u\|t^b;$$

(ii) *The operators $\{C_b(t), t \geq 0\}$, $\{K_b(t), t \geq 0\}$, and $\{t^{b-1}T_b(t), t \geq 0\}$ are strongly continuous*;

(iii) *For any $t \geq 0$, $C_b(t)$, $K_b(t)$, and $T_b(t)$ are also compact operators if $T(t)$ is compact.*

Lemma 2.10 ([34]) *The following results are true*:

(i) *There exist $P \geq 1$ and $\omega \geq 0$ such that $\|C(t)\|_{L_c(U)} \leq Pe^{\omega|t|}$, for all $t \in \mathbb{R}$*;

(ii) $\|S(t_2) - S(t_1)\|_{L_c(U)} \leq P|\int_{t_1}^{t_2} e^{\omega|s|}\,ds|$ *for all $t_2, t_1 \in \mathbb{R}$*;

(iii) *If $u \in \mathcal{A}$, then $S(t)u \in D(A)$ and $\frac{d}{dt}C(t)u = AS(t)u$.*

Lemma 2.11 ([41]) *Let $\{C(t)\}_{t\in\mathbb{R}}$ be a strongly continuous cosine family in U, and A be the infinitesimal generator of $\{C(t)\}_{t\in\mathbb{R}}$. Then*

$$\lim_{t\to 0} \frac{1}{t} S(t)u = u, \quad \forall u \in U.$$

Lemma 2.12 ([34]) *Let $\{C(t)\}_{t\in\mathbb{R}}$ be a strongly continuous cosine family in U satisfying $\|C(t)\|_{L_c(U)} \leq Pe^{\omega|t|}$, $t \in \mathbb{R}$. Then for* $\operatorname{Re}\Lambda > \omega$, $\Lambda^2 \in \rho(A)$ *and*

$$\Lambda R(\Lambda^2; A)u = \int_0^\infty e^{-\Lambda t}C(t)u\,dt, \qquad R(\Lambda^2; A)u = \int_0^\infty e^{-\Lambda t}S(t)u\,dt, \quad \text{for } u \in U.$$

Theorem 2.13 ([22]) *Let X is a Banach space and $\mathcal{H} : X \to 2^X$ be a compact convex valued, upper semicontinuous multivalued map such that there exists a closed neighborhood $\mathbb{V}$ of zero for which $\mathcal{H}(\mathbb{V})$ is relatively compact set. If*

$$\Phi = \{u \in X : \lambda u \in \mathcal{H}(u) \text{ for some } \lambda > 1\}$$

is bounded, then $\mathcal{H}$ has a fixed point.

3 Existence results

In this section, we study the existence of mild solutions of system (1.1). Before starting and proving the main results of this section, we impose the following hypotheses:

(H_1) The operator $\{C(t)\}$ is compact for all $t \geq 0$.

(H_2) $\mathcal{G} : W \times U \to \mathbb{R}$ is a function such that

(i) the function $t \mapsto \mathcal{G}(t,u)$ is measurable for all $u \in U$;

(ii) the function $u \mapsto \mathcal{G}(t,u)$ is locally Lipschitz for all $t \in W$;

(iii) there exists a function $\beta \in L^2(W,\mathbb{R}^+)$ and $j > 0$ such that

$$\|\partial\mathcal{G}(t,u)\|_U = \sup\{\|g\|_U : g \in \partial\mathcal{G}(t,u)\} \leq \beta(t) + j\|u\|_U,$$

for every $u \in U$ and for all $t \in W$.

Now, we consider the operator $\mathcal{V} : L^2(W,U) \to 2^{L^2(W,U)}$ as follows:

$$\mathcal{V}(u) = \{z \in L^2(W,U) : z(t) \in \partial\mathcal{G}(t;u(t)), \text{for any } t \in W\}, \quad \text{for all } u \in L^2(W,U).$$

Lemma 3.1 ([25]) *If* (H_2) *holds, then for every* $u \in L^2(W,U)$, *the set* $\mathcal{V}(u)$ *has nonempty, convex, and weakly compact values.*

Lemma 3.2 ([24]) *If* (H_2) *holds, the operator* $\mathcal{P}$ *has the property that if* $u_k \to u \in L^2(W,U)$, $z_k \to z$ *weakly in* $L^2(W,U)$ *and* $z_k \in \mathcal{P}(u_k)$, *then we have* $z \in \mathcal{P}(u)$.

Theorem 3.3 *If* (H_1) *and* (H_2) *are satisfied, then system* (1.1) *has a mild solution on* W.

Proof For each $x \in L^2(W,U)$ and all $u \in \mathbb{C}(W,U) \subset L^2(W,U)$, from Definition 2.7, we introduce the multivalued map $\mathcal{G} : \mathbb{C}(W,U) \to 2^{\mathbb{C}(W,U)}$ as follows:

$$\mathcal{G}(u) = \left\{ y \in \mathbb{C}(W,U) : y(t) = C_b(t)u_0 + K_b(t)u_1 + \int_0^t (t-s)^{b-1} T_b(t-s)g(s)\,ds \right.$$
$$\left. + \int_0^t (t-s)^{b-1} T_b(t-s)\mathrm{B}x(s)\,ds, g \in \mathcal{V}(u) \right\}, \quad \text{for } u \in \mathbb{C}(W,U).$$

Then our problem is reduced to finding the fixed point of $\mathcal{H}$. For this, we shall check that $\mathcal{H}$ satisfies all the assumptions of Theorem 2.13. Now, $\mathcal{H}(u)$ is convex by the convexity of $\mathcal{V}(u)$. Now, we split our discussion into following steps for ease of exposition.

Step 1: $\mathcal{H}$ maps bounded subsets into bounded subsets in $\mathbb{C}(W,U)$.

For any $u \in \mathcal{B}_p = \{u \in \mathbb{C}(W,U) : \|u\|_{\mathbb{C}} \leq p\}$, $p > 0$, $\vartheta \in \mathcal{H}(u)$, we obtain $g \in \mathcal{V}(u)$ such that

$$\vartheta(t) = C_b(t)u_0 + K_b(t)u_1 + \int_0^t (t-s)^{b-1} T_b(t-s)g(s)\,ds$$
$$+ \int_0^t (t-s)^{b-1} T_b(t-s)\mathrm{B}x(s)\,ds, \quad t \in W. \tag{3.1}$$

Using (H_2)(iii), Lemma 2.10, and Hölder inequality, we get

$$\|\vartheta(t)\|_U$$
$$\leq \|C_b(t)u_0\|_U + \|K_b(t)u_1\|_U + \int_0^t (t-s)^{b-1} \|T_b(t-s)g(s)\|_U\,ds$$

$$
\begin{aligned}
&+ \int_0^t (t-s)^{b-1} \| T_b(t-s)\mathrm{B}x(s)\|_U \, ds \\
&\leq P\|u_0\|_U + Pc\|u_1\|_U + \frac{P}{\Gamma(2b)} \int_0^t (t-s)^{2b-1} \| g(s)\|_U \, ds \\
&+ \frac{P}{\Gamma(2b)} \int_0^t (t-s)^{2b-1} \| \mathrm{B}x(s)\|_U \, ds \\
&\leq P\|u_0\|_U + Pc\|u_1\|_U + \frac{P}{\Gamma(2b)} \int_0^t (t-s)^{2b-1} \big(\beta(s) + j\| u(s)\|_U\big) \, ds \\
&+ \frac{P}{\Gamma(2b)} \int_0^t (t-s)^{2b-1} \|\mathrm{B}\| \| x(s)\|_U \, ds \\
&\leq P\|u_0\|_U + Pc\|u_1\|_U + \frac{P\|\beta\|_{L^2(W,\mathbb{R}^+)}}{\Gamma(2b)} \left(\int_0^t (t-s)^{\frac{2b-1}{1-b_1}} \, ds \right)^{1-b_1} + \frac{Pjc^{2b}p}{\Gamma(2b+1)} \\
&+ \frac{P}{\Gamma(2b)} \left(\int_0^t (t-s)^{\frac{2b-1}{1-b_1}} \, ds \right)^{1-b_1} \|\mathrm{B}\| \|x\|_{L^2(W,Y)} \\
&\leq P\|u_0\|_U + Pc\|u_1\|_U + \frac{Pc^{2b-\frac{1}{2}}}{\sqrt{4b-1}\,\Gamma(2b)} \big[\|\beta\|_{L^2(W,\mathbb{R}^+)} + \|\mathrm{B}\| \|x\|_{L^2(W,Y)} \big] + \frac{Pjc^{2b}p}{\Gamma(2b+1)}.
\end{aligned}
$$

Hence, $\mathcal{H}(\mathcal{B}_p)$ is bounded in $\mathbb{C}(W, U)$.

Step 2: $\{\mathcal{H}(u) : u \in \mathcal{B}_p\}$ is equicontinuous for every $p > 0$.

For any $u \in \mathcal{B}_p$, $\vartheta \in \mathcal{H}(u)$ there exists $g \in \mathcal{V}(u)$ such that (3.1) holds true. Now, for every $\epsilon > 0$, from (H_2)(iii), we have for all $t \in W$,

$$
\begin{aligned}
&\| \vartheta(t) - \vartheta(0)\|_U \\
&\quad \leq \| C_b(t)u_0 - u_0\|_U + \| K_b(t)u_1 - u_1\|_U + \int_0^t (t-s)^{b-1} \| T_b(t-s)g(s)\|_U \, ds \\
&\quad + \int_0^t (t-s)^{b-1} \| T_b(t-s)\mathrm{B}x(s)\|_U \, ds \\
&\quad \leq \| C_b(t)u_0 - u_0\|_U + \| K_b(t)u_1 - u_1\|_U + \frac{Pc^{2b-\frac{1}{2}}}{\sqrt{4b-1}\,\Gamma(2b)} \\
&\quad \times \big(\|\beta\|_{L^2(W,\mathbb{R}^+)} + \|\mathrm{B}\| \|x\|_{L^2(W,Y)} \big) + \frac{Pjc^{2b}p}{\Gamma(2b+1)}.
\end{aligned}
$$

Then, there exists $\eta_1 > 0$ sufficiently small such that for all $0 < t \leq \eta_1$,

$$
\| \vartheta(t) - \vartheta(0)\|_U < \frac{\epsilon}{2}.
$$

Since, for each $\epsilon > 0$, all $\varpi_1, \varpi_2 \in [0, \eta_1]$, and every $\vartheta \in \mathcal{H}(\mathcal{B}_p)$, we obtain

$$
\| \vartheta(\varpi_2) - \vartheta(\varpi_1)\|_U < \epsilon
$$

independently of $u \in \mathcal{B}_p$. Next, for all $u \in \mathcal{B}_p$, and $\frac{\eta_1}{2} \leq \varpi_1 < \varpi_2 \leq c$, we get

$$
\begin{aligned}
\| \vartheta(\varpi_2) - \vartheta(\varpi_1)\|_U \leq{} & \| C_b(\varpi_2)u_0 - C_b(\varpi_1)u_0\|_U + \| K_b(\varpi_2)u_1 - K_b(\varpi_1)u_1\|_U \\
& + \int_0^{\varpi_1} \big[(\varpi_2 - s)^{b-1} - (\varpi_1 - s)^{b-1} \big] \| T_b(\varpi_2 - s)g(s)\|_U \, ds
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^{\varpi_1}(\varpi_1-s)^{b-1}\big\|\big[T_b(\varpi_2-s)-T_b(\varpi_1-s)\big]g(s)\big\|_U\,ds\\
&+\int_{\varpi_1}^{\varpi_2}(\varpi_2-s)^{b-1}\big\|T_b(\varpi_2-s)g(s)\big\|_U\,ds\\
&+\int_0^{\varpi_1}\big[(\varpi_2-s)^{b-1}-(\varpi_1-s)^{b-1}\big]\big\|T_b(\varpi_2-s)\mathrm{B}x(s)\big\|_U\,ds\\
&+\int_0^{\varpi_1}(\varpi_1-s)^{b-1}\big\|\big[T_b(\varpi_2-s)-T_b(\varpi_1-s)\big]\mathrm{B}x(s)\big\|_U\,ds\\
&+\int_{\varpi_1}^{\varpi_2}(\varpi_2-s)^{b-1}\big\|T_b(\varpi_2-s)\mathrm{B}x(s)\big\|_U\\
:=&\sum_{k=1}^{8}\mathcal{E}_k.
\end{aligned}
$$

Using Lemma 2.9, (H_2)(iii), and Hölder inequality, we get

$$
\begin{aligned}
\mathcal{E}_3 &\le \frac{P}{\Gamma(2b)}\int_0^{\varpi_1}\big[(\varpi_2-s)^{2b-1}-(\varpi_1-s)^{2b-1}\big]\big\|g(s)\big\|_U\,ds\\
&\le \frac{P}{\Gamma(2b)}\int_0^{\varpi_1}\big[(\varpi_2-s)^{2b-1}-(\varpi_1-s)^{2b-1}\big]\big(\beta(s)+j\big\|u(s)\big\|_U\big)\,ds\\
&\le \frac{P\|\beta\|_{L^2(W,\mathbb{R}^+)}}{\Gamma(2b)}\bigg(\int_0^{\varpi_1}\big[(\varpi_2-s)^{2b-1}-(\varpi_1-s)^{2b-1}\big]^{\frac{1}{1-b_1}}\,ds\bigg)^{1-b_1}\\
&\quad+\frac{Pjp}{\Gamma(2b)}\int_0^{\varpi_1}\big[(\varpi_2-s)^{2b-1}-(\varpi_1-s)^{2b-1}\big]\,ds\\
&\le \frac{P\|\beta\|_{L^2(W,\mathbb{R}^+)}}{\sqrt{4b-1}\,\Gamma(2b)}\big(\varpi_2^{4b-1}-(\varpi_2-\varpi_1)^{4b-1}-\varpi_1^{4b-1}\big)^{\frac{1}{2}}\\
&\quad+\frac{Pjp}{\Gamma(2b+1)}\big(\varpi_2^{2b}-(\varpi_2-\varpi_1)^{2b}-\varpi_1^{2b}\big)^{\frac{1}{2}}.
\end{aligned}
$$

Let $M_b(t)=t^{b-1}T_b(t)$ for all $t\in W$. From Lemma 2.9(ii), we get that $M_b(t)$ is a strongly continuous operator. Choosing $\eta_2>0$, we have

$$
\begin{aligned}
\mathcal{E}_4 &\le \int_0^{\varpi_1-\eta_2}\big\|\big[M_b(\varpi_2-s)-M_b(\varpi_1-s)\big]\big\|\,\big\|g(s)\big\|_U\,ds\\
&\quad+\int_{\varpi_1-\eta_2}^{\varpi_1}\big\|\big[M_b(\varpi_2-s)-M_b(\varpi_1-s)\big]\big\|\,\big\|g(s)\big\|_U\,ds\\
&\le \sup_{s\in[0,\varpi_1-\eta_2]}\big\|\big[M_b(\varpi_2-s)-M_b(\varpi_1-s)\big]\big\|\int_0^{\varpi_1-\eta_2}\big(\beta(s)+j\big\|u(s)\big\|_U\big)\,ds\\
&\quad+\frac{2P(\varpi_2-\varpi_1+\eta_2)^{2b-1}}{\Gamma(2b)}\int_{\varpi_1-\eta_2}^{\varpi_1}\big(\beta(s)+j\big\|u(s)\big\|_U\big)\,ds\\
&\le \sup_{s\in[0,\varpi_1-\eta_2]}\big\|\big[M_b(\varpi_2-s)-M_b(\varpi_1-s)\big]\big\|\big(\|\beta\|_{L^2(W,\mathbb{R}^+)}+jp\big)\\
&\quad+\frac{2P(\varpi_2-\varpi_1+\eta_2)^{2b-1}}{\Gamma(2b)}\bigg(\int_{\varpi_1-\eta_2}^{\varpi_1}\beta(s)\,ds+pj\bigg),
\end{aligned}
$$

$$\begin{aligned}
\mathcal{E}_5 &\le \frac{P}{\Gamma(2b)}\int_{\varpi_1}^{\varpi_2}(\varpi_2-s)^{2b-1}\|g(s)\|_U\,ds\\
&\le \frac{P}{\Gamma(2b)}\int_{\varpi_1}^{\varpi_2}(\varpi_2-s)^{2b-1}\big(\beta(s)+j\|u(s)\|_U\big)\,ds\\
&\le \frac{P\|\beta\|_{L^2(W,\mathbb{R}^+)}}{\Gamma(2b)}\left(\int_{\varpi_1}^{\varpi_2}(\varpi_2-s)^{\frac{2b-1}{1-b_1}}\,ds\right)^{1-b_1}+\frac{Pjp}{\Gamma(2b)}\int_{\varpi_1}^{\varpi_2}(\varpi_2-s)^{2b-1}\,ds\\
&\le \frac{P\|\beta\|_{L^2(W,\mathbb{R}^+)}}{\sqrt{4b-1}\,\Gamma(2b)}(\varpi_2-\varpi_1)^{2b-\frac{1}{2}}+\frac{Pjp}{\Gamma(2b+1)}(\varpi_2-\varpi_1)^{2b},\\
\mathcal{E}_6 &\le \frac{P}{\Gamma(2b)}\int_0^{\varpi_1}\big[(\varpi_2-s)^{2b-1}-(\varpi_1-s)^{2b-1}\big]\|\mathrm{B}x\|\,ds\\
&\le \frac{P}{\Gamma(2b)}\left(\int_0^{\varpi_1}\big[(\varpi_2-s)^{2b-1}-(\varpi_1-s)^{2b-1}\big]^{\frac{1}{1-b_1}}\,ds\right)^{1-b_1}\|\mathrm{B}\|\|x\|_{L^2(W,Y)}\\
&\le \frac{P\|\mathrm{B}\|\|x\|_{L^2(W,Y)}}{\sqrt{4b-1}\,\Gamma(2b)}\big(\varpi_2^{4b-1}-(\varpi_2-\varpi_1)^{4b-1}-\varpi_1^{4b-1}\big)^{\frac{1}{2}}.
\end{aligned}$$

Let $M_b(t)=t^{b-1}T_b(t)$ for all $t\in W$. From Lemma 2.9(ii), we get that $M_b(t)$ is a strongly continuous operator. Choosing $\eta_2>0$, we have

$$\begin{aligned}
\mathcal{E}_7 &\le \int_0^{\varpi_1-\eta_2}\big\|\big[M_b(\varpi_2-s)-M_b(\varpi_1-s)\big]\big\|\|\mathrm{B}x\|\,ds\\
&\quad+\int_{\varpi_1-\eta_2}^{\varpi_1}\big\|\big[M_b(\varpi_2-s)-M_b(\varpi_1-s)\big]\big\|\|\mathrm{B}x\|\,ds\\
&\le \sup_{s\in[0,\varpi_1-\eta_2]}\big\|\big[M_b(\varpi_2-s)-M_b(\varpi_1-s)\big]\big\|\int_0^{\varpi_1-\eta_2}\|\mathrm{B}x\|\,ds\\
&\quad+\frac{2P(\varpi_2-\varpi_1+\eta_2)^{2b-1}}{\Gamma(2b)}\int_{\varpi_1-\eta_2}^{\varpi_1}\|\mathrm{B}x\|\,ds\\
&\le \sup_{s\in[0,\varpi_1-\eta_2]}\big\|\big[M_b(\varpi_2-s)-M_b(\varpi_1-s)\big]\big\|\sqrt{\varpi_1}\|\mathrm{B}\|\|x\|_{L^2(W,Y)}\\
&\quad+\frac{2P(\varpi_2-\varpi_1+\eta_2)^{2b-1}}{\Gamma(2b)}\sqrt{\eta_2}\|\mathrm{B}\|\|x\|_{L^2(W,Y)},\\
\mathcal{E}_8 &\le \frac{P}{\Gamma(2b)}\int_{\varpi_1}^{\varpi_2}(\varpi_2-s)^{2b-1}\|\mathrm{B}x\|\,ds\\
&\le \frac{P}{\Gamma(2b)}\left(\int_{\varpi_1}^{\varpi_2}(\varpi_2-s)^{\frac{2b-1}{1-b_1}}\,ds\right)^{1-b_1}\|\mathrm{B}\|\|x\|_{L^2(W,Y)}\\
&\le \frac{P\|\mathrm{B}\|\|x\|_{L^2(W,Y)}}{\sqrt{4b-1}\,\Gamma(2b)}(\varpi_2-\varpi_1)^{2b-\frac{1}{2}}.
\end{aligned}$$

Now Lemma 2.9 and the compactness of $T(t)$ $(t>0)$ gives the continuity of $T_b(t)$ $(t>0)$ in the uniform operator topology. Then $\mathcal{E}_4$, $\mathcal{E}_7\to 0$ independently of $u\in\mathcal{B}_p$ as $\varpi_2\to\varpi_1$, $\eta_2\to 0$, and $\mathcal{E}_1-\mathcal{E}_3$, $\mathcal{E}_5$, $\mathcal{E}_6$, $\mathcal{E}_8\to 0$ as $\varpi_2\to\varpi_1$ does not depend on a particular choice of u. Hence,

$$\big\|\vartheta(\varpi_2)-\vartheta(\varpi_1)\big\|_U\to 0$$

independently of $u\in\mathcal{B}_p$ as $\varpi_2\to\varpi_1$ and $\eta_2\to 0$.

Let $\eta = \min\{\eta_1, \eta_2\}$. In addition, for all $\epsilon > 0$, all $\varpi_1, \varpi_2 \in [0, c]$, $|\varpi_2 - \varpi_1| < \eta$, and all $\vartheta \in \mathcal{H}(\mathcal{B}_p)$, we easily prove that $\|\vartheta(\varpi_2) - \vartheta(\varpi_1)\|_U < \epsilon$ independently of $u \in \mathcal{B}_p$. Hence $\{\mathcal{H}(u) : u \in \mathcal{B}_p\}$ is an equicontinuous family of functions in $\mathbb{C}(W, U)$.

Step 3: For each positive constant p, set $\Upsilon_p = \{u \in U : |u| \leq p\}$. Obviously, Υ_p a bounded subset in U. We need to verify that $\forall\, p > 0$ and $t > 0$,

$$\mathcal{T}(t) = \left\{ \int_0^\infty b\tau S_b(\tau) S\big(t^b \tau\big) u \, d\tau, u \in \Upsilon_p \right\}$$

is relatively compact in U.

Let $t > 0$ be fixed. For every $\eta > 0$ and $0 < \epsilon \leq t$, define the subset in U by

$$\mathcal{T}_{\epsilon,\eta}(t) = \left\{ \frac{S(\epsilon^b \eta)}{\epsilon^b \eta} \int_\eta^\infty b\tau S_b(\tau) S\big(t^b \tau - \epsilon^b \eta\big) u \, d\tau, u \in \Upsilon_p \right\}.$$

Clearly, for each fixed $t > 0$, $\mathcal{T}_{\epsilon,\eta}(t)$ is well-defined. In fact, by the uniformly boundedness of cosine family $\tau \in (\eta, \infty)$, we have for every $u \in \Upsilon_p$,

$$\begin{aligned} \left| \frac{S(\epsilon^b \eta)}{\epsilon^b \eta} \int_\eta^\infty b\tau S_b(\tau) S\big(t^b \tau - \epsilon^b \eta\big) u \, d\tau \right| &\leq P^2 |u| \int_\eta^\infty b\tau S_b(\tau) \big(t^b \tau + \epsilon^b \eta\big) \, d\tau \\ &\leq 2P^2 |u| t^b \int_\eta^\infty b\tau^2 S_b(\tau) \, d\tau \leq \frac{2P^2}{\Gamma(2b)} |u| t^b. \end{aligned}$$

Hence, the set $\mathcal{T}_{\epsilon,\eta}(t)$ is relatively compact since $S(\epsilon^b \eta)$ is compact for $\epsilon^b \eta > 0$.

Moreover, we have

$$\begin{aligned} &\left| \frac{S(\epsilon^b \eta)}{\epsilon^b \eta} \int_\eta^\infty b\tau S_b(\tau) S\big(t^b \tau - \epsilon^b \eta\big) u \, d\tau - \int_0^\infty b\tau S_b(\tau) S\big(t^b \tau\big) u \, d\tau \right| \\ &\quad \leq \left| \frac{S(\epsilon^b \eta)}{\epsilon^b \eta} \int_\eta^\infty b\tau S_b(\tau) S\big(t^b \tau - \epsilon^b \eta\big) u \, d\tau - \int_\eta^\infty b\tau S_b(\tau) S\big(t^b \tau\big) u \, d\tau \right| \\ &\qquad + \left| \int_\eta^\infty b\tau S_b(\tau) S\big(t^b \tau\big) u \, d\tau - \int_0^\infty b\tau S_b(\tau) S\big(t^b \tau\big) u \, d\tau \right| \\ &\quad \leq \int_\eta^\infty b\tau S_b(\tau) \left| \frac{S(\epsilon^b \eta)}{\epsilon^b \eta} S\big(t^b \tau - \epsilon^b \eta\big) u - S\big(t^b \tau\big) u \right| d\tau + \int_0^\eta b\tau S_b(\tau) \big| S\big(t^b \tau\big) u \big| \, d\tau \\ &\quad := l_1 + l_2. \end{aligned}$$

Since

$$b\tau S_b(\tau) \left| \frac{S(\epsilon^b \eta)}{\epsilon^b \eta} S\big(t^b \tau - \epsilon^b \eta\big) u - S\big(t^b \tau\big) u \right| \leq 2P^2 t^b b\tau^2 S_b(\tau) |u|$$

and

$$\int_0^\infty b\tau^2 S_b(\tau) \, d\tau = \frac{2}{\Gamma(1 + 2b)},$$

we can see that

$$\int_0^\infty b\tau S_b(\tau) \left| \frac{S(\epsilon^b \eta)}{\epsilon^b \eta} S\big(t^b \tau - \epsilon^b \eta\big) u - S\big(t^b \tau\big) u \right| d\tau$$

is uniformly convergent. Further, due to the strong continuity of $\{S(t)\}_{t>0}$, for $\tau \in (\eta, \infty)$, using Lemma 2.9, we have

$$\begin{aligned}&\left|\frac{S(\epsilon^b\eta)}{\epsilon^b\eta}S(t^b\tau - \epsilon^b\eta)u - S(t^b\tau)u\right| \\ &\quad \le \left|\frac{S(\epsilon^b\eta)}{\epsilon^b\eta}S(t^b\tau - \epsilon^b\eta)u - S(t^b\tau - \epsilon^b\eta)u\right| + \left|S(t^b\tau - \epsilon^b b)u - S(t^b\tau)u\right| \to 0,\end{aligned}$$

as $b \to 0$. Hence, we have

$$l_1 \le \int_0^\infty b\tau S_b(\tau)\left|\frac{S(\epsilon^b\eta)}{\epsilon^b\eta}S(t^b\tau - \epsilon^b\eta)u - S(t^b\tau)u\right| d\tau \to 0, \quad \text{when } \eta \to 0.$$

On the other hand, since $\int_0^b b\tau^2 S_b(\tau)\,d\tau \to 0$ as $b \to 0$, we have

$$l_2 \le P|u|t^b \int_0^\eta b\tau^2 S_b(\tau)\,d\tau \to 0, \quad \text{as } \eta \to 0.$$

Hence, there are relatively compact sets arbitrarily close to $\mathcal{T}(t)$ for every $t > 0$. Therefore, the set $\mathcal{T}(t)$ is relatively compact in U for every $t > 0$. By Arzela–Ascoli's theorem, $\mathcal{H}$ is completely continuous.

Step 4: We have to prove that $\mathcal{H}$ is upper semicontinuous. First, we prove that $\mathcal{H}$ has a closed graph.

Let $u_k \to u_* \in \mathbb{C}(W, U)$, $\vartheta_k \in \mathcal{H}(u_k)$ and $\vartheta_k \to \vartheta_* \in \mathbb{C}(W, U)$. We prove that $\vartheta_* \in \mathcal{H}(u_*)$. In fact, $\vartheta_k \in \mathcal{H}(u_k)$ implies there exists $g_k \in \mathcal{V}(u_k)$ such that

$$\begin{aligned}\vartheta_k(t) = {}& C_b(t)u_0 + K_b(t)u_1 + \int_0^t (t-s)^{b-1}T_b(t-s)g_k(s)\,ds \\ & + \int_0^t (t-s)^{b-1}T_b(t-s)\mathrm{B}x(s)\,ds, \quad t \in W.\end{aligned} \tag{3.2}$$

Using (H_2)(iii), $\{g_k\}_{k\ge1} \in L^2(W, U)$ is bounded. Then, moving to a subsequence, if necessary, we obtain that

$$g_k \to g_*, \quad \text{weakly in } L^2(W, U). \tag{3.3}$$

From (3.2), (3.3), and the compactness of the operator T_b, we see that

$$\begin{aligned}\vartheta_k(t) \to {}& C_b(t)u_0 + K_b(t)u_1 + \int_0^t (t-s)^{b-1}T_b(t-s)g_*(s)\,ds \\ & + \int_0^t (t-s)^{b-1}T_b(t-s)\mathrm{B}x(s)\,ds, \quad t \in W.\end{aligned} \tag{3.4}$$

Note that $\vartheta_k \to \vartheta_*$ in $\mathbb{C}(W, U)$ and $g_k \in \mathcal{V}(u_k)$. From Lemma 3.2 and (3.4), we have $g_* \in \mathcal{V}(u_*)$. Therefore, we get $\vartheta_* \in \mathcal{H}(u_*)$, thus $\mathcal{H}$ has a closed graph. By [25], $\mathcal{H}$ is upper semicontinuous.

From the above steps, we get that $\mathcal{H}$ is upper semicontinuous, compact and convex valued, and $\mathcal{H}(\mathcal{B}_p)$ is relatively compact.

Step 5: A priori estimate.

We now prove that Φ is bounded to obtain that $\mathcal{H}$ has a fixed point, where Φ is

$$\Phi = \{u \in \mathbb{C}(W,U) : \varphi u \in \mathcal{H}(u), \varphi > 1\}.$$

For any $u \in \Phi$, there exists $g \in \mathcal{V}(u)$ such that

$$\begin{aligned} u(t) &= \varphi^{-1}C_b(t)u_0 + \varphi^{-1}K_b(t)u_1 + \varphi^{-1}\int_0^t (t-s)^{b-1}T_b(t-s)g(s)\,ds \\ &\quad + \varphi^{-1}\int_0^t (t-s)^{b-1}T_b(t-s)\mathrm{B}x(s)\,ds. \end{aligned}$$

Using (H_2)(iii), we obtain

$$\begin{aligned} &\|u(t)\|_U \\ &\quad\le \|C_b(t)u_0\|_U + \|K_b(t)u_1\|_U + \int_0^t (t-s)^{b-1}\|T_b(t-s)g(s)\|_U\,ds \\ &\qquad + \int_0^t (t-s)^{b-1}\|T_b(t-s)\mathrm{B}x(s)\|_U\,ds \\ &\quad\le P\|u_0\|_U + Pc\|u_1\|_U + \frac{P}{\Gamma(2b)}\int_0^t (t-s)^{2b-1}\|g(s)\|_U\,ds \\ &\qquad + \frac{P}{\Gamma(2b)}\int_0^t (t-s)^{2b-1}\|\mathrm{B}x(s)\|_U\,ds \\ &\quad\le P\|u_0\|_U + Pc\|u_1\|_U + \frac{P}{\Gamma(2b)}\int_0^t (t-s)^{2b-1}\big(\beta(s) + j\|u(s)\|_U\big)\,ds \\ &\qquad + \frac{P}{\Gamma(2b)}\int_0^t (t-s)^{2b-1}\|\mathrm{B}\|\,\|x(s)\|_U\,ds \\ &\quad\le P\|u_0\|_U + Pc\|u_1\|_U + \frac{P\|\beta\|_{L^2(W,\mathbb{R}^+)}}{\Gamma(2b)}\left(\int_0^t (t-s)^{\frac{2b-1}{1-b_1}}\,ds\right)^{1-b_1} \\ &\qquad + \frac{Pj}{\Gamma(2b)}\int_0^t (t-s)^{2b-1}\|u(s)\|\,ds + \frac{P}{\Gamma(2b)}\left(\int_0^t (t-s)^{\frac{2b-1}{1-b_1}}\,ds\right)^{1-b_1}\|\mathrm{B}\|\,\|x\|_{L^2(W,Y)} \\ &\quad\le P\|u_0\|_U + Pc\|u_1\|_U + \frac{Pc^{2b-\frac{1}{2}}}{\sqrt{4b-1}\,\Gamma(2b)}\big[\|\beta\|_{L^2(W,\mathbb{R}^+)} + \|\mathrm{B}\|\,\|x\|_{L^2(W,Y)}\big] \\ &\qquad + \frac{Pj}{\Gamma(2b)}\int_0^t (t-s)^{2b-1}\|u(s)\|_U\,ds \\ &\quad\le \wp + \frac{Pj}{\Gamma(2b)}\int_0^t (t-s)^{2b-1}\|u(s)\|_U\,ds, \end{aligned} \tag{3.5}$$

where

$$\wp = P\|u_0\|_U + Pc\|u_1\|_U + \frac{Pc^{2b-\frac{1}{2}}}{\sqrt{4b-1}\,\Gamma(2b)}\big[\|\beta\|_{L^2(W,\mathbb{R}^+)} + \|\mathrm{B}\|\,\|x\|_{L^2(W,Y)}\big].$$

From (3.5) and [39], we easily conclude that

$$\|u(t)\|_U \le \wp E_b\big(Pjt^{2b}\big).$$

Therefore, Φ is bounded. Then by Theorem 2.13, $\mathcal{H}$ has a fixed point u, which is a mild solution of system (3.1). The proof is finished. □

4 Approximate controllability of the linear system

Our focus here is on the approximate controllability of the following linear fractional systems:

$$\begin{cases} {}^{C}D_t^r u(t) = Au(t) + \mathrm{B}x(t), & t \in W := [0,c], 1 < r < 2, \\ u(0) = u_0, \qquad u'(0) = u_1 \in U. \end{cases} \tag{4.1}$$

In the continuation, let us define the bounded linear operator $\mathcal{Q} : L^2(W,U) \to U$ as follows:

$$\mathcal{Q}j = \int_0^c (c-s)^{b-1} T_b(c-s)j(s)\,ds, \quad j(\cdot) \in L^2(W,U).$$

Now, we examine the approximate controllability of (4.1) and introduce the following hypothesis:

(H$_3$) For any $j(\cdot) \in L^2(W,U)$, there exists a function $f(\cdot) \in \overline{R(\mathrm{B})}$ such that $\mathcal{Q}j = \mathcal{Q}f$, where $R(\mathrm{B})$ denotes the range of operator B and $\overline{R(\mathrm{B})}$ is the closure of $R(\mathrm{B})$.

Theorem 4.1 *If* (H$_3$) *is satisfied, then system* (4.1) *is approximately controllable on W if $C(t)$ is a differentiable cosine family.*

Proof Our main idea of the proof comes from the paper [17]. Since the domain $D(A)$ of the operator A is dense in U, it is sufficient to show that $D(A) \subset \overline{N_c(0)}$, that is, for any $\epsilon > 0$ and $\hbar \in D(A)$, there exists $x \in L^2(W,Y)$ such that

$$\left\| \hbar - C_b(c)u_0 - K_b(c)u_1 - \mathcal{Q}\mathrm{B}x \right\|_U < \epsilon. \tag{4.2}$$

Firstly, for any $u_0, u_1 \in U$, we know that $C_b(c)u_0, K_b(c)u_1 \in D(A)$, since $C(t)$ is a differentiable cosine family. Letting $\hbar \in D(A)$, we can be seen that there exists a function $j(\cdot) \in L^2(W,U)$ such that $\mathcal{Q}j = \hbar - C_b(c)u_0 - K_b(c)u_1$.

For example, we have

$$j(t) = \frac{(\Gamma(2b))^2(c-t)^{1-2b}}{c}\left(T_b(c-t) + 2t\frac{dT_b(c-t)}{dt}\right)\left[\hbar - C_b(c)u_0 - K_b(c)u_1\right],$$

with $t \in (0,c)$.

Next, we prove that one can get a control function $x_\alpha \in L^2(W,Y)$ such that the inequality (4.2) holds. From (H$_3$), we know that for $j(\cdot) \in L^2(W,U)$, there exists $f \in \overline{R(\mathrm{B})}$ such that the following equality holds:

$$\mathcal{Q}j = \int_0^c (c-s)^{b-1} T_b(c-s)j(s)\,ds = \int_0^c (c-s)^{b-1} T_b(c-s)f(s)\,ds.$$

Since $f \in \overline{R(\mathrm{B})}$, given $\alpha > 0$, there exists a control function $x_\alpha \in L^2(W,Y)$ such that

$$\|\mathrm{B}x_\alpha - f\|_{L^2} < \frac{\Gamma(2b)}{P}\sqrt{4b-1}c^{\frac{1}{2}-2b}\alpha.$$

Then for $\alpha > 0$ and $x_\alpha \in L^2(W, Y)$ from the above arguments, we obtain

$$\begin{aligned}\left\| \hbar - C_b(c)u_0 - K_b(c)u_1 - \mathcal{Q}\mathrm{B}x \right\|_U &= \| \mathcal{Q}j - \mathcal{Q}\mathrm{B}x_\alpha \|_U \\ &= \| \mathcal{Q}f - \mathcal{Q}\mathrm{B}x_\alpha \|_U \\ &\leq \int_0^c (c-s)^{b-1} \left\| T_b(c-s) \right\| \left\| \mathrm{B}x_\alpha(s) - f(s) \right\|_U ds \\ &\leq \frac{P}{\Gamma(2b)} \left(\int_0^c (c-s)^{\frac{2b-1}{1-b_1}} ds \right)^{1-b_1} \| \mathrm{B}x_\alpha - f \|_U \\ &\leq \frac{Pc^{2b-\frac{1}{2}}}{\sqrt{4b-1}\Gamma(2b)} \| \mathrm{B}x_\alpha - f \|_U < \alpha.\end{aligned}$$

Since α is arbitrary, we can deduce $D(A) \subset \overline{N_c(0)}$. The density of $D(A)$ in U implies the approximate controllability of system (4.1) on W and the proof is finished. □

We consider the two relevant operators connected with (4.1):

$$\Gamma_0^c = \int_0^c (c-s)^{b-1} T_b(c-s)\mathrm{B}\mathrm{B}^* T_b^*(c-s)\, ds$$

and

$$R(\alpha, \Gamma_0^c) = \frac{1}{(\alpha I, \Gamma_0^c)}, \quad \alpha > 0,$$

where B^*, $T_b^*(t)$ denote the adjoints of B and $T_b(t)$, respectively.

Theorem 4.2 (Lemma 2.10 of [21]) *The linear fractional control system* (4.1) *is approximately controllable on* W *if and only if* $\alpha R(\alpha, \Gamma_0^c) \to 0$ *as* $\alpha \to 0^+$ *in the strong operator topology.*

5 Approximate controllability for the semilinear case

In this section, we present our main result on approximate controllability of control system (1.1). Firstly, for any $u \in \mathbb{C}(W, U) \subset L^2(W, U)$, from Lemma 3.1, we know that $\mathcal{V}(u) \neq \emptyset$. Therefore, for every $\alpha > 0$, let us introduce the multivalued map $\mathcal{G}_\alpha : \mathbb{C}(W, U) \to 2^{\mathbb{C}(W,U)}$ as follows:

$$\begin{aligned}\mathcal{G}_\alpha(u) = \biggl\{ & j \in \mathbb{C}(W, U) : j(t) = C_b(t)u_0 + K_b(t)u_1 + \int_0^t (t-s)^{b-1} T_b(t-s)g(s)\, ds \\ & + \int_0^t (t-s)^{b-1} T_b(t-s)\mathrm{B}x_\alpha(s)\, ds, g \in \mathcal{V}(z) \biggr\},\end{aligned}$$

where

$$x_\alpha(t) = \mathrm{B}^* T_b^*(c-t) R(\alpha, \Gamma_0^c) \left[u_c - C_b(t)u_0 - K_b(t)u_1 - \int_0^c (c-\upsilon)^{b-1} T_b(c-\upsilon)g(\upsilon)\, d\upsilon \right].$$

Theorem 5.1 *If the hypotheses* (H_1), (H_2)(i), *and* (H_2)(ii) *are satisfied and, in addition, if there exists* $\phi \in L^2(W, \mathbb{R}^+)$ *such that*

$$\left\| \partial \mathcal{G}(t,u) \right\|_U \le \phi(t), \quad \textit{for a.e. } t \in W, \textit{ for all } u \in U,$$

then $\mathcal{G}_\alpha$ *has a fixed point on the interval* W.

Proof The verification is similar to that of Theorem 3.3. To finish the work, we introduce the straightforward interpretation of our proof. Obviously, for all $u \in \mathbb{C}(W, U), \mathcal{G}_\alpha(u)$ is convex due to $\mathcal{V}(u)$. Now, we split our discussion into the following steps for clarity of exposition:

Step 1: $\mathcal{H}_\alpha$ maps bounded subsets into bounded subsets in $\mathbb{C}(W, U)$.

For any $u \in \mathcal{B}_\ell = \{u \in \mathbb{C}(W, U) : \|u\|_{\mathbb{C}} \le \ell\}$, $\ell > 0$, $\vartheta \in \mathcal{H}_\alpha(u)$, we have $g \in \mathcal{V}(u)$ such that

$$\begin{aligned} \vartheta(t) = {} & C_b(t)u_0 + K_b(t)u_1 + \int_0^t (t-s)^{b-1} T_b(t-s) g(s)\, ds \\ & + \int_0^t (t-s)^{b-1} T_b(t-s) \mathrm{B} x_\alpha(s)\, ds, \quad t \in W. \end{aligned} \tag{5.1}$$

Because $\|\partial \mathcal{G}(t,u)\|_U \le \phi(t)$, and from Hölder inequality, we get

$$\begin{aligned} & \left\| x_\alpha(t) \right\|_U \\ & \quad = \left\| \mathrm{B}^* T_b^*(c-t) R\left(\alpha, \Gamma_0^c\right) \left[u_c - C_b(t)u_0 - K_b(t)u_1 - \int_0^c (c-\upsilon)^{b-1} T_b(c-\upsilon) g(\upsilon)\, d\upsilon \right] \right\|_U \\ & \quad \le \left\| \mathrm{B}^* \right\| \left\| T_b^*(c-t) \right\|_U \left\| R\left(\alpha, \Gamma_0^c\right) \right\|_U \\ & \qquad \times \left\| \left[u_c - C_b(t)u_0 - K_b(t)u_1 - \int_0^c (c-\upsilon)^{b-1} T_b(c-\upsilon) g(\upsilon)\, d\upsilon \right] \right\|_U \\ & \quad \le \left\| \mathrm{B}^* \right\| \frac{P}{\Gamma(2b)} \frac{1}{\alpha} \left[\|u_c\|_U + \left\| C_b(t)u_0 \right\|_U + \left\| K_b(t)u_1 \right\|_U \right. \\ & \qquad \left. + \int_0^c (c-\upsilon)^{b-1} \left\| T_b(c-\upsilon) g(\upsilon) \right\|_U d\upsilon \right] \\ & \quad \le \left\| \mathrm{B}^* \right\| \frac{P}{\alpha \Gamma(2b)} \left[\|u_c\|_U + P\|u_0\|_U + Pc\|u_1\|_U + \frac{P}{\Gamma(2b)} \int_0^c (c-\upsilon)^{2b-1} \left\| g(\upsilon) \right\|_U d\upsilon \right] \\ & \quad \le \left\| \mathrm{B}^* \right\| \frac{P}{\alpha \Gamma(2b)} \left[\|u_c\|_U + P\|u_0\|_U + Pc\|u_1\|_U \right. \\ & \qquad \left. + \frac{P}{\Gamma(2b)} \left(\int_0^c (c-\upsilon)^{\frac{2b-1}{1-b_1}} d\upsilon \right)^{1-b_1} \|\phi\|_{L^2(W,\mathbb{R}^+)} \right] \\ & \quad \le \left\| \mathrm{B}^* \right\| \frac{P}{\alpha \Gamma(2b)} \left[\|u_c\|_U + P\|u_0\|_U + Pc\|u_1\|_U + \frac{P\|\phi\|_{L^2(W,\mathbb{R}^+)}}{\sqrt{4b-1}\,\Gamma(2b)} c^{2b-\frac{1}{2}} \right] := \Theta. \end{aligned} \tag{5.2}$$

Using (5.2), we have

$$\begin{aligned} \left\| \vartheta(t) \right\|_U \le {} & \left\| C_b(t)u_0 \right\|_U + \left\| K_b(t)u_1 \right\|_U + \int_0^t (t-s)^{b-1} \left\| T_b(t-s) g(s) \right\|_U ds \\ & + \int_0^t (t-s)^{b-1} \left\| T_b(t-s) \mathrm{B} x_\alpha(s) \right\|_U ds \end{aligned}$$

$$\begin{aligned}
&\leq P\|u_0\|_U + Pc\|u_1\|_U + \frac{P}{\Gamma(2b)}\int_0^t (t-s)^{2b-1}\big\|g(s)\big\|\,ds \\
&\quad + \frac{P}{\Gamma(2b)}\int_0^t (t-s)^{2b-1}\big\|\mathrm{B}x_\alpha(s)\big\|_U\,ds \\
&\leq P\|u_0\|_U + Pc\|u_1\|_U + \frac{P\|\phi\|_{L^2(W,\mathbb{R}^+)}}{\Gamma(2b)}\biggl(\int_0^t (t-s)^{\frac{2b-1}{1-b_1}}\,ds\biggr)^{1-b_1} \\
&\quad + \frac{P}{\Gamma(2b)}\int_0^t (t-s)^{2b-1}\,ds\|\mathrm{B}\|\big\|x_\alpha(t)\big\|_U \\
&\leq P\|u_0\|_U + Pc\|u_1\|_U + \frac{P}{\Gamma(2b)}\biggl(\frac{\|\phi\|_{L^2(W,\mathbb{R}^+)}c^{2b-\frac{1}{2}}}{\sqrt{4b-1}} + \frac{\|\mathrm{B}\|\Theta c^{2b}}{2b}\biggr).
\end{aligned}$$

Then, we get that $\mathcal{H}_\alpha(\mathcal{B}_\ell)$ is bounded in $\mathbb{C}(W,U)$.

Step 2: Next, we prove that $\{\mathcal{H}_\alpha(u) : u \in \mathcal{B}_\ell\}$ is equicontinuous.

Firstly, for any $u \in \mathcal{B}_\ell$, $\vartheta \in \mathcal{H}_\alpha$, there exists $g \in \mathcal{V}(u)$ such that

$$\begin{aligned}
\vartheta(t) &= C_b(t)u_0 + K_b(t)u_1 + \int_0^t (t-s)^{b-1}T_b(t-s)g(s)\,ds \\
&\quad + \int_0^t (t-s)^{b-1}T_b(t-s)\mathrm{B}x_\alpha(s)\,ds, \quad t \in W.
\end{aligned}$$

From the value of $\|x_\alpha(t)\|$ as in (5.2) and thus *Step 2* of Theorem 3.3, it follows that $\{\mathcal{H}_\alpha(u) : u \in \mathcal{B}_\ell\}$ is equicontinuous.

Step 3: For each positive constant ℓ, set $\Upsilon_\ell = \{u \in U : |u| \leq \ell\}$. Obviously, Υ_ℓ a bounded subset in U. We need to check that for every $\ell > 0$ and $t > 0$,

$$\mathcal{T}(t) = \biggl\{\int_0^\infty b\tau S_b(\tau)S\bigl(t^b\tau\bigr)u\,d\tau, u \in \Upsilon_\ell\biggr\}$$

is relatively compact in U.

Let $t > 0$ be determined. For all $\eta > 0$ and $0 < \epsilon \leq t$, define a subset in U by

$$\mathcal{T}_{\epsilon,\eta}(t) = \biggl\{\frac{S(\epsilon^b\eta)}{\epsilon^b\eta}\int_\eta^\infty b\tau S_b(\tau)S\bigl(t^b\tau - \epsilon^b\eta\bigr)u\,d\tau, u \in \Upsilon_\ell\biggr\}.$$

Clearly, for each fixed $t > 0$, $\mathcal{T}_{\epsilon,\eta}(t)$ is well-defined. Indeed, by using the uniform boundedness of the cosine family $\tau \in (\eta,\infty)$, we obtain that for any $u \in \Upsilon_p$,

$$\begin{aligned}
\biggl|\frac{S(\epsilon^b\eta)}{\epsilon^b\eta}\int_\eta^\infty b\tau S_b(\tau)S\bigl(t^b\tau - \epsilon^b\eta\bigr)u\,d\tau\biggr| &\leq P^2|u|\int_\eta^\infty b\tau S_b(\tau)\bigl(t^b\tau + \epsilon^b\eta\bigr)\,d\tau \\
&\leq 2P^2|u|t^b\int_\eta^\infty b\tau^2 S_b(\tau)\,d\tau \leq \frac{2P^2}{\Gamma(2b)}|u|t^b.
\end{aligned}$$

Hence, the set $\mathcal{T}_{\epsilon,\eta}(t)$ is relatively compact since $S(\epsilon^b\eta)$ is compact for $\epsilon^b\eta > 0$.

Moreover, we have

$$\begin{aligned}
&\biggl|\frac{S(\epsilon^b\eta)}{\epsilon^b\eta}\int_\eta^\infty b\tau S_b(\tau)S\bigl(t^b\tau - \epsilon^b\eta\bigr)u\,d\tau - \int_0^\infty b\tau S_b(\tau)S\bigl(t^b\tau\bigr)u\,d\tau\biggr| \\
&\quad\leq \biggl|\frac{S(\epsilon^b\eta)}{\epsilon^b\eta}\int_\eta^\infty b\tau S_b(\tau)S\bigl(t^b\tau - \epsilon^b\eta\bigr)u\,d\tau - \int_\eta^\infty b\tau S_b(\tau)S\bigl(t^b\tau\bigr)u\,d\tau\biggr|
\end{aligned}$$

$$\begin{aligned}
&+ \left| \int_\eta^\infty b\tau S_b(\tau) S(t^b\tau) u \, d\tau - \int_0^\infty b\tau S_b(\tau) S(t^b\tau) u \, d\tau \right| \\
&\leq \int_\eta^\infty b\tau S_b(\tau) \left| \frac{S(\epsilon^b\eta)}{\epsilon^b\eta} S(t^b\tau - \epsilon^b\eta) u - S(t^b\tau) u \right| d\tau + \int_0^\eta b\tau S_b(\tau) \left| S(t^b\tau) u \right| d\tau \\
&:= l_1 + l_2.
\end{aligned}$$

Since

$$b\tau S_b(\tau) \left| \frac{S(\epsilon^b\eta)}{\epsilon^b\eta} S(t^b\tau - \epsilon^b\eta) u - S(t^b\tau) u \right| \leq 2P^2 t^b b\tau^2 S_b(\tau) |u|$$

and

$$\int_0^\infty b\tau^2 S_b(\tau) \, d\tau = \frac{2}{\Gamma(1+2b)},$$

we can see that

$$\int_0^\infty b\tau S_b(\tau) \left| \frac{S(\epsilon^b\eta)}{\epsilon^b\eta} S(t^b\tau - \epsilon^b\eta) u - S(t^b\tau) u \right| d\tau$$

is uniformly convergent. Further, from the strong continuity of $\{S(t)\}_{t>0}$, for $\tau \in (\eta, \infty)$, using Lemma 2.9, we get

$$\begin{aligned}
&\left| \frac{S(\epsilon^b\eta)}{\epsilon^b\eta} S(t^b\tau - \epsilon^b\eta) u - S(t^b\tau) u \right| \\
&\quad \leq \left| \frac{S(\epsilon^b\eta)}{\epsilon^b\eta} S(t^b\tau - \epsilon^b\eta) u - S(t^b\tau - \epsilon^b\eta) u \right| + \left| S(t^b\tau - \epsilon^b b) u - S(t^b\tau) u \right| \to 0,
\end{aligned}$$

as $b \to 0$. Hence, we get

$$l_1 \leq \int_0^\infty b\tau S_b(\tau) \left| \frac{S(\epsilon^b\eta)}{\epsilon^b\eta} S(t^b\tau - \epsilon^b\eta) u - S(t^b\tau) u \right| d\tau \to 0, \quad \text{when } \eta \to 0.$$

On the other hand, since $\int_0^b b\tau^2 S_b(\tau) \, d\tau \to 0$ as $b \to 0$, we have

$$l_2 \leq P|u| t^b \int_0^\eta b\tau^2 S_b(\tau) \, d\tau \to 0, \quad \text{as } \eta \to 0.$$

Therefore, there are relatively compact sets arbitrarily close to $\mathcal{T}(t)$ for every $t > 0$. Hence, $\mathcal{T}(t)$ is relatively compact in U for any $t > 0$. From Arzela–Ascoli's theorem, we get that, for $\alpha > 0$, $\mathcal{H}_\alpha$ is completely continuous.

Step 4: We have to verify $\mathcal{H}_\alpha$ is upper semicontinuous.

First, we need to verify $\mathcal{H}_\alpha$ has a closed graph.

Let $u_k \to u_* \in \mathbb{C}(W,U)$, $\vartheta_k \in \mathcal{H}_\alpha(u_k)$ and $\vartheta_k \to \vartheta_* \in \mathbb{C}(W,U)$. We prove that $\vartheta_* \in \mathcal{H}_\alpha(u_*)$. In fact, $\vartheta_k \in \mathcal{H}_\alpha(u_k)$ means that there exists $g_k \in \mathcal{V}(u_k)$ such that

$$\begin{aligned}\vartheta_k(t) = {} & C_b(t)u_0 + K_b(t)u_1 + \int_0^t (t-s)^{b-1} T_b(t-s) g_k(s)\,ds \\ & + \int_0^t (t-s)^{b-1} T_b(t-s)\mathrm{BB}^* T_b^*(c-t) R\bigl(\alpha, \Gamma_0^c\bigr) \\ & \times \left[u_c - C_b(t)u_0 - K_b(t)u_1 - \int_0^c (c-\upsilon)^{b-1} T_b(c-\upsilon) g_k(\upsilon)\,d\upsilon\right] ds. \end{aligned} \tag{5.3}$$

Using $\|\partial\mathcal{G}(t,u)\|_U \le \phi(t)$, $\{g_k\}_{k\ge1} \subseteq L^2(W,U)$ is bounded. Therefore, we may assume, passing to a subsequence if necessary, that

$$g_k \to g_* \quad \text{weakly in } L^2(W,U). \tag{5.4}$$

From (5.3), (5.4), and the compactness of the operator $T_b(t)$, we have

$$\begin{aligned}\vartheta_k(t) \to {} & C_b(t)u_0 + K_b(t)u_1 + \int_0^t (t-s)^{b-1} T_b(t-s) g_*(s)\,ds \\ & + \int_0^t (t-s)^{b-1} T_b(t-s)\mathrm{BB}^* T_b^*(c-t) R\bigl(\alpha, \Gamma_0^c\bigr) \\ & \times \left[u_c - C_b(t)u_0 - K_b(t)u_1 - \int_0^c (c-\upsilon)^{b-1} T_b(c-\upsilon) g_*(\upsilon)\,d\upsilon\right] ds, \\ & t \in W. \end{aligned} \tag{5.5}$$

Note that $\vartheta_k \to \vartheta_*$ in $\mathbb{C}(W,U)$ and $g_k \in \mathcal{V}(u_k)$. From Lemma 3.2 and (5.5), we have $g_* \in \mathcal{V}(u_*)$. Therefore, we get $\vartheta_* \in \mathcal{H}_\alpha(u_*)$, and so $\mathcal{H}_\alpha$ has a closed graph. From [25], $\mathcal{H}_\alpha$ is upper semicontinuous.

From the above steps, we get that $\mathcal{H}_\alpha$ is upper semicontinuous, compact and convex valued, and $\mathcal{H}_\alpha(\mathcal{B}_\ell)$ is relatively compact.

Step 5: A priori estimate.

We show that Φ is bounded to get that $\mathcal{H}_\alpha$ has a fixed point, where Φ is

$$\Phi = \bigl\{u \in \mathbb{C}(W,U) : u \in \varphi\mathcal{H}_\alpha(u), \varphi > 1\bigr\}.$$

Let for any $u \in \Phi$, there exists $g \in \mathcal{V}(u)$ such that

$$\begin{aligned}u(t) = {} & \varphi^{-1}C_b(t)u_0 + \varphi^{-1}K_b(t)u_1 + \varphi^{-1}\int_0^t (t-s)^{b-1} T_b(t-s) g(s)\,ds \\ & + \varphi^{-1}\int_0^t (t-s)^{b-1} T_b(t-s)\mathrm{B}\Bigl(\mathrm{B}^* T_b^*(c-t) R\bigl(\alpha, \Gamma_0^c\bigr) \\ & \times \left[u_c - C_b(t)u_0 - K_b(t)u_1 - \int_0^c (c-\upsilon)^{b-1} T_b(c-\upsilon) g(\upsilon)\,d\upsilon\right]\Bigr) ds. \end{aligned}$$

Using (H_2)(iii), we have

$$\begin{aligned}\|u(t)\|_U &\le \|C_b(t)u_0\|_U + \|K_b(t)u_1\|_U + \int_0^t (t-s)^{b-1}\|T_b(t-s)g(s)\|_U\,ds \\ &\quad + \int_0^t (t-s)^{b-1}\|T_b(t-s)\mathrm{B}\|_U \bigg\| \Big(\mathrm{B}^* T_b^*(c-t)R\big(\alpha,\Gamma_0^c\big) \\ &\quad \times \Big[u_c - C_b(t)u_0 - K_b(t)u_1 - \int_0^c (c-\upsilon)^{b-1}T_b(c-\upsilon)g(\upsilon)\,d\upsilon\Big]\Big)\bigg\|_U\,ds.\end{aligned}$$

Using (5.2), we have

$$\begin{aligned}\|u(t)\|_U &= P\|u_0\|_U + Pc\|u_1\|_U + \frac{P\|\phi\|_{L^2(W,\mathbb{R}^+)}}{\sqrt{4b-1}\Gamma(2b)}c^{2b-\frac{1}{2}} + \frac{P^2\|\mathrm{B}\|^2c^{2b}}{\alpha\Gamma(2b+1)\Gamma(2b)} \\ &\quad \times \Big[\|u_c\|_U + P\|u_0\|_U + Pc\|u_1\|_U + \frac{P\|\phi\|_{L^2(W,\mathbb{R}^+)}}{\sqrt{4b-1}\Gamma(2b)}c^{2b-\frac{1}{2}}\Big].\end{aligned}$$

Therefore, Φ is bounded. Then, by Theorem 2.13, one can get that $\mathcal{H}_\alpha$ has a fixed point and the proof is finished. □

Next, we will prove the main results of this article.

Theorem 5.2 *If the hypotheses of Theorem 5.1 are satisfied, then (1.1) is approximately controllable on the interval W if the system (4.1) is approximately controllable on the same interval W.*

Proof In the previous Theorem 5.1, we proved that the operator $\mathcal{H}_\alpha$ has a fixed point in $\mathbb{C}(W,U)$ for any $\alpha > 0$. Let u^α be a fixed point of $\mathcal{H}_\alpha$ in $\mathbb{C}(W,U)$. Hence, there exists $g^\alpha \in \mathcal{V}(u^\alpha)$ such that for any $t \in W$,

$$\begin{aligned}u^\alpha(t) &= C_b(t)u_0 + K_b(t)u_1 + \int_0^t (t-s)^{b-1}T_b(t-s)g^\alpha(s)\,ds \\ &\quad + \int_0^t (t-s)^{b-1}T_b(t-s)\mathrm{B}\Big(\mathrm{B}^* T_b^*(c-t)R\big(\alpha,\Gamma_0^c\big) \\ &\quad \times \Big[u_c - C_b(t)u_0 - K_b(t)u_1 - \int_0^c (c-\upsilon)^{b-1}T_b(c-\upsilon)g^\alpha(\upsilon)\,d\upsilon\Big]\Big)\,ds.\end{aligned}$$

Considering $I \to \Gamma_0^c R(\alpha,\Gamma_0^c) = \alpha R(\alpha,\Gamma_0^c)$, we obtain

$$u^\alpha(c) = u_c - \alpha R\big(\alpha,\Gamma_0^c\big)\mathcal{P}\big(g^\alpha\big),$$

where

$$\mathcal{P}\big(g^\alpha\big) = u_c - C_b(c)u_0 - K_b(c)u_1 - \int_0^c (c-\upsilon)^{b-1}T_b(c-\upsilon)g^\alpha(\upsilon)\,d\upsilon.$$

Because $\|\partial\mathcal{G}(t,u)\|_U \le \phi(t)$, we get

$$\int_0^c \|g^\alpha(s)\|\,ds \le \|\phi\|_{L^2(W,\mathbb{R}^+)}\sqrt{c}.$$

Further, the sequence $\{g^\alpha\}$ is bounded in $L^2(W,U)$. Hence, there is a subsequence, still denoted by $\{g^\alpha\}$, which converges weakly to $g \in L^2(W,U)$. Denote

$$j = u_c - C_b(c)u_0 - K_b(c)u_1 - \int_0^c (c-\upsilon)^{b-1} T_b(c-\upsilon) g^\alpha(\upsilon)\,d\upsilon.$$

Because the corresponding linear system (4.1) is approximately controllable, by Theorem 4.2, we have

$$\alpha R(\alpha, \Gamma_0^c) \to 0 \quad \text{as } \alpha \to 0.$$

Then

$$\begin{aligned} \|\mathcal{P}(g^\alpha) - j\| &= \left\| \int_0^c (c-\upsilon)^{b-1} T_b(c-\upsilon)\big[g^\alpha(\upsilon) - g(\upsilon)\big]\,d\upsilon \right\| \\ &\leq \sup_{t\in W} \left\| \int_0^t (t-\upsilon)^{b-1} T_b(t-\upsilon)\big[g^\alpha(\upsilon) - g(\upsilon)\big]\,d\upsilon \right\| \to 0, \end{aligned}$$

as $\alpha \to 0^+$, due to the compactness of the operator

$$g \to \int_0^{\cdot} (\cdot - \upsilon)^{b-1} T_b(\cdot - \upsilon) g(\upsilon)\,d\upsilon : L^1(W,U) \to \mathbb{C}(W,U).$$

Therefore, we get by the previous arguments

$$\begin{aligned} \|u^\alpha(c) - u_c\| &\leq \|\alpha R(\alpha, \Gamma_0^c)\mathcal{P}(g^\alpha)\| \\ &\leq \|\alpha R(\alpha, \Gamma_0^c)(j)\| + \|\alpha R(\alpha, \Gamma_0^c)[\mathcal{P}(g^\alpha) - j]\| \\ &\leq \|\alpha R(\alpha, \Gamma_0^c)(j)\| + \|\mathcal{P}(g^\alpha) - j\|, \quad \text{as } \alpha \to 0^+. \end{aligned}$$

Hence, the system (1.1) is approximately controllable on W and the proof is finished. □

Remark 5.3 The idea of "nonlocal conditions" has been introduced by Byszewski [8] for the augmentation of issues dependent on classical conditions. When comparing nonlocal and classical initial conditions, which are more precise to depict the nature marvels, since more information is considered, along these lines we lessen the negative impacts initiated by a potentially incorrect single estimation taken toward the beginning time. For exceptionally valuable discussion about differential systems under nonlocal conditions, one can refer to [5, 8, 11, 12, 16, 26–29, 33, 35–38]. The nonlocal term h has a superior impact on the solution and is more accurate for physical measurements than the classical condition $u(0) = u_0$ alone. For example, $h(u)$ can be presented by $h(u) = \sum_{i=1}^{n} c_i u(t_i)$, where c_i $(i = 1, 2, 3, \ldots, m)$ are the constants and $0 < t_1 < \cdots < t_m \leq c$. Inspired by this fact and [5], we may extend our current work to the fractional system with nonlocal conditions too, by replacing the equation $u(0) = u_0 + h(u)$ and introducing the required hypothesis related to the function h.

6 Application

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with sufficiently smooth boundary and $U = Y = L^2(\Omega)$. As an application of the obtained theory, we consider a control system which is represented by fractional partial differential systems of the form

$$\begin{cases} \partial_t^r u(t,\nu) = \Delta u(t,\nu) + \mu(t,\nu) + E(t,\nu), & t \in W, \nu \in \Omega, \\ u(t,\nu) = 0, \quad t \in W, \nu \in \partial\Omega, \\ u(0,\nu) = u_0(\nu), \qquad u'(0,\nu) = u_1(\nu), & \nu \in \Omega, \end{cases} \tag{6.1}$$

where ∂_t^r is the Caputo fractional partial derivative of order $1 < r < 2$ and $x(t)(\nu) = \mu(t,\nu)$. The bounded linear operator B is defined by $\mathrm{B}x = \mu(t,\nu)$, $t \in W$, $0 \leq \nu \leq 1$, $x \in U$. Let us consider $E(t,\nu) = \overline{E}(t,\nu) + \overline{\overline{E}}(t,\nu)$, where $\overline{\overline{E}}(t,\nu)$ is provided and $\overline{E}(t,\nu)$ is a known function of the temperature of the form

$$-\overline{E}(t,\nu) \in \partial\mathcal{G}\big(t,\nu,u(t,\nu)\big), \quad (t,\nu) \in W \times \Omega.$$

In this place, $\mathcal{G} = \mathcal{G}(t,\nu,\zeta)$ is a locally Lipschitz energy function which is generally nonsmooth and nonconvex; $\partial\mathcal{G}$ denotes the generalized Clarke's gradient in the third variable ζ [9]. A simple example of a function $\mathcal{G}$ which satisfies hypothesis (H_2) is $\mathcal{G}(\zeta) = \min\{j_1(\zeta), j_2(\zeta)\}$, where $j_k : \mathbb{R} \to \mathbb{R}$ $(k = 1,2)$ are convex quadratic functions [23]. Dynamic systems modeled by (6.1) arise in the theory of contact mechanics for elastic bodies in many engineering applications. In such a framework, the set Ω stands for a planar deformable purely elastic body which remains in contact with another medium introducing frictional effects. In the system of small deformations, the body is subjected to nonmonotone friction skin effects (skin friction, adhesion, etc.), E is the reaction force of the constraint introducing the skin effect (e.g., due to the gluing material), u is the displacement field.

Let A be the Laplace operator with Dirichlet boundary conditions given by $A = \Delta$ and

$$D(A) = \big\{g \in H_0^1(\Omega), Ag \in L^2(\Omega)\big\}.$$

Clearly, we have $D(A) = H_0^1(\Omega) \cap H^2(\Omega)$. An operator A produces $C(t)$ for $t \geq 0$ (refer to [7]).

Indeed, let $\psi_k = k^2\pi^2$ and $\phi_k(\nu) = \sqrt{(2/\pi)}\sin(k\pi\nu)$, $\forall k \in \mathbb{N}$. Let $\{-\psi_k, \phi_k\}_{k=1}^\infty$ be the eigensystem of operator A, then $0 < \psi_1 \leq \psi_2 \leq \cdots$, $\psi_k \to \infty$ as $k \to \infty$, and $\{\phi_k\}_{k=1}^\infty$ form an orthonormal basis of U. Next,

$$Au = -\sum_{k=1}^{\infty} \psi_k \langle u, \phi_k\rangle \phi_k, \quad u \in D(A),$$

where $\langle \cdot,\cdot \rangle$ denotes an inner product in U. Then the cosine family is given by

$$C(t)u = \sum_{k=1}^{\infty} \cos(\sqrt{\psi_k}t)\langle u, \phi_k\rangle \phi_k, \quad u \in U,$$

and the sine family $S(t)$ associated with cosine family is given by

$$S(t)u = \sum_{k=1}^{\infty} \frac{1}{\sqrt{\psi_k}} \sin(\sqrt{\psi_k} t)\langle u, \phi_k \rangle \phi_k, \quad u \in U.$$

Clearly, $C(t)$ is compact for $t \geq 0$ and $\|C(t)\|_{L_c(U)} \leq 1$, for all $t \in \mathbb{R}$. Let the infinite-dimensional Hilbert space Y be defined by

$$Y := \left\{ x : x = \sum_{k=2}^{\infty} x_k \phi_k, \sum_{k=2}^{\infty} x_k^2 < \infty \right\}.$$

The norm in Y is defined by $\|x\|_Y = (\sum_{k=2}^{\infty} x_k^2)^{\frac{1}{2}}$. Define the mapping $\mathrm{B} \in \mathcal{L}(W, U)$ as follows:

$$\mathrm{B}x = 2x_2\phi_1 + \sum_{k=2}^{\infty} x_k \phi_k \quad \text{for } x = \sum_{k=2}^{\infty} x_k \phi_k \in Y,$$

also let $y = \sum_{k=1}^{\infty} y_k \phi_k \in U$, and consider the inner product $\langle \mathrm{B}x, y \rangle = \langle x, \mathrm{B}^* y \rangle$. Then

$$\mathrm{B}^* y = (2y_1 + y_2)\phi_2 + \sum_{k=3}^{\infty} y_k \phi_k,$$

and

$$\mathrm{B}^* C^*(t) u = \left(2u_1 \phi^{-t} + u_2 \phi^{-4t}\right)\phi_2 + \sum_{k=3}^{\infty} \cos(\sqrt{\phi_k} t) u_k \phi_k.$$

It follows that having $\|\mathrm{B}C^*(t)u\|_U = 0$ for some $t \in W$, implies $u = 0$. Hence, the linear control system corresponding to (6.1) is approximately controllable on W.

Thus, all the assumptions of Theorem 5.2 are satisfied. Therefore (6.1) is approximately controllable on W.

7 Conclusion

In our paper, we discussed the approximate controllability of fractional evolution inclusions with hemivariational inequalities of order $1 < r < 2$. Initially, we presented the existence of the mild solution for the class of fractional systems. After that, we established the approximate controllability of linear and semilinear control systems. In the end, an application was presented to illustrate our theoretical results. In our future work, the focus will be on the hemivariational inequalities for the exact controllability of fractional differential system having order $r \in (1, 2)$ via a measure of noncompactness.

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Authors' contributions
The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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