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Solution of time fractional Black-Scholes European option pricing equation arising in financial market

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Abstract: In this paper, we present fractional differential transform method (FDTM) and modified fractional differential transform method (MFDTM) for the solution of time fractional Black-Scholes European option pricing equation. The method finds the solution without any discretization, transformation, or restrictive assumptions with the use of appropriate initial or boundary conditions. The efficiency and exactitude of the proposed methods are tested by means of three examples.

Keywords: Fractional Black-Scholes equation, Fractional differential transform method, Modified fractional differential transform method

1 Introduction

In the past few decades, financial securities became significant tools for corporates and investors. A principal problem in financial investment is the pricing of options for example, to hedge assets and portfolios in order to control the risk due to the movement in stock prices. The famous theoretical valuation formula for options derived by Fischer Black and Myron Scholes [1] in 1973. The central theoretical idea of Black and Scholes lie in the construction of a riskless portfolio taking positions in bonds (cash), option and the underlying stock. This methodology reinforces the use of the no-arbitrage principle as well. Thus, the Black-Scholes formula is used as a model for valuing European or American call and put options on a non-dividend paying stock [2]. The major difference between the European and American option is that American option can be exercised at any time up to the date while the European option can be exercised only on a specified future date. In [3–11], many researchers premeditated the existence of solutions of the Black-Scholes equation.

In recent past, the glorious developments have been envisaged in the field of fractional calculus and fractional differential equations. Differential equations involving fractional order derivatives are used to model a variety of systems, of which the important applications lie in field of viscoelasticity, electrode–electrolyte polarization, heat conduction, electromagnetic waves, diffusion equations and so on [12, 13]. Several definitions of a fractional derivative of order $\alpha > 0$ [14] such as, Riemann-Liouville, Grunwald-Letnikow, Caputo and generalised functions approach. The most commonly used definitions are the Riemann-Liouville and Caputo. Readers can refer the basic definitions and properties of fractional calculus theory in [13, 14]. In recent times, fractional partial differential equation was presented further into financial theory. In [15] presented the fractional Black-Scholes equation with a time-fractional derivative to price European call option. Several fractional diffusion models of option prices in markets with jumps and priced barrier option using fractional partial differential equation given in [16]. Jumarie [17, 18] derived the time- and space-fractional Black-Scholes equations and obtained optimal fractional Merton's portfolio.

Consider the time fractional Black-Scholes equation

$$\frac{\partial^\alpha v(x,t)}{\partial t^\alpha} + \frac{\sigma^2 x^2}{2}\frac{\partial^2 v(x,t)}{\partial x^2} + r(t)x\frac{\partial v(x,t)}{\partial x} - r(t)v(x,t) = 0, \quad 0 < \alpha \le 1 \tag{1}$$

Subject to the conditions

$$v(x,T) = \max(x - E, 0), \quad x \in R^+, \quad v(0,t) = 0, \quad t \in [0,T] \tag{2}$$

where $v(x,t)$ is the European call option price at asset price x and at time t, $\sigma(x,t)$ represents the volatility function of underlying asset, $r(t)$ is the risk free interest rate, T is the maturity and E denotes the expiration price. A wide range of research has been carried out for analytical and semi-analytical methods to study the fractional Black-Scholes equation and it plays a noticeable role in financial marketing. Due to its remarkable scope and applications in several disciplines, a considerable attention has been

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given to exact and numerical solutions of fractional Black-Scholes equation. Some of the methods such as Laplace transform [19], Finite difference method [20], Adomian decomposition method(ADM) [21], Homotopy perturbation method (HPM) and Homotopy analysis method(HAM) [22]. The proposed FDTM and MFDTM do not require linearization, discretization or perturbation unlike the method discussed in the literature. The main drawback of the ADM is to calculate Adomian polynomials for a nonlinear operator where the procedure is very complex. The difficulty in VIM has an inherent inaccuracy in identifying the Lagrange multiplier, correctional functional and stationary conditions for the fractional order. The disadvantage of the Homotopy perturbation method is to solve functional equation in each iteration, which is sometimes complicated and unattainable. Therefore, the proposed FDTM and MFDTM are much easier when compared with ADM, VIM and HPM.

The main aim of this paper is to extend the FDTM and MFDTM to obtain analytic and approximate solution of time fractional Black-Scholes equations. To the best of author's knowledge no paper has been reported yet for the solution of time fractional Black-Scholes equation using FDTM and MFDTM. The differential transform method obtains an analytical solution in the form of a polynomial. It is different from the traditional high order Taylor's series method, which requires symbolic competition of the necessary derivatives of the data functions. The Taylor series method is computationally taken long time for large orders. With this method, it is possible to obtain highly accurate results or exact solutions for differential equations. The use of differential transform method in electric circuit analysis was pioneered by Zhou [23]. Since then, differential transform method was successfully applied for large variety of problems such as partial differential equations [24, 25], solitary wave solutions for the KdV and mKdV equations [26], linear and nonlinear Schrodinger equations [27], linear and nonlinear Klein-Gordon equations [28], nonlinear oscillators with fractional nonlinearities [29], fractional linear and nonlinear schrodinger equation [30], nonlinear fractional Klein-Gordon Equation [31], (*1*+*n*)-dimensional Burger's equation [32], HIV infection of CD4+T cells mathematical model [33], Black-Scholes pricing model of European option valuation [34, 35] and references therein. Recently, in [36] two dimensional extended differential transform method has been used for solving the local fractional diffusion equation.

As we know that, FDTM is based on Taylor series for all variables. Even though the proposed FDTM does not require linearization, discretization or perturbation it also encounters difficulties while handling with the non-linear functions. For example, let us consider the fractional differential transform for $u^3(x, t)$ involves four summations i.e.

$$u^3(x,t)=\sum_{r=0}^{k}\sum_{q=0}^{k-r}\sum_{s=0}^{h}\sum_{p=0}^{h-s}U_{\alpha,1}(r,h-s-p)\,U_{\alpha,1}(q,s)U_{\alpha,1}(k-r-q,p) \tag{3}$$

Thus it is necessary to have a lot of computational work to calculate such differential transform $U_{\alpha,1}(k, h)$ for the large number of (k, h). Hence, we introduce the modified version of the standard FDTM. Instead of considering the Taylor series of $u(x, t)$ for all variables x and t, in MFDTM, we considered the Taylor's series of the function$u(x, t)$ with respect to the specific variable x or t. The MFDTM of $u^3(x, t)$ for the specific variable t as follows

$$u^3(x,t)=\sum_{m=0}^{h}\sum_{l=0}^{m}U_{\alpha,1}(x,h-m)U_{\alpha,1}(x,l)U_{\alpha,1}(x,m-l) \tag{4}$$

It is observed that MFDTM of $u^3(x, t)$ involves only two summations therefore it minimizes the computation cost and effective method compared with the FDTM.

The outline of this paper is as follows. Two-dimensional FDTM are discussed in section 2. The MFDTM and its definitions presented in section 3. In section 4 applications of FDTM and MFDTM via time fractional Black-Scholes equation are given to elucidate the proposed methods. Conclusions of this work are given in section 5.

2 Two-Dimensional Fractional Differential Transform Method

Consider a function of two variables $u(x, t)$ and suppose that it can be represented as a product of two single variable functions i.e., $u(x, t) = f(x)g(t)$. Based on the properties of two- dimensional fractional differential transform, the function $u(x, t)$ can be represented as

$$u(x,t)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}U_{\alpha,1}(k,h)(x-x_0)^k(t-t_0)^{h\alpha} \tag{5}$$

where $0 < \alpha$, $U_{\alpha,1}(k, h)$ is called the spectrum of $u(x, t)$. The generalized two-dimensional fractional differential transform of the function $u(x, t)$ is given by

$$U_{\alpha,1}=\frac{1}{\Gamma(k+1)\Gamma(\alpha h+1)}\left[\left(D^{1}_{\star x_0}\right)^k\left(D^{\alpha}_{\star t_0}\right)^h u(x,t)\right]_{x_0,t_0} \tag{6}$$

where $\left(D^{\alpha}_{\star t_0}\right)^h=\underbrace{D^{\alpha}_{\star t_0}D^{\alpha}_{\star t_0}\ldots D^{h}_{\star t_0}}_{h}$. In real applications the function $u(x, t)$ is represented by a finite series of (5) can be written as

$$u(x,t)=\sum_{k=0}^{l}\sum_{h=0}^{n}U_{\alpha,1}(k,h)x^k t^{\alpha h}+R_{\ln}(x,t) \tag{7}$$

and (5) implies that $R_{\ln}(x,t)=\sum\limits_{k=l+1}^{\infty}\sum\limits_{h=n+1}^{\infty}U_{\alpha,1}(k,h)x^k t^{\alpha h}$ is negligibly small. Usually, the values of l and n are decided by convergence of the series solution. In case of$\alpha=1$, the generalized two-dimensional fractional differential transform method (5) reduces to classical two-dimensional differential transform [24–29]. The fundamental mathematical operations performed by two-dimensional FDTM are listed in Table 1.

3 Modified Fractional Differential Transform Method

We consider the Taylor series of $u(x,t)$ with respect to the specific variable t then, the Taylor series expansion of the function $u(x,t)$ with respect to the specific variable $t=t_0$ is

$$u(x,t)=\sum_{h=0}^{\infty}\frac{1}{\Gamma(\alpha h+1)}\left(\frac{\partial^{\alpha h}u(x,t)}{\partial t^{\alpha h}}\right)_{t=t_0}(t-t_0)^{\alpha h} \tag{8}$$

The modified fractional differential transform $U_{\alpha,1}(x,h)$ of $u(x,t)$ with respect to the variable t at t_0 is defined by

$$U_{\alpha,1}(x,h)=\frac{1}{\Gamma(\alpha h+1)}\left(\frac{\partial^{\alpha h}u(x,t)}{\partial t^{\alpha h}}\right)_{t=t_0} \tag{9}$$

The modified fractional differential inverse differential transform $U_{\alpha,1}(x,h)$ of $u(x,t)$ with respect to the variable$t$ at t_0 is defined by

$$u(x,t)=\sum_{h=0}^{\infty}U_{\alpha,1}(x,h)(t-t_0)^{\alpha h} \tag{10}$$

In real application, the function $u(x,t)$ is expressed by a finite series and eq. (10) can be written as

$$u(x,t)=\sum_{h=0}^{m}U_{\alpha,1}(x,h)(t-t_0)^{\alpha h}+R_m(x,t) \tag{11}$$

which means that $R_m(x,t)=\sum\limits_{h=m+1}^{\infty}U(x,h)(t-t_0)^h$ is small and negligible. Usually the value of m decided by the convergence of the series.

Since the MFDTM results from the Taylor's series of the function with respect to the specific variable it is expected that the corresponding algebraic equation from the given problem is much simpler than the result obtained by the standard FDTM. The fundamental mathematical operations performed by MFDTM are listed in Table 2.

4 Applications

In this section, three examples are tested to validate the proposed FDTM and MFDTM for solving fractional Black-Scholes equation.

Example 1: First consider the fractional Black-Scholes equation [11],

$$\frac{\partial^\alpha v}{\partial t^\alpha}=\frac{\partial^2 v}{\partial x^2}+(k-1)\frac{\partial v}{\partial x}-kv,\quad 0<\alpha\le 1 \tag{12}$$

Subject to the initial condition

$$v(x,0)=\max(e^x-1,0) \tag{13}$$

where $k=\frac{2r}{\sigma^2}$ and it represents the balance between the rate of interests and the variability of stock returns and the dimensionless time to expiry $\frac{1}{2}\sigma^2 T$, even though there are four dimensional parameters, E, T,σ^2 and r, in the original statement of the problem.

FDTM: The transformed version of eq. (12) is

$$\frac{\Gamma(\alpha(h+1)+1)}{\Gamma(\alpha h+1)}V_{\alpha,1}(m,h+1)=(m+1)(m+2)V_{\alpha,1}(m+2,h)$$
$$+(k-1)(m+1)V_{\alpha,1}(m+1,h)-kV_{\alpha,1}(m,h) \tag{14}$$

The transformed version of eq. (13) is

$$V_{\alpha,1}(m,0)=\max\left(\frac{1}{m!}-\delta(m),0\right),\quad m=0,1,2,\ldots \tag{15}$$

Substituting eq. (15) into eq. (14), yields the $V_{\alpha,1}(m,h)$ values,

$$V_{\alpha,1}(0,1)=\frac{k}{\Gamma(\alpha+1)},\quad V_{\alpha,1}(1,1)=V_{\alpha,1}(2,1)=\ldots=0$$

$$V_{\alpha,1}(0,2)=\frac{-k^2}{\Gamma(2\alpha+1)},\quad V_{\alpha,1}(1,2)=V_{\alpha,1}(2,2)=\ldots=0$$

Using $V_{\alpha,1}(m,h)$ values in (5), we obtained the series solution as

$$v(x,t)=\sum_{m=0}^{\infty}\sum_{h=0}^{\infty}U_{\alpha,1}(m,h)x^m t^{\alpha h}$$
$$=\left(x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\right)+\frac{kt^\alpha}{\Gamma(\alpha+1)}-\frac{k^2t^{2\alpha}}{\Gamma(2\alpha+1)}+\ldots \tag{16}$$

MFDTM: The transformed version of eq. (12) with respect to t is

$$\frac{\Gamma(\alpha(h+1)+1)}{\Gamma(\alpha h+1)}V_{\alpha,1}(x,h+1)=\frac{\partial^2 V_{\alpha,1}(x,h)}{\partial x^2}$$
$$+(k-1)\frac{\partial V_{\alpha,1}(x,h)}{\partial x}-kV_{\alpha,1}(x,h) \tag{17}$$

Table 1: The operations for the two-dimensional FDTM.

Original function	Transformed function
$w(x,t) = u(x,t) \pm v(x,t)$	$W_{\alpha,1}(k,h) = U_{\alpha,1}(k,h) \pm V_{\alpha,1}(k,h)$
$w(x,t) = \mu u(x,t)$	$W_{\alpha,1}(k,h) = \mu U_{\alpha,1}(k,h)$
$w(x,t) = \frac{\partial u(x,t)}{\partial x}$	$W_{\alpha,1}(k,h) = (k+1)U_{\alpha,1}(k+1,h)$
$w(x,t) = D^{\alpha}_{*t_0} u(x,t),\ 0<\alpha\le 1$	$W_{\alpha,1}(k,h) = \frac{\Gamma(\alpha(h+1)+1)}{\Gamma(\alpha h+1)} U_{\alpha,1}(k,h+1)$
$w(x,t) = (x-x_0)^m (t-t_0)^{n\alpha}$	$W_{\alpha,1}(k,h) = \delta(k-m, h\alpha-n) = \begin{cases} 1, & k=m,\ h=n \\ 0, & otherwise \end{cases}$
$w(x,t) = u^2(x,t)$	$W_{\alpha,1}(k,h) = \sum_{m=0}^{k}\sum_{n=0}^{h} U_{\alpha,1}(m,h-n)U_{\alpha,1}(k-m,n)$
$w(x,t) = u^3(x,t)$	$W_{\alpha,1}(k,h) = \sum_{r=0}^{k}\sum_{q=0}^{k-r}\sum_{s=0}^{h}\sum_{p=0}^{h-s} U_{\alpha,1}(r,h-s-p)U_{\alpha,1}(q,s)U_{\alpha,1}(k-r-q,p)$

Table 2: The operations for the two-dimensional MFDTM.

Original function	Transformed function
$w(x,t) = u(x,t) \pm v(x,t)$	$W_{\alpha,1}(x,h) = U_{\alpha,1}(x,h) \pm V_{\alpha,1}(x,h)$
$w(x,t) = \mu u(x,t)$	$W_{\alpha,1}(x,h) = \mu U_{\alpha,1}(x,h)$
$w(x,t) = \frac{\partial u(x,t)}{\partial x}$	$W_{\alpha,1}(x,h) = \frac{\partial U_{\alpha,1}(x,h)}{\partial x}$
$w(x,t) = D^{\alpha}_{*t_0} u(x,t),\ 0<\alpha\le 1$	$W_{\alpha,1}(x,h) = \frac{\Gamma(\alpha(h+1)+1)}{\Gamma(\alpha h+1)} U_{\alpha,1}(x,h+1)$
$w(x,t) = (x-x_0)^m (t-t_0)^{n\alpha}$	$W_{\alpha,1}(x,h) = (x-x_0)^m \delta(h\alpha-n)$
$w(x,t) = u^2(x,t)$	$W_{\alpha,1}(x,h) = \sum_{m=0}^{h} U_{\alpha,1}(x,m)U_{\alpha,1}(x,h-m)$
$w(x,t) = u^3(x,t)$	$W_{\alpha,1}(x,h) = \sum_{m=0}^{h}\sum_{l=0}^{m} U_{\alpha,1}(x,h-m)U_{\alpha,1}(x,l)U_{\alpha,1}(x,m-l)$

The transformed version of eq. (13) is

$$V_{\alpha,1}(x,0) = \max(e^x - 1, 0) \tag{18}$$

The MFDTM recurrence equation (17) yields the $V_{\alpha,1}(x,h)$ values

$$V_{\alpha,1}(x,1) = \frac{k}{\Gamma(\alpha+1)}\left(\max(e^x,0) - \max(e^x-1,0)\right),$$

$$V_{\alpha,1}(x,2) = \frac{k^2}{\Gamma(2\alpha+1)}\left(\max(e^x,0) - \max(e^x-1,0)\right),\ldots$$

Substituting $V_{\alpha,1}(x,h)$'s into (10) , we obtained solution in the following form

$$\begin{aligned} v(x,t) &= \max(e^x-1,0) \\ &\quad + \frac{k}{\Gamma(\alpha+1)}\left(\max(e^x,0) - \max(e^x-1,0)\right)t^{\alpha} \\ &\quad + \frac{k^2}{\Gamma(2\alpha+1)}\left(-\max(e^x,0) + \max(e^x-1,0)\right)t^{2\alpha} + \ldots \end{aligned}$$

$$\begin{aligned} v(x,t) &= \max(e^x,0) \\ &\quad - \max(e^x,0)\sum_{h=0}^{\infty}\frac{(-kt^{\alpha})^h}{\Gamma(h\alpha+1)} \\ &\quad + \max(e^x-1,0)\sum_{h=0}^{\infty}\frac{(-kt^{\alpha})^h}{\Gamma(h\alpha+1)} \end{aligned}$$

$$\begin{aligned} v(x,t) &= \max(e^x,0)(1-E_{\alpha}(-kt^{\alpha})) \\ &\quad + \max(e^x-1,0)E_{\alpha}(-kt^{\alpha}) \end{aligned} \tag{19}$$

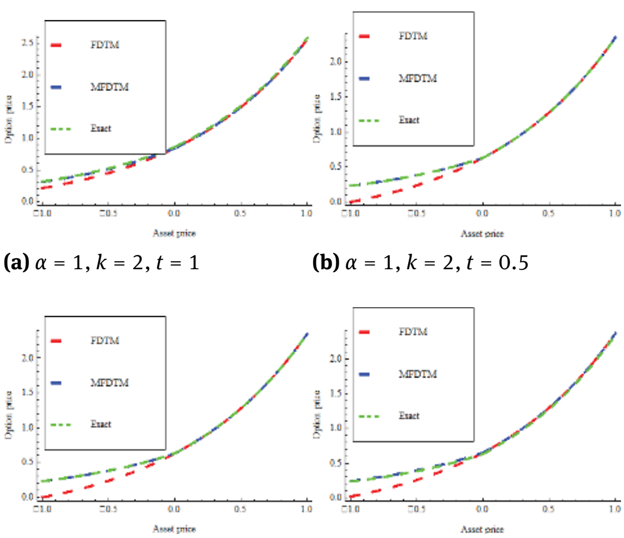
where $E_{\alpha}(-kt^{\alpha})$ is the Mittag-Leffler function defined as [37]. The MFDTM solution obtained in eq. (19) is same as the solution obtained in [22] and it is the exact solution of eqs. (12)–(13). It is well known from the FDTM solution in eq. (16) and MFDTM solution in eq. (19) the FDTM needs more terms in the series to obtain the exact solution. Fig. 1(a)–1(d) presents the comparison of the approximate solution obtained using FDTM and MFDTM with the exact solution for different values of fractional order α for fixed t and k.

Example 2: Consider the generalized Black-Scholes equation [5]

$$\frac{\partial^{\alpha} v}{\partial t^{\alpha}} + 0.08(2+\sin x)^2 x^2 \frac{\partial^2 v}{\partial x^2} + 0.06x\frac{\partial v}{\partial x} - 0.06v = 0, \quad 0<\alpha\le 1 \tag{20}$$

Subject to the initial condition

$$v(x,0) = \max(x - 25e^{-0.06}, 0) \tag{21}$$



(a) $\alpha = 1, k = 2, t = 1$ (b) $\alpha = 1, k = 2, t = 0.5$

(c) $\alpha = 0.7, k = 2, t = 0.5$ (d) $\alpha = 0.8, k = 2, t = 0.5$

Fig. 1: Comparison of the approximate solution obtained using FDTM and MFDTM with the exact solution.

FDTM: The transformed version of eq. (20) is

$$
\begin{aligned}
&\frac{\Gamma(\alpha(h+1)+1)}{\Gamma(\alpha h+1)} V_{\alpha,1}(m, h+1) \\
&+ 0.36 \sum_{l=0}^{m} \sum_{n=0}^{h} \delta(l-2, h-n)(m-l+1)(m-l+2) \\
&\times V_{\alpha,1}(m-l+2, n) \\
&- 0.04 \sum_{l=0}^{m} \sum_{q=0}^{m-l} \sum_{n=0}^{h} \sum_{p=0}^{h-n} \delta(l-2, h-n-p) \frac{2^q \cos\left(\frac{q\pi}{2}\right)}{q!} \\
&\times (m-l-q+1)(m-l-q+2) V_{\alpha,1}(m-l-q+2, p) \\
&+ 0.32 \sum_{l=0}^{m} \sum_{n=0}^{h} \delta(l-2, h-n)(m-l+1)(m-l+2) \\
&\times V_{\alpha,1}(m-l+2, n) \\
&+ 0.06 \sum_{l=0}^{m} \sum_{n=0}^{h} \delta(l-1, h-n)(m-l+1) \times V_{\alpha,1}(m-l+1, h) \\
&- 0.06 V_{\alpha,1}(m, h) = 0 \qquad (22)
\end{aligned}
$$

The transformed version of eq. (21) is

$$
V_{\alpha,1}(m, 0) = \max\left(\delta(m-1) - 25e^{-0.06}, 0\right), \quad m = 0, 1, 2, \ldots \qquad (23)
$$

Substituting eq. (23) into eq. (22), yields the $V_{\alpha,1}(m, h)$ values,

$$
\begin{aligned}
&V_{\alpha,1}(0, 1) = 0, \quad V_{\alpha,1}(1, 1) = -\frac{0.06}{\Gamma(\alpha+1)}, \\
&V_{\alpha,1}(2, 1) = V_{\alpha,1}(3, 1) = \ldots = 0, \\
&V_{\alpha,1}(0, 2) = 0, \quad V_{\alpha,1}(1, 2) = -\frac{(0.06)^2}{\Gamma(2\alpha+1)}, \\
&V_{\alpha,1}(2, 2) = V_{\alpha,1}(3, 2) = \ldots = 0, \ldots
\end{aligned}
$$

Using $V_{\alpha,1}(m, h)$ values in (5), we obtained the series solution as

$$
\begin{aligned}
v(x, t) &= \sum_{m=0}^{\infty} \sum_{h=0}^{\infty} U_{\alpha,1}(m, h) x^m t^{\alpha h} \\
&= -x \left(\frac{0.06 t^{\alpha}}{\Gamma(\alpha+1)} + \frac{(0.06)^2 t^{2\alpha}}{\Gamma(2\alpha+1)} + \ldots \right) \qquad (24)
\end{aligned}
$$

MFDTM: The transformed version of eq. (20) with respect to t is

$$
\begin{aligned}
&\frac{\Gamma(\alpha(h+1)+1)}{\Gamma(\alpha h+1)} V_{\alpha,1}(x, h+1) \\
&+ 0.08(2+\sin x)^2 x^2 \frac{\partial^2 V_{\alpha,1}(x, h)}{\partial x^2} \\
&+ 0.06x \frac{\partial V_{\alpha,1}(x, h)}{\partial x} - 0.06 V_{\alpha,1}(x, h) = 0 \qquad (25)
\end{aligned}
$$

The transformed version of eq. (21) is

$$
V_{\alpha,1}(x, 0) = \max(x - 25e^{-0.06}, 0) \qquad (26)
$$

The MFDTM recurrence equation (25) yields the $V_{\alpha,1}(x, h)$values

$$
\begin{aligned}
V_{\alpha,1}(x, 1) &= \frac{0.06}{\Gamma(\alpha+1)} \left(\max(x - 25e^{-0.06}, 0) - x \right), \\
V_{\alpha,1}(x, 2) &= \frac{(0.06)^2}{\Gamma(2\alpha+1)} \left(\max(x - 25e^{-0.06}, 0) - x \right), \ldots
\end{aligned}
$$

Substituting $V_{\alpha,1}(x, h)$'s into (10) , we obtained solution in the following form

$$
\begin{aligned}
v(x, t) = {} & \max(x - 25e^{-0.06}, 0) \\
& + \frac{0.06}{\Gamma(\alpha+1)} \left(\max(x - 25e^{-0.06}, 0) - x \right) t^{\alpha} \\
& + \frac{(0.06)^2}{\Gamma(2\alpha+1)} \left(\max(x - 25e^{-0.06}, 0) - x \right) t^{2\alpha} + \ldots
\end{aligned} \qquad (27)
$$

The approximate solution obtained in eq. (24) and eq. (27) is same as the solution obtained in [22]. When $\alpha = 1$ eq. (24) and eq. (27) takes the following form $v(x, t) = \max(x - 25e^{-0.06}, 0)e^{0.06t} + x(1 - e^{0.06t})$ and $v(x, t) = -x\left(\frac{0.06t}{1!} + \frac{(0.06t)^2}{2!} + \ldots\right)$ respectively.

Example 3: Finally, consider the following fractional Black-Scholes option pricing equation [35]

$$
\frac{\partial^{\alpha} v}{\partial t^{\alpha}} + \frac{\sigma^2}{2} x^2 \frac{\partial^2 v}{\partial x^2} + (r - \tau) x \frac{\partial v}{\partial x} - r v = 0, \quad 0 < \alpha \leq 1 \qquad (28)
$$

Subject to the initial condition

$$
v(x, 0) = \max(Ax - B, 0) \qquad (29)
$$

FDTM: The transformed version of eq. (28) is

$$\begin{aligned}&\frac{\Gamma(\alpha(h+1)+1)}{\Gamma(\alpha h+1)}V_{\alpha,1}(m,h+1)\\&+\frac{\sigma^2}{2}\sum_{l=0}^{m}\sum_{n=0}^{h}\delta(l-2,h-n)(m-l+1)(m-l+2)\\&\times V_{\alpha,1}(m-l+2,n)\\&+(r-\tau)\sum_{l=0}^{m}\sum_{n=0}^{h}\delta(l-1,h-n)(m-l+1)\\&\times V_{\alpha,1}(m-l+1,n)-rV_{\alpha,1}(m,h)=0\end{aligned}\tag{30}$$

The transformed version of eq. (29) is

$$V_{\alpha,1}(m,0)=\max\left(A\delta(m-1)-B\delta(m),0\right),\quad m=0,1,2,\ldots\tag{31}$$

Substituting eq. (31) into eq. (30), yields the $V_{\alpha,1}(m,h)$ values,

$$V_{\alpha,1}(0,1)=0,\quad V_{\alpha,1}(1,1)=\frac{\tau\, max(A,0)}{\Gamma(\alpha+1)},$$
$$V_{\alpha,1}(2,1)=V_{\alpha,1}(3,1)=\ldots=0,$$

$$V_{\alpha,1}(0,2)=0,\quad V_{\alpha,1}(2,1)=\frac{\tau^2\, max(A,0)}{\Gamma(2\alpha+1)},$$
$$V_{\alpha,1}(2,2)=V_{\alpha,1}(3,2)=\ldots=0,\ldots$$

Using $V_{\alpha,1}(m,h)$ values in (5), we obtained the series solution as

$$\begin{aligned}v(x,t)&=\sum_{m=0}^{\infty}\sum_{h=0}^{\infty}U_{\alpha,1}(m,h)x^m t^{\alpha h}\\&=x\max(A,0)\left(1+\frac{\tau t^{\alpha}}{\Gamma(\alpha+1)}+\frac{\tau^2t^{2\alpha}}{\Gamma(2\alpha+1)}+\ldots\right)\end{aligned}\tag{32}$$

MFDTM: The transformed version of eq. (28) with respect to t is

$$\begin{aligned}&\frac{\Gamma(\alpha(h+1)+1)}{\Gamma(\alpha h+1)}V_{\alpha,1}(x,h+1)+\sigma^2x^2\frac{\partial^2V_{\alpha,1}(x,h)}{\partial x^2}\\&+(r-\tau)x\frac{\partial V_{\alpha,1}(x,h)}{\partial x}-rV_{\alpha,1}(x,h)=0\end{aligned}\tag{33}$$

The transformed version of eq. (29) is

$$V_{\alpha,1}(x,0)=\max(Ax-B,0)\tag{34}$$

The MFDTM recurrence equation (33) yields the $V_{\alpha,1}(x,h)$ values

$$V_{\alpha,1}(x,1)=\frac{1}{\Gamma(\alpha+1)}\left(r\max(Ax-B,0)-(r-\tau)x\max(A,0)\right),$$

$$V_{\alpha,1}(x,2)=\frac{1}{\Gamma(2\alpha+1)}\left(r^2\max(Ax-B,0)-(r^2-\tau^2)x\max(A,0)\right),\ldots$$

Substituting $V_{\alpha,1}(x,h)$'s into (10) , we obtained solution in the following form

$$\begin{aligned}v(x,t)=&\max(Ax-B,0)\left(1+\frac{rt^{\alpha}}{\Gamma(\alpha+1)}+\frac{r^2t^{2\alpha}}{\Gamma(2\alpha+1)}\right.\\&\left.+\frac{r^3t^{3\alpha}}{\Gamma(3\alpha+1)}+\ldots\right)-\max(A,0)x\left(\frac{(r-\tau)}{\Gamma(\alpha+1)}t^{\alpha}\right.\\&\left.+\frac{(r^2-\tau^2)}{\Gamma(2\alpha+1)}t^{2\alpha}+\frac{(r^3-\tau^3)}{\Gamma(3\alpha+1)}t^{3\alpha}+\ldots\right)\end{aligned}\tag{35}$$

The MFDTM solution obtained in eq. (35) is same as the solution obtained in [35].

Fig. 2(a–c), 3(a–c) presents the comparison of the approimate solution obtained by FDTM, MFDTM with the solution in [35] for different values of fractional order.

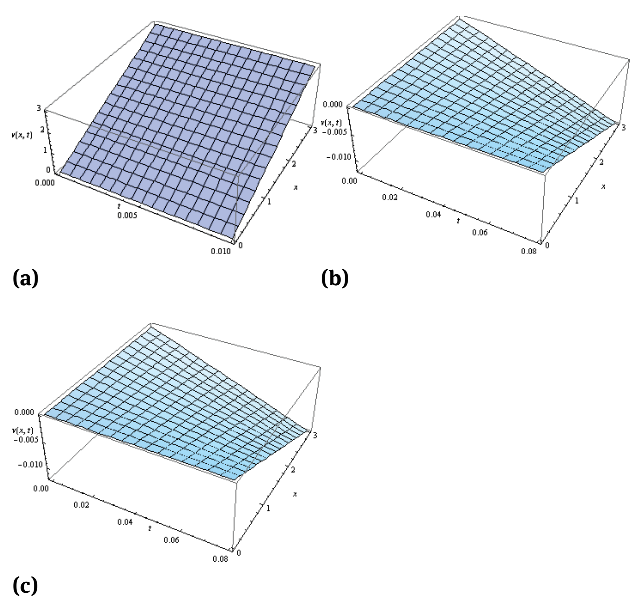


Fig. 2: $v(x,t)$ obtained by (a) FDTM, (b) MFDTM and (c) Solution in [35] when $\alpha=1$, $r=0.25$, $\tau=0.2$, $A=1$ and $B=10$.

5 Conclusions

In this paper, we implemented the two-dimensional FDTM and MFDTM for solving time fractional Black-Scholes equation. DTM is an attractive tool for solving linear and nonlinear partial differential equations and it does not require linearization, discretization or perturbation. But it also faces some difficulties while constructing recursive

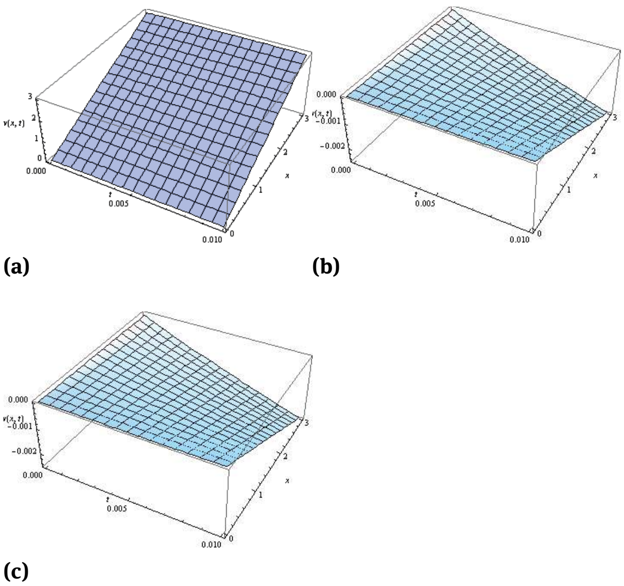


Fig. 3: $v(x, t)$ obtained by (a) FDTM, (b) MFDTM and (c) Solution in [35] when $\alpha = 0.9$, $r = 0.25$, $\tau = 0.2$, $A = 1$ and $B = 10$.

equation for the function of three or more variables and it requires an expensive computational cost to solve the algebraic recursive equation. The proposed MFDTM for the specific variable can obtain the simple recursive equation. Thus it is concluded that MFDTM enhances the effectiveness of the computational work when compared with the FDTM. The proposed methods are simpler in its principles and effective in solving linear and nonlinear differential equations of fractional order and promising tool for solving wider class of nonlinear fractional models in mathematical physics and financial theory with high accuracy.

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