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# Solution of a fractal energy integral operator without body force using measure of noncompactness

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**Abstract** In this paper, we study the solution of fractal energy integral equation for one-dimensional compressible flows without body force using measure of noncompactness. We also discuss the solution of the local fractal equation of losing energy system using the notion of local fractal differential idea. For this, a new notion of $\chi$-$\Delta$-set contraction condition under simulation function is defined and two main fixed point and coupled fixed point results are obtained.



## 1. Introduction and preliminaries

Compressible flow (or gas dynamics) is a subarea of fluid mechanics that studies flows consuming important fluctuations in fluid density. Whereas all flows are compressible, flows regularly preserved as organism incompressible when the Mach numeral is less than 0.3. The investigation of compressible flow is related to high-speed aircraft, rocket engines, plane engines, high-speed entrance into an environmental atmosphere, gas tubes, profit-making applications such as rough carpeting, and many other areas [7–14]. One-dimensional $(1-D)$ flow rises to flow of gas through a duct or canal in which the flow parameters are considered for adjustment suggestively lengthways-solitary one spatial dimension, namely, the duct length.

The idea is based on imposing the energy integral equation for 1-D compressible flows without body force. Our methodology is based on the locally arbitrary calculus [16]. We consider the integral operator of the energy given by the formula of locally arbitrary integral

$$\Lambda^{(\sigma)}\Psi(\upsilon) = \frac{1}{\Gamma(1+\sigma)} \int_{\alpha}^{\beta} \Psi(\upsilon)(d\upsilon)^{\sigma}, \tag{1.1}$$

where $\Psi(\upsilon) = \psi_{\sigma}^{2}(\upsilon)$, where $\psi_{\sigma}$ represents the fractal sub-band signal, which is a local fractional continuous in $J = [\alpha, \beta]$. In this connection, we use the concept of fixed point theory under measure of noncompactness (MNC, for short) to get the solution of Eq. (1.1).

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In 1930, Kuratowski [6] discussed a new direction of research with the notion of MNC that combines with some algebraic arguments are useful for studying the mathematical formulations, particularly for solving the existence of solutions of some nonlinear problems under certain conditions. The Kuratowskii and Hausdorff MNC in a metric space are well-known in the literature. Fixed point theory has two main branches: Constructive fixed point theorems in the line of Banach Contraction Principle, and nonconstructive fixed point theorems, where results are obtained by using topological properties in the direction of Brouwer's/Schauder's/Darbo's fixed point theorem. Schauder discussed the convexity of domains and the compactness of operators. Darbo relaxed the strong condition of compactness of operators with the use of MNC and defined appropriate classes of operators [3].

Throughout the paper, $\mathbb{R}$ = the set of real numbers, $\mathbb{N}$ = the set of natural numbers, $\mathbb{R}_+ = [0, +\infty)$ and $\mathbb{N}^* = \mathbb{N} \cup \{0\}$.

Let $(\mathscr{X}, \|\cdot\|)$ be a real Banach space and $\theta$ be its zero element. $\mathscr{B}(\vartheta, \zeta)$ will denote the closed ball with center $\vartheta$ are radius $\zeta$ and $\mathscr{B}_\zeta$ will stand for $\mathscr{B}(\theta, \zeta)$. Moreover, $\mathfrak{M}_{\mathscr{X}}$ will denote the family of nonempty bounded subsets of $\mathscr{X}$ and $\mathfrak{N}_{\mathscr{X}}$ its subfamily consisting of all relatively compact sets.

**Definition 1.1.** [1] A mapping $\chi : \mathfrak{M}_{\mathscr{X}} \to \mathbb{R}_+$ is said to be a MNC in $\mathscr{X}$ if it satisfies the following conditions $(\mathscr{Y}, \mathscr{Y}_1, \mathscr{Y}_2 \in \mathfrak{M}_{\mathscr{X}})$:

(1°) $ker\,\chi := \{\mathscr{Y} \in \mathfrak{M}_{\mathscr{X}} : \chi(\mathscr{Y}) = 0\} \neq \emptyset$ and $ker\,\chi \subset \mathfrak{N}_{\mathscr{X}}$,
(2°) $\mathscr{Y}_1 \subseteq \mathscr{Y}_2 \Rightarrow \chi(\mathscr{Y}_1) \leqslant \chi(\mathscr{Y}_2)$,
(3°) $\chi(\overline{\mathscr{Y}}) = \chi(\mathscr{Y})$,
(4°) $\chi(\overline{Conv\,\mathscr{Y}}) = \chi(\mathscr{Y})$,
(5°) $\chi(\lambda\mathscr{Y}_1 + (1-\lambda)\mathscr{Y}_2) \leqslant \lambda\chi(\mathscr{Y}_1) + (1-\lambda)\chi(\mathscr{Y}_2)$ for $\lambda \in [0, 1]$,
(6°) $\chi(\mathscr{Y}_1 \cup \mathscr{Y}_2) = \max\{\chi(\mathscr{Y}_1), \chi(\mathscr{Y}_2)\}$,
(7°) If $(\mathscr{Y}_n)$ is a decreasing sequence of non-empty closed sets in $\mathfrak{M}_{\mathscr{X}}$ and $\lim_{n\to\infty}\chi(\mathscr{Y}_n) = 0$, then the set $\mathscr{Y}_\infty = \bigcap_{n=1}^{\infty}\mathscr{Y}_n$ is non-empty and compact.

A map $\alpha : \mathfrak{M}_{\mathscr{X}} \to \mathbb{R}_+$ is said to be a Kuratowski MNC [6] if

$$\alpha(\mathscr{Y}) = \inf\left\{\epsilon > 0 : \mathscr{Y} \subset \bigcup_{k=1}^{n}\mathscr{S}_k, \mathscr{S}_k \subset \mathscr{X}, diam(\mathscr{S}_k) < \epsilon\,(k \in \mathbb{N})\right\}. \quad (1.2)$$

The following extensions of topological Schauder fixed point theorem and classical Banach fixed point theorem were proved by Darbo (resp. Sadovskii) in 1955 (resp. 1972).

We denote $\Lambda(\mathscr{X})$ a nonempty, bounded, closed and convex set on Banach space $\mathscr{X}$.

**Theorem 1.2.** [3] *Let $\mathscr{X}$ be a Banach space, $\mathscr{Y} \in \Lambda(\mathscr{X})$ and $\mathscr{T} : \mathscr{Y} \to \mathscr{Y}$ be a continuous operator such that there exists a $\lambda \in [0, 1)$ with*

$$\chi(\mathscr{T}(\mathscr{A})) \leqslant \lambda\chi(\mathscr{A})$$

*for any $\emptyset \neq \mathscr{A} \subset \mathscr{Y}$, where $\chi$ is the Kuratowski MNC on $\mathscr{X}$. Then we have $\mathscr{T}$ has a fixed point.*

**Theorem 1.3.** [15] *Let $\mathscr{X}$ be a Banach space, $\mathscr{Y} \in \Lambda(\mathscr{X})$ and $\mathscr{T} : \mathscr{Y} \to \mathscr{Y}$ be a continuous operator such that*

$$\chi(\mathscr{A}) > 0 \Rightarrow \chi(\mathscr{T}(\mathscr{A})) < \chi(\mathscr{A}),$$

for any $\emptyset \neq \mathscr{A} \subset \mathscr{Y}$, where $\chi$ is the Kuratowski MNC on $\mathscr{X}$. Then we can conclude that $\mathscr{T}$ has a fixed point.

The paper is organized as follows. In Section 1, we give some preliminaries. In Section 2, we give a new $\chi$-$\Delta$-set contraction condition under simulation function and derive two main fixed point while Section 3 is devoted for coupled fixed point results. In the final Section 4, we study the solution of fractal energy integral equation for one-dimensional compressible flows without body force using measure of noncompactness while in SubSection 4.2, we discuss the solution of the local fractal equation of losing energy system using the notion of local fractal differential.

## 2. Main results

**Definition 2.1.** [2] A continuous mapping $F : \mathbb{R}_+^2 \to \mathbb{R}$ is called a $\mathscr{C}$-class function if it is satisfies

(1) $F(s, t) \leqslant s$,
(2) $F(s, t) = s$ implies that either $s = 0$ or $t = 0$, for all $s, t \in \mathbb{R}_+$.

**Definition 2.2.** [2] A $\mathscr{C}$-class function has a property $\mathscr{C}_F$, if there exists a $\mathscr{C}_F \geqslant 0$ such that

(1) $F(s, t) > \mathscr{C}_F \Rightarrow s > t$,
(2) $F(t, t) \leqslant \mathscr{C}_F$, for all $s, t \in \mathbb{R}_+$.

**Definition 2.3.** [2]. Let $\Delta(\Theta, \mathscr{C}_F)$ be the family of extended $\mathscr{C}_F$-simulation functions $\Theta : \mathbb{R}_+^2 \to \mathbb{R}$ satisfying following conditions:

($\Delta_1$) $\Theta(s, t) < F(t, s)$ for all $s, t > 0$, where $F \in \mathscr{C}$ with property $\mathscr{C}_F$;
($\Delta_2$) if $\{s_n\}, \{t_n\} \in (0, +\infty)$such that $\lim_{n\to\infty}s_n = \lim_{n\to\infty}t_n = \ell$, where $\ell \in (0, +\infty)$ and $t_n > \ell$ for all $n \in \mathbb{N}$, then $\limsup_{n\to\infty}\Theta(s_n, t_n) < \mathscr{C}_F$;
($\Delta_3$) if $\{s_n\} \in (0, +\infty)$ such that $\lim_{n\to\infty}s_n = \ell \in (0, +\infty)$, $\Theta(s_n, \ell) \geqslant \mathscr{C}_F$ implies $\ell = 0$.

Denote $\Psi$ by a collection of continuous and strictly increasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$.

We define that a notion of $\chi$-$\Delta$-set contractive operator in $\Lambda$.

**Definition 2.4.** A self operator $\mathscr{T}$ on $\mathscr{U} \in \Lambda(\mathscr{X})$ is said to be $\chi$-$\Delta$-set contractive if there exist $\Theta \in \Delta(\Theta, \mathscr{C}_F)$ and the function $\psi \in \Psi$ such that

$$\Theta(\chi(\mathscr{T}\mathscr{V}) + \psi(\chi(\mathscr{T}\mathscr{V})), \chi(\mathscr{V}) + \psi(\chi(\mathscr{V}))) \geqslant \mathscr{C}_F$$

for every $\emptyset \neq \mathscr{V} \subseteq \mathscr{U}$, where $\chi$is an arbitrary MNC.

**Theorem 2.5.** *A continuous $\chi$-$\Delta$-set contractive self operator $\mathscr{T}$ on $\mathscr{U} \in \Lambda$ has at least one fixed point in $\mathscr{U}$.*

**Proof.** Starting with the assumption $\mathscr{U}_0 = \mathscr{U}$, we define a sequence $\{\mathscr{U}_n\}$ such that $\mathscr{U}_{n+1} = \overline{Conv}(\mathscr{T}\mathscr{U}_n)$, for $n \in \mathbb{N}^*$. If $\chi(\mathscr{U}_{n_0}) + \psi(\chi(\mathscr{U}_{n_0})) = 0$, that is, $\chi(\mathscr{U}_{n_0}) = 0$ for some natural number $n_0 \in \mathbb{N}$, then $\mathscr{U}_{n_0}$ is compact and since $\mathscr{T}(\mathscr{U}_{n_0}) \subseteq \overline{Conv}(\mathscr{T}\mathscr{U}_{n_0}) = \mathscr{U}_{n_0+1} \subseteq \mathscr{U}_{n_0}$. Thus we conclude the result from Schauder theorem. For all $n \in \mathbb{N}^*$, we assume that $\chi(\mathscr{U}_n) + \psi(\chi(\mathscr{U}_n)) > 0$. Then from $\chi$-$\Delta$-set contractivity and Definition 1.1 ($4^0$), we get

$$\begin{aligned} &\Theta(\chi(\mathscr{U}_{n+1}) + \psi(\chi(\mathscr{U}_{n+1})), \chi(\mathscr{U}_n) + \psi(\chi(\mathscr{U}_n))) \\ = \ &\Theta(\chi(\overline{Conv}(\mathscr{T}\mathscr{U}_n)) + \psi(\chi(\overline{Conv}(\mathscr{T}\mathscr{U}_n))), \\ &\chi(\mathscr{U}_n) + \psi(\chi(\mathscr{U}_n))) \\ = \ &\Theta(\chi(\mathscr{T}\mathscr{U}_n) + \psi(\chi(\mathscr{T}\mathscr{U}_n)), \chi(\mathscr{U}_n) + \psi(\chi(\mathscr{U}_n))) \\ \geqslant \ &\mathscr{C}_F, \end{aligned} \tag{2.1}$$

that is,

$$\begin{aligned} \mathscr{C}_F &\leqslant \Theta(\chi(\mathscr{U}_{n+1}) + \psi(\chi(\mathscr{U}_{n+1})), \chi(\mathscr{U}_n) + \psi(\chi(\mathscr{U}_n))) \\ &\leqslant F(\chi(\mathscr{U}_n) + \psi(\chi(\mathscr{U}_n)), \chi(\mathscr{U}_{n+1}) + \psi(\chi(\mathscr{U}_{n+1}))) \\ &\chi(\mathscr{U}_n) + \psi(\chi(\mathscr{U}_n)) > \chi(\mathscr{U}_{n+1}) + \psi(\chi(\mathscr{U}_{n+1})) \end{aligned} \tag{2.2}$$

for all $n \in \mathbb{N}$.

This indicate that $\{\chi(\mathscr{U}_n) + \psi(\chi(\mathscr{U}_n))\}$ is a decreasing sequence of positive real numbers. Thus there exists $\gamma \geqslant 0$ such that $\{\chi(\mathscr{U}_n) + \psi(\chi(\mathscr{U}_n))\} \to \gamma$ as $n \to \infty$. Assume $\gamma > 0$. Using property of Definition 2.3 ($\Delta_2$), for sequence $\zeta_n = \chi(\mathscr{U}_{n+1}) + \psi(\chi(\mathscr{U}_{n+1}))$ and $\xi_n = \chi(\mathscr{U}_n) + \psi(\chi(\mathscr{U}_n))$ then $\zeta_n, \xi_n \to \gamma$ and $\xi_n > \gamma$,

$$\begin{aligned} \mathscr{C}_F &\leqslant \limsup_{n\to\infty} \Theta(\chi(\mathscr{U}_{n+1}) + \psi(\chi(\mathscr{U}_{n+1})), \chi(\mathscr{U}_n) + \psi(\chi(\mathscr{U}_n))) \\ &= \limsup_{n\to\infty} \Theta(\zeta_n, \xi_n) < \mathscr{C}_F, \end{aligned}$$

a contradiction, thus $\gamma = 0$ and $\chi(\mathscr{U}_n) + \psi(\chi(\mathscr{U}_n)) \to 0$ as $n \to \infty$. That is, $\lim_{n\to\infty} \chi(\mathscr{U}_n) = 0, \lim_{n\to\infty} \psi(\chi(\mathscr{U}_n)) = 0$. Since $\mathscr{U}_n \supseteq \mathscr{U}_{n+1}$ and $\mathscr{T}\mathscr{U}_n \subseteq \mathscr{U}_n$ for all $n = 1, 2, \ldots$, then by ($7^0$) of Definition 1.1, $\mathscr{U}_\infty = \bigcap_{n=1}^{\infty} \mathscr{U}_n$ is nonempty convex closed set, invariant under $\mathscr{T}$ and belongs to $ker\chi$. So, Schauder's fixed point theorems gives the requested result. □

**Corollary 2.6.** *A continuous self operator $\mathscr{T}$ on $\mathscr{U} \in \Lambda(\mathscr{X})$ satisfying*

$$\Theta(\chi(\mathscr{T}\mathscr{V}) + \psi(\chi(\mathscr{T}\mathscr{V})), \chi(\mathscr{V}) + \psi(\chi(\mathscr{V}))) \geqslant 0$$

for every $\varnothing \neq \mathscr{V} \subseteq \mathscr{U}$, where $\Psi \ni \psi$ has at least one fixed point in $\mathscr{U}$ and $\chi$is an arbitrary MNC, $\Theta \in \Delta(\Theta, \mathscr{C}_F)$ .

**Proof.** If we set $\Delta(\Theta, \mathscr{C}_F)$ with $C_F = 0$ in Theorem 2.5, we get the result. □

**Proposition 2.7.** *A continuous self operator $\mathscr{T}$ on $\mathscr{U} \in \Lambda(\mathscr{X})$ satisfying*

$$\begin{aligned} &\Theta(diam(\mathscr{T}(\mathscr{V})) + \psi(diam(\mathscr{T}(\mathscr{V}))), \\ &diam(\mathscr{V}) + \psi(diam(\mathscr{V}))) \geqslant \mathscr{C}_F, \end{aligned} \tag{2.3}$$

for any $\varnothing \neq \mathscr{V}$ of $\mathscr{U}$, where $\Theta \in \Delta(\Theta, \mathscr{C}_F)$ and $\psi \in \Psi$, then $\mathscr{T}$ admits a unique fixed point in $\mathscr{U}$.

**Proof.** Theorem 2.5 and Proposition 3.2 [4] claim the existence of a $\mathscr{T}$-invariant nonempty closed convex subset $\mathscr{U}$ with $diam(\mathscr{U}_\infty) = 0$, that is, $\mathscr{U}_\infty$ has singleton element, hence fixed point of $\mathscr{T} \neq \varnothing$.

Next suppose $v \neq \vartheta \in \Omega$ is different fixed point, then we define that the set $\mathscr{U} := \{v, \vartheta\}$. Moreover, in this case $diam(\mathscr{U}) = diam(\mathscr{T}(\mathscr{U})) = \|v - \vartheta\| > 0$, by using (2.3), we have

$$\begin{aligned} \mathscr{C}_F &\leqslant \Theta(diam(\mathscr{T}(\mathscr{U})) + \psi(diam(\mathscr{T}(\mathscr{U}))), diam(\mathscr{U}) + \psi(diam(\mathscr{U}))) \\ &\leqslant F(diam(\mathscr{U}) + \psi(diam(\mathscr{U})), diam(\mathscr{U}) + \psi(diam(\mathscr{U}))), \end{aligned}$$

a contradiction from Definition 2.2 and hence the result.

Next, we show a classical fixed point theorem in the following theorem.

**Theorem 2.8.** *A continuous self operator $\mathscr{T}$ on $\mathscr{U} \in \Lambda(\mathscr{X})$ satisfying*

$$\Theta(\|\mathscr{T}\omega - \mathscr{T}\hat{\omega}\| + \psi(\|\mathscr{T}\omega - \mathscr{T}\hat{\omega}\|), \|\omega - \hat{\omega}\| + \psi(\|\omega - \hat{\omega}\|)) \geqslant \mathscr{C}_F \tag{2.4}$$

for all $\omega, \hat{\omega} \in \mathscr{U}$, where $\Theta \in \Delta(\Theta, \mathscr{C}_F)$ and $\psi \in \Psi$. Then $\mathscr{T}$ admits a unique fixed point.

**Proof.** Suppose that $\chi(\mathscr{U}) = diam\mathscr{U}$, where $diam\mathscr{U} = \sup\{\|\omega - \hat{\omega}\| : \omega, \hat{\omega} \in \mathscr{U}\}$ is the diameter of $\mathscr{U}$. Clearly, in the sense of Definition 1.1, we have $\chi$ is a MNC in a space $\mathscr{X}$. Therefore, since (2.4) we obtain

Proposition 2.7 implies that $\mathscr{T}$ has a unique fixed point.

$$\begin{aligned} \mathscr{C}_F \leqslant \ &\sup_{\omega,\hat{\omega}\in\mathscr{U}} \Theta(\|\mathscr{T}\omega - \mathscr{T}\hat{\omega}\| + \psi(\|\mathscr{T}\omega - \mathscr{T}\hat{\omega}\|), \|\omega - \hat{\omega}\| + \psi(\|\omega - \hat{\omega}\|)) \\ \leqslant \ &\Theta\left(\sup_{\omega,\hat{\omega}\in\mathscr{U}} \|\mathscr{T}\omega - \mathscr{T}\hat{\omega}\| + \sup_{\omega,\hat{\omega}\in\mathscr{U}} \psi(\|\mathscr{T}\omega - \mathscr{T}\hat{\omega}\|), \sup_{\omega,\hat{\omega}\in\mathscr{U}} \|\omega - \hat{\omega}\| + \sup_{\omega,\hat{\omega}\in\mathscr{U}} \psi(\|\omega - \hat{\omega}\|)\right) \\ = \ &\Theta(diam(\mathscr{T}(\mathscr{U})) + \psi(diam(\mathscr{T}(\mathscr{U}))), diam(\mathscr{U}) + \psi(diam(\mathscr{U}))). \end{aligned}$$

## 3. Coupled fixed point results

**Definition 3.1.** [5] An argument $(v,\vartheta)\in\mathscr{X}^2$ is said to be a coupled fixed point (CFP) of a mapping $\mathscr{H}:\mathscr{X}^2\to\mathscr{X}$ if $\mathscr{H}(v,\vartheta)=v$ and $\mathscr{H}(\vartheta,v)=\vartheta$.

**Theorem 3.2.** *Let $\mathscr{U}\in\Lambda(\mathscr{X})$ and $\mathscr{H}:\mathscr{U}^2\to\mathscr{U}$ is continuous operator satisfying*

$$2\Theta\big(\chi\big(\mathscr{H}\big(\mathscr{W}_i\times\mathscr{W}_j\big)\big),\chi(\mathscr{W}_i)+\chi\big(\mathscr{W}_j\big)\big)\geqslant\mathscr{C}_F, i\neq j\in\{1,2\} \tag{3.1}$$

for all $\mathscr{W}_1,\mathscr{W}_2$ in $\mathscr{U}$, where $\Theta\in\Delta(\Theta,\mathscr{C}_F)$ with $\Theta(P+Q,R)\geqslant\Theta(P,R)+\Theta(Q,R)$, for $P,Q,R\in\mathbb{R}_+$, then $\mathscr{H}$ admits at least a CFP.

**Proof.** Consider the map $\widehat{\mathscr{H}}:\mathscr{U}^2\to\mathscr{U}^2$ having the definition $\widehat{\mathscr{H}}(\zeta,\xi)=(\mathscr{H}(\zeta,\xi),\mathscr{H}(\xi,\zeta))$. Define MNC in $\mathscr{U}^2$ by $\widehat{\chi}(\mathscr{W})=\chi(\mathscr{W}_1)+\chi(\mathscr{W}_2)$,where $\mathscr{W}=\mathscr{W}_1\times\mathscr{W}_2$ and $\mathscr{W}_i, i=1,2$ denote the natural projections of $\mathscr{U}$. Assume $\varnothing\neq\mathscr{W}$, from (3.1) and Definition 1.1 ($2^0$),

$$\begin{aligned}\Theta\Big(\widehat{\chi}\Big(\widehat{\mathscr{H}}(\mathscr{W})\Big),\widehat{\chi}(\mathscr{W})\Big) &= \Theta(\widehat{\chi}(\mathscr{H}(\mathscr{W}_1\times\mathscr{W}_2)\times\mathscr{H}(\mathscr{W}_2\times\mathscr{W}_1)),\chi(\mathscr{W}_1)+\chi(\mathscr{W}_2))\\ &= \Theta(\chi(\mathscr{H}(\mathscr{W}_1\times\mathscr{W}_2)+\chi(\mathscr{H}(\mathscr{W}_2\times\mathscr{W}_1)),\chi(\mathscr{W}_1)+\chi(\mathscr{W}_2)))\\ &\geqslant \Theta(\chi(\mathscr{H}(\mathscr{W}_1\times\mathscr{W}_2),\chi(\mathscr{W}_1)+\chi(\mathscr{W}_2)))\\ &\quad+\Theta(\chi(\mathscr{H}(\mathscr{W}_2\times\mathscr{W}_1)),\chi(\mathscr{W}_1)+\chi(\mathscr{W}_2)))\\ &\geqslant \mathscr{C}_F,\end{aligned}$$

that is,

$$\Theta\Big(\widehat{\chi}\Big(\widehat{\mathscr{H}}(\mathscr{W})\Big),\widehat{\chi}(\mathscr{W})\Big)\geqslant\mathscr{C}_F.$$

Hence, from Theorem 2.5 for $\psi(t)=0,\widehat{\mathscr{H}}$ has at least one fixed point, i.e., $\mathscr{H}$ admits a CFP.

**Theorem 3.3.** Let $\mathscr{U}\in\Lambda(\mathscr{X})$ and $\mathscr{H}:\mathscr{U}^2\to\mathscr{U}$ is continuous operator satisfying

$$\begin{aligned}&\Theta\big(\chi\big(\mathscr{H}\big(\mathscr{W}_i\times\mathscr{W}_j\big)\big),\max\{\chi(\mathscr{W}_i),\chi(\mathscr{W}_j)\}\big)\\ &\quad\geqslant\mathscr{C}_F, i\neq j\in\{1,2\}\end{aligned} \tag{3.2}$$

for all $\mathscr{W}_1,\mathscr{W}_2$ in $\mathscr{U}$, where $\Theta\in\Delta(\Theta,\mathscr{C}_F)$, then $\mathscr{H}$ admits at least a CFP.

**Proof.** Consider the map $\mathscr{H}:\mathscr{U}^2\to\mathscr{U}^2$ having the definition $\widehat{\mathscr{H}}(v,v)=(\mathscr{H}(v,v),\mathscr{H}(v,v))$. Define MNC in $\mathscr{U}^2$ by $\widehat{\chi}(\mathscr{W})=\max\{\chi(\mathscr{W}_1),\chi(\mathscr{W}_2)\}$, where $\mathscr{W}_i, i=1,2$ denote the natural projections of $\mathscr{U}$. Assume $\varnothing\neq\mathscr{W}$, from (3.1) and Definition 1.1 ($2^0$),

$$\begin{aligned}\Theta\Big(\widehat{\chi}\Big(\widehat{\mathscr{H}}(\mathscr{W})\Big),\widehat{\chi}(\mathscr{W})\Big) &= \Theta(\widehat{\chi}(\mathscr{H}(\mathscr{W}_1\times\mathscr{W}_2)\times\mathscr{H}(\mathscr{W}_2\times\mathscr{W}_1)),\max\{\chi(\mathscr{W}_1),\chi(\mathscr{W}_2)\})\\ &= \Theta(\max\{\chi(\mathscr{H}(\mathscr{W}_1\times\mathscr{W}_2)),\chi(\mathscr{H}(\mathscr{W}_2\times\mathscr{W}_1))\},\max\{\chi(\mathscr{W}_1),\chi(\mathscr{W}_2)\})\\ &= \max\left\{\begin{array}{l}\Theta(\chi(\mathscr{H}(\mathscr{W}_1\times\mathscr{W}_2)),\max\{\chi(\mathscr{W}_1),\chi(\mathscr{W}_2)\}),\\ \Theta(\chi(\mathscr{H}(\mathscr{W}_2\times\mathscr{W}_1)),\max\{\chi(\mathscr{W}_2),\chi(\mathscr{W}_1)\})\end{array}\right\}\\ &\geqslant \mathscr{C}_F,\end{aligned}$$

that is,

$$\Theta\Big(\widehat{\chi}\Big(\widehat{\mathscr{H}}(\mathscr{W})\Big),\widehat{\chi}(\mathscr{W})\Big)\geqslant\mathscr{C}_F.$$

Hence, from Theorem 2.5 for $\psi(t)=0,\widehat{\mathscr{H}}$ has at least one fixed point, namely, $\mathscr{H}$ admits a CFP.

## 4. Applications

This section is devoted into two parts. The first part is regarding the existence solution of local fractional integral equation. While, the second is about the solvability of local fractional differential equation.

### *4.1. Local fractional integral equation*

We consider the integral operator of the energy given by the formula of locally arbitrary integral

$$\Lambda^{(\sigma)}\Psi(v)=\frac{1}{\Gamma(1+\sigma)}\int_\alpha^\beta\Psi(v)(dv)^\sigma, \tag{4.1}$$

where $\Psi(\upsilon) = \psi_\sigma^2(\upsilon)$, where $\psi_\sigma$ represents the fractal sub-band signal, which is a local fractional continuous in $J = [\alpha, \beta]$. Assume that the local fractional integral of $\Psi(\upsilon)$ on the closed interval $J$ be equal to $\Sigma$. And for each $\rho > 0$ there occurs $\varrho > 0$ such that $|\upsilon_1 - \upsilon_2| < \varrho^\sigma$ implies

$$|\Sigma - \Lambda^{(\sigma)}\Psi(\upsilon)| = |\Sigma - \frac{1}{\Gamma(1+\sigma)} \int_\alpha^\beta \Psi(\upsilon)(d\upsilon)^\sigma| < \rho^\sigma, \quad 0 < \sigma \leqslant 1. \tag{4.2}$$

In the following theorem, we present a result about a unique fixed point.

**Theorem 4.1.** Let $\tau : \mathscr{X} \to \mathscr{X}$ be a self mapping achieving the following assumption with $(\mathscr{X}, \|.\|)$ be a complete metric space

$$\frac{1}{\Gamma(1+\sigma)} \int_0^{\|\tau\chi - \tau y\|} \Psi(\upsilon)(d\upsilon)^\sigma \leqslant \|\chi - y\|, \tag{4.3}$$

where $\Psi : [0, \infty) \to [0, \infty)$ is an integrable function fulfilling the integral inequality

$$\int_0^{\|\tau\chi - \tau y\|} \Psi(\upsilon)(d\upsilon)^\sigma := \int_0^{\rho^\sigma} \Psi(\upsilon)(d\upsilon)^\sigma > \rho^\sigma. \tag{4.4}$$

This leads that $\tau$ admits a unique fixed point.

**Proof.** Define the function $\Upsilon : [0, \infty) \times [0, \infty) \to \mathbb{R}$ by

$$\Upsilon(\upsilon, \varsigma) = \varsigma - \frac{1}{\Gamma(1+\sigma)} \int_0^\upsilon \Psi(v)(dv)^\sigma, \quad \upsilon, \varsigma \in [0, \infty).$$

Consequently, we get the following facts

- $\Upsilon(0, 0) = 0$;
- $\Upsilon(\upsilon, \varsigma) < \varsigma - \frac{\rho^\sigma}{\Gamma(\sigma+1)}$;
- $\Upsilon(\varsigma, \varsigma) < 0, \varsigma < \frac{\rho^\sigma}{\Gamma(\sigma+1)}$.

Thus, $\Upsilon$ is a simulation function. Now, we proceed to complete all the conditions of Theorem 2.8. A computation implies that

$$\begin{aligned}
&\sup_{\omega,\hat\omega\in\mathscr{U}} \Upsilon(\|\tau\omega - \tau\hat\omega\| + \psi(\|\tau\omega - \tau\hat\omega\|), \|\omega - \hat\omega\| + \psi(\|\omega - \hat\omega\|)) \\
&= \sup_{\omega,\hat\omega\in\mathscr{U}} \left(\|\tau\omega - \tau\hat\omega\| + \psi(\|\tau\omega - \tau\hat\omega\|) - \tfrac{1}{\Gamma(1+\sigma)} \int_0^{\|\omega-\hat\omega\|+\psi(\|\omega-\hat\omega\|\Psi(v)(dv)^\sigma)}\right. \\
&= \left(\sup_{\omega,\hat\omega\in\mathscr{U}} \|\tau\omega - \tau\hat\omega\| + \psi(\|\tau\omega - \tau\hat\omega\|) - \sup_{\omega,\hat\omega\in\mathscr{U}} \tfrac{1}{\Gamma(1+\sigma)} \int_0^{\|\omega-\hat\omega\|+\psi(\|\omega-\hat\omega\|\Psi(v)(dv)^\sigma)}\right. \\
&\leqslant \Upsilon\left(\sup_{\omega,\hat\omega\in\mathscr{U}} \|\tau\omega - \tau\hat\omega\| + \sup_{\omega,\hat\omega\in\mathscr{U}} \psi(\|\tau\omega - \tau\hat\omega\|), \sup_{\omega,\hat\omega\in\mathscr{U}} \|\omega - \hat\omega\| + \sup_{\omega,\hat\omega\in\mathscr{U}} \psi(\|\omega - \hat\omega\|)\right) \\
&\qquad = \Upsilon(diam(\tau(\mathscr{U})) + \psi(diam(\tau(\mathscr{U}))), diam(\mathscr{U}) + \psi(diam(\mathscr{U}))) > 0.
\end{aligned}$$

Therefore, $\tau$ has a unique fixed point.

**Example 4.2.**

- Consider $\Psi(\upsilon) = C$, where $C$ is a constant such that $diam(\mathscr{U}) > \frac{C\upsilon^\sigma}{\Gamma(1+\sigma)}$. This implies that

$$\frac{1}{\Gamma(1+\sigma)} \int_0^{\upsilon=\tau\chi} C(d\upsilon)^\sigma = \frac{C\upsilon^\sigma}{\Gamma(1+\sigma)}$$

  and hence,

$$\Upsilon(\upsilon, \varsigma) = \|\omega - \hat\omega\| - \frac{C\upsilon^\sigma}{\Gamma(1+\sigma)}.$$

  Consequently, we obtain $\Upsilon(\upsilon, \|\omega - \hat\omega\|) > 0$. In view of Theorem 4.1, a self map $\tau\chi = \chi$. has a unique fixed point.
- We have the following data $\Psi(\upsilon) = \sin_\sigma(\upsilon^\sigma)$ where $diam(\mathscr{U}) \geqslant 1$. This yields that

$$\frac{1}{\Gamma(1+\sigma)} \int_0^{\upsilon=\tau\chi} \sin_\sigma(v^\sigma)(dv)^\sigma = 1 - \cos_\sigma(\upsilon^\sigma)$$

  and thus,

$$\Upsilon(\upsilon, \varsigma) = \|\omega - \hat\omega\| - (1 - \cos_\sigma(v^\sigma)) > 0, \quad v \in [0, \infty).$$

  From the above results, we can conclude that $\tau\chi = \chi$ also has a fixed point and it is unique.
- Assume the following data $\Psi(\upsilon) = \cos_\sigma(\upsilon^\sigma)$ where $diam(\mathscr{U}) \geqslant 1$. This gives that

$$\frac{1}{\Gamma(1+\sigma)} \int_0^{\tau\chi} \cos_\sigma(v^\sigma)(dv)^\sigma = \sin_\sigma(\upsilon^\sigma).$$

  Therefore, we obtain

$$\Upsilon(\upsilon, \varsigma) = \|\omega - \hat\omega\| - \sin_\sigma(\upsilon^\sigma) > 0, \quad v \in [0, \infty).$$

  This leads that $\tau\chi = \chi$ also has a fixed point and it is unique.

### 4.2. Local fractional differential equation

In this place, we consider a local fractional differential equation by using the following local derivative (see [16], P18) for a function $\varphi$ in the continuous fractal space:

$$\Delta^\sigma f(\upsilon) = \frac{\Gamma(\sigma+1)[f(\upsilon) - f(0)]}{(\upsilon - \upsilon_0)^\sigma}, \quad \sigma \in (0, 1].$$

By using the local fractal differential idea, we suggest the local fractal equation of losing energy as follows:

$$\Delta^\sigma f(\upsilon) = F(\upsilon, f(\upsilon)), \tag{4.5}$$

where $f$ is the power function in the interval $[0, j]$ such that $f(0) = f_0$. And $F$ is the conductance of the dielectric material. The variation of $F$ can be recognized from $f$ (power function) limits to be autonomous of frequency that is there occurs a positive constant $\delta$ such that

$$\delta \leqslant \sup_{\upsilon\in[0,j]} |F(\upsilon, f)| \leqslant \delta|f|, \quad \delta \in (0, \infty).$$

We have the following result:

**Theorem 4.3.** Define the self mapping $P : \mathscr{X} \to \mathscr{X}$, where $(\mathscr{X}, \|.\|)$ is a complete metric space fulfilling the conditions

$$\frac{1}{\Gamma(1+\sigma)} \int_0^{\|Px - Py\|} F(\upsilon, f)(d\upsilon)^\sigma \leqslant \|x - y\|, \tag{4.6}$$

and

$$\int_0^{\|Px - Py\|} F(\upsilon, f)(d\upsilon)^\sigma := \int_0^{\delta^\sigma} F(\upsilon, f)(d\upsilon)^\sigma > \delta. \tag{4.7}$$

Then $P$ indicates a unique fixed point.

**Proof.** In the same manner of Theorem 4.1, we define the function $\Bbbk : [0, \infty) \times [0, \infty) \to \mathbb{R}$ by

$$\Bbbk(\upsilon, t) = t - \frac{1}{\Gamma(1+\sigma)} \int_0^{\upsilon} F(v, f(v))(dv)^{\sigma}, \quad \upsilon, t \in [0, \infty).$$

It is clear that

- $\Bbbk(0, 0) = 0$;
- $\Bbbk(\upsilon, t) < t - \frac{\delta}{\Gamma(\sigma+1)}$;
- $\Bbbk(t, t) < 0,\ t < \frac{\delta}{\Gamma(\sigma+1)}$.

Thus, $\Bbbk$ is a simulation function. A calculation leads to

$$\begin{aligned}
&\sup_{\omega,\hat{\omega}\in\mathcal{U}} \Bbbk(\|P\omega - P\hat{\omega}\| + \psi(\|P\omega - P\hat{\omega}\|), \|\omega - \hat{\omega}\| + \psi(\|\omega - \hat{\omega}\|)) \\
&= \sup_{\omega,\hat{\omega}\in\mathcal{U}} \Big( \|P\omega - P\hat{\omega}\| + \psi(\|P\omega - P\hat{\omega}\|) - \tfrac{1}{\Gamma(1+\sigma)} \int_0^{\|\omega-\hat{\omega}\|+\psi(\|\omega-\hat{\omega}\| F(v,f(v))(dv)^{\sigma})} \\
&= \Bigg( \sup_{\omega,\hat{\omega}\in\mathcal{U}} \|P\omega - P\hat{\omega}\| + \psi(\|P\omega - P\hat{\omega}\|) - \sup_{\omega,\hat{\omega}\in\mathcal{U}} \tfrac{1}{\Gamma(1+\sigma)} \int_0^{\|\omega-\hat{\omega}\|+\psi(\|\omega-\hat{\omega}\| F(v,f(v))(dv)^{\sigma})} \\
&= \Bigg( \sup_{\omega,\hat{\omega}\in\mathcal{U}} \|P\omega - P\hat{\omega}\| + \psi(\|P\omega - P\hat{\omega}\|) - \sup_{\omega,\hat{\omega}\in\mathcal{U}} \tfrac{\delta}{\Gamma(1+\sigma)} \int_0^{\|\omega-\hat{\omega}\|+\psi(\|\omega-\hat{\omega}\|(dv)^{\sigma})} \\
&\leqslant \Bbbk\Bigg( \sup_{\omega,\hat{\omega}\in\mathcal{U}} \|P\omega - P\hat{\omega}\| + \sup_{\omega,\hat{\omega}\in\mathcal{U}} \psi(\|P\omega - P\hat{\omega}\|), \sup_{\omega,\hat{\omega}\in\mathcal{U}} \|\omega - \hat{\omega}\| + \sup_{\omega,\hat{\omega}\in\mathcal{U}} \psi(\|\omega - \hat{\omega}\|) \Bigg) \\
&= \Bbbk(diam(\tau(\mathcal{U})) + \psi(diam(\tau(\mathcal{U}))), diam(\mathcal{U}) + \psi(diam(\mathcal{U}))) > 0.
\end{aligned}$$

Therefore, in view of Theorem 2.8, $P$ has a unique fixed point. □

## 5. Conclusion

In this work, we consider the fractal energy integral equation for one-dimensional compressible flows without body force using measure of noncompactness followed by an example. We also use local fractal differential idea, to solve the local fractal equation of losing energy system. For this we introduce a new notion of $\mu$-$\Delta$-set contraction condition under the simulation function. Moreover, we also show two main fixed point and coupled fixed point results.

## Declaration of Competing Interest

None.

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