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Research Article

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Solution to Fredholm integral inclusions via (F, δ_b)-contractions

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Abstract: We present sufficient conditions for the existence of solutions of Fredholm integral inclusion equations using new sort of contractions, named as multivalued almost F-contractions and multivalued almost F-contraction pairs under δ-distance, defined in b-metric spaces. We give its relevance to fixed point results in orbitally complete b-metric spaces. To rationalize the notions and outcome, we illustrate the appropriate examples.

Keywords: Almost contraction, Multivalued mapping, b-metric space, F-contraction, Integral inclusion equation

MSC: 47H10, 54H25, 45B99

1 Introduction

Integral equations appear in numerous scientific and engineering problems. A large class of initial and boundary value problems can be transformed to Volterra or Fredholm integral equations. Mathematical physics models, such as diffraction problems, scattering in quantum mechanics, conformal mapping, and water waves contributed to the creation of integral equations, as well. These equations represent a significant part of mathematical analysis and have various applications in real-world problems. Numerous studies have considered the integral inclusions that arise in the study of problems in applied mathematics, engineering and economics, since some mathematical models utilize multivalued maps instead of single-valued maps, see, e.g., [1–4] and references cited therein.

The advancement of geometric fixed point theory for multivalued mappings was initiated in the work of Nadler, Jr. in 1969 [5]. He used the concept of Hausdorff-Pompeiu metric to establish the multivalued contraction principle containing the Banach contraction principle as a special case. Since then, this discipline has been more developed, and many profound concepts and results have been set up in more generalized spaces.

We construct in this paper a new notion—almost F-contraction for multivalued mappings, by considering the δ-distance in the frame-work of b-metric spaces [6–8] and a concept of F-contractions which was introduced by Wardowski [9]. The paper is organized as follows. In Section 3, we introduce the notion of almost F-contraction for a multivalued mapping $\mathcal{J}$ under δ-distance in a b-metric space and originate fixed point results in orbitally complete b-metric spaces. In Section 4 we introduce the concept of almost F-contraction pair of multivalued mappings $\mathcal{J}_2$ and $\mathcal{J}_1$ under δ-distance. The existence and uniqueness of their common fixed point is obtained under additional assumptions on the mappings. We also furnish suitable examples to demonstrate the validity of our results and

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to distinguish them from some known ones. In Section 5 we deal with solutions of a Fredholm integral inclusion equation, based on the results of Section 3.

Our work improves and extends the works done in the papers [10–13] with the consideration of orbitally complete b-metric space.

2 Preliminaries

The notion of b-metric space as an extension of metric space was introduced by Bakhtin in [6] and then extensively used by Czerwik in [7, 8, 14]. Since then, a lot of papers have been published on the fixed point theory of various classes of single-valued and multi-valued operators in this type of spaces. We recall here just some basic definitions and notation that we are going to use. $\mathbb{R}^+$ and $\mathbb{R}_0^+$ will denote the set of all positive, resp. nonnegative real numbers and $\mathbb{N}$ will be the set of positive integers.

A b-metric on a nonempty set $\mathcal{E}$ is a function $d_b : \mathcal{E}\times\mathcal{E} \to \mathbb{R}_0^+$ such that for a constant $s \geq 1$ and all $x, y, z \in \mathcal{E}$ the following three conditions hold true:

(M1) $d_b(x, y) = 0 \iff x = y$,

(M2) $d_b(x, y) = d_b(y, x)$,

(M3) $d_b(x, y) \leq s(d_b(x, z) + d_b(z, y))$.

The triple $(\mathcal{E}, d_b, s)$ is called a b-metric space.

Obviously, each metric space is a b-metric space (for $s = 1$), but the converse need not be true. Standard examples of b-metric spaces that are not metric spaces are the following:

1. $\mathcal{E} = \mathbb{R}$ and $d_b : \mathcal{E} \times \mathcal{E} \to \mathbb{R}$ defined by $d_b(x, y) = |x - y|^2$ for all $x, y \in \mathcal{E}$, with $s = 2$.
2. $\ell^p(\mathbb{R}) := \{\{x_n\} \subset \mathbb{R} : \sum_{n=1}^{\infty} |x_n|^p < \infty\}$, $0 < p < 1$, $d_b : \ell^p(\mathbb{R}) \times \ell^p(\mathbb{R}) \to \mathbb{R}$ given by
$$d_b(\{x_n\}, \{y_n\}) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^p\right)^{1/p}$$
for all $\{x_n\}, \{y_n\} \in \ell^p(\mathbb{R})$. Here $s = 2^{1/p}$.
3. $L^p([0, 1]) \ni f : [0, 1] \to \mathbb{R}$ such that $\int_0^1 |f(t)|^p dt < \infty$, $p > 1$, $d_b : L^p([0, 1]) \times L^p([0, 1]) \to \mathbb{R}$ given by
$$d_b(f, g) = \int_0^1 |f(t) - g(t)|^p$$
for all $f, g \in L^p([0, 1])$; here $s = 2^{p-1}$.

The topology on b-metric spaces and the notions of convergent and Cauchy sequences, as well as the completeness of the space are defined similarly as for standard metric spaces. However, one has to be aware of some differences. For instance, a b-metric need not be a continuous mapping in both variables (see, e.g., [15]).

Now, we give a brief background for multivalued mappings defined in a b-metric space $(\mathcal{E}, d_b, s)$.

We denote the class of non-empty and bounded subsets of $\mathcal{E}$ by $\mathcal{P}_b(\mathcal{E})$, and the class of non-empty, closed and bounded subsets of $\mathcal{E}$ by $\mathcal{P}_{cb}(\mathcal{E})$. For $\mathcal{U}, \mathcal{V}, \mathcal{W} \in \mathcal{P}_b(\mathcal{E})$, we define:

$$\mathcal{D}_b(\mathcal{U}, \mathcal{V}) = \inf\{d_b(u, v) : u \in \mathcal{U},\ v \in \mathcal{V}\} \text{ and}$$
$$\delta_b(\mathcal{U}, \mathcal{V}) = \sup\{d_b(u, v) : u \in \mathcal{U},\ v \in \mathcal{V}\}$$

with $\mathcal{D}_b(w, \mathcal{W}) = \mathcal{D}_b(\{w\}, \mathcal{W}) = \inf\{d_b(w, x) : x \in \mathcal{W}\}$.

The following are some easy properties of $\mathcal{D}_b$ and δ_b (see, e.g. [7, 8, 14]):

(i) if $\mathcal{U} = \{u\}$ and $\mathcal{V} = \{v\}$ then $\mathcal{D}_b(\mathcal{U}, \mathcal{V}) = \delta_b(\mathcal{U}, \mathcal{V}) = d_b(u, v)$;

(ii) $\mathcal{D}_b(\mathcal{U}, \mathcal{V}) \leq \delta_b(\mathcal{U}, \mathcal{V})$;
(iii) $\mathcal{D}_b(x, \mathcal{V}) \leq d_b(x, b)$ for any $b \in \mathcal{V}$;
(iv) $\delta_b(\mathcal{U}, \mathcal{V}) \leq s[\delta_b(\mathcal{U}, \mathcal{W}) + \delta_b(\mathcal{W}, \mathcal{V})]$;
(v) $\delta_b(\mathcal{U}, \mathcal{V}) = 0$ iff $\mathcal{U} = \mathcal{V} = \{v\}$.

Moreover, we will always suppose that
(vi) the function $\mathcal{D}_b$ is continuous in its variables.

Recall that $z \in \mathcal{E}$ is called a fixed point of a multivalued mapping $\mathcal{J} : \mathcal{E} \to \mathcal{P}_b(\mathcal{E})$ if $z \in \mathcal{J}z$.

The concepts of orbit, orbitally complete space and orbitally continuous mapping given in [16–18] for metric spaces can be extended to the case of b-metric spaces, as follows:

Definition 2.1. *Let $(\mathcal{E}, d_b, s)$ be a b-metric space and $\mathcal{J}, \mathcal{J}_1, \mathcal{J}_2 : \mathcal{E} \to \mathcal{P}_b(\mathcal{E})$ be three mappings.*

1. *An orbit $\mathcal{O}(x_0; \mathcal{J})$ of $\mathcal{J}$ at a point $x_0 \in \mathcal{E}$ is any sequence $\{x_n\}$ such that $x_n \in \mathcal{J}x_{n-1}$ for $n = 1, 2, \dots$.*
2. *If for a point $x_0 \in \mathcal{E}$, there exists a sequence $\{x_n\}$ in $\mathcal{E}$ such that $x_{2n+1} \in \mathcal{J}_2 x_{2n}$, $x_{2n+2} \in \mathcal{J}_1 x_{2n+1}$, $n = 0, 1, 2, \dots$, then the set $\mathcal{O}(x_0; \mathcal{J}_1, \mathcal{J}_2) = \{x_n : n = 1, 2, \dots\}$ is called an orbit of $(\mathcal{J}_1, \mathcal{J}_2)$ at x_0.*
3. *The space $(\mathcal{E}, d_b, s)$ is said to be $(\mathcal{J}_1, \mathcal{J}_2)$-orbitally complete if any Cauchy subsequence $\{x_{n_i}\}$ of $\mathcal{O}(x_0; \mathcal{J}_1, \mathcal{J}_2)$ (for some x_0 in $\mathcal{E}$) converges in $\mathcal{E}$. In particular, for $\mathcal{J}_1 = \mathcal{J}_2 = \mathcal{J}$, we say that $\mathcal{E}$ is $\mathcal{J}$-orbitally complete.*
4. *The mapping $\mathcal{J}$ is said to be orbitally continuous at a point $x_0 \in \mathcal{E}$ if for any sequence $\{x_n\}_{n \geq 0} \subset \mathcal{O}(x_0; \mathcal{J})$ and $z \in \mathcal{E}$, $d(x_n, z) \to 0$ as $n \to \infty$ implies $\delta_b(\mathcal{J}x_n, \mathcal{J}z) \to 0$ as $n \to \infty$. $\mathcal{J}$ is called orbitally continuous in $\mathcal{E}$ if it is orbitally continuous at every point of $\mathcal{E}$.*
5. *The graph $G(\mathcal{J})$ of $\mathcal{J}$ is defined as $G(\mathcal{J}) = \{(x, y) : x \in \mathcal{E}, y \in \mathcal{J}x\}$. The graph $G(\mathcal{J})$ of $\mathcal{J}$ is called $\mathcal{J}$-orbitally closed if, for any sequence $\{x_n\}$, we have $(x, x) \in G(\mathcal{J})$ whenever $(x_n, x_{n+1}) \in G(\mathcal{J})$ and $\lim_{n\to\infty} x_n = x$.*

In his paper [9], Wardowski introduced a new type of contractions which he called F-contractions. Several authors proved various variants of fixed point results using such contractions. In particular, Acar and Altun proved in [10] a fixed point theorem for multivalued mappings under δ-distance.

Adapting Wardowski's approach to b-metric space, Cosentino et al. used in [13] the set of functions $\mathfrak{F}_s$ defined as follows

Definition 2.2. *Let $s \geq 1$ be a real number. We denote by $\mathfrak{F}_s$ the family of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ with the following properties:*
(F1) F is strictly increasing;
(F2) for each sequence $\{\alpha_n\}$ of positive numbers, $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$;
(F3) for each sequence $\{\alpha_n\}$ of positive numbers with $\lim_{n\to\infty} \alpha_n = 0$, there exists $k \in (0, 1)$ such that $\lim_{n\to\infty} \alpha_n^k F(\alpha_n) = 0$;
(F4) there exists $\tau \in \mathbb{R}^+$ such that for each sequence $\{\alpha_n\}$ of positive numbers, if $\tau + F(s\alpha_n) \leq F(\alpha_{n-1})$ for all $n \in \mathbb{N}$, then $\tau + F(s^n\alpha_n) \leq F(s^{n-1}\alpha_{n-1})$ for all $n \in \mathbb{N}$.

Example 2.3. *Let $F : \mathbb{R}^+ \to \mathbb{R}$ be defined by $F(\alpha) = \ln\alpha$ or $F(\alpha) = \alpha + \ln\alpha$. It can be easily checked [13, Example 3.2] that F satisfies the properties (F1)–(F4).*

They proved the following (note that $\mathcal{H}_b$ here denotes the b-Hausdorff-Pompeiu metric).

Theorem 2.4. *[13, Theorem 3.4] Let $(\mathcal{E}, d_b, s)$ be a complete b-metric space and let $\mathcal{J} : \mathcal{E} \to \mathcal{P}_{cb}(\mathcal{E})$. Assume that there exists a continuous from the right function $F \in \mathfrak{F}_s$ and $\tau \in \mathbb{R}^+$ such that*

$$2\tau + F(s\mathcal{H}_b(\mathcal{J}x, \mathcal{J}y)) \leq F(d_b(x, y)), \tag{1}$$

for all $x, y \in \mathcal{E}$, $\mathcal{J}x \neq \mathcal{J}y$. Then $\mathcal{J}$ has a fixed point.

3 Multivalued almost (F, δ_b)-contractions and relevance to fixed point results

We first introduce the notion of multivalued almost (F, δ_b)-contraction in a b-metric space and give relevance to fixed point results.

Definition 3.1. *Let $(\mathcal{E}, d_b, s)$ be a b-metric space with $s > 1$. We say that a multivalued mapping $\mathcal{J}: \mathcal{E} \to \mathcal{P}_b(\mathcal{E})$ is a multivalued almost (F, δ_b)-contraction if $F \in \mathfrak{F}_s$ (with parameter τ) and there exists $\lambda \geq 0$ such that*

$$\tau + F(s\delta_b(\mathcal{J}x, \mathcal{J}y)) \leq F(\Theta_1(x, y) + \lambda\Theta_2(x, y)), \tag{2}$$

for all $x, y \in \mathcal{E}$ with $\min\{\delta_b(\mathcal{J}x, \mathcal{J}y), d_b(x, y)\} > 0$, where

$$\Theta_1(x, y) = \max\left\{d_b(x, y), \mathcal{D}_b(x, \mathcal{J}x), \mathcal{D}_b(y, \mathcal{J}y), \frac{\mathcal{D}_b(x, \mathcal{J}y) + \mathcal{D}_b(y, \mathcal{J}x)}{2s}\right\} \tag{3}$$

and

$$\Theta_2(x, y) = \min\{\mathcal{D}_b(x, \mathcal{J}x), \mathcal{D}_b(y, \mathcal{J}y), \mathcal{D}_b(x, \mathcal{J}y), \mathcal{D}_b(y, \mathcal{J}x)\}.$$

If (2) is satisfied just for $x, y \in \overline{\mathcal{O}(x_0; \mathcal{J})}$ (for some $x_0 \in \mathcal{E}$), we say that $\mathcal{J}$ is a multivalued almost orbitally (F, δ_b)-contraction.

We are equipped now to state our first main result.

Theorem 3.2. *Let $(\mathcal{E}, d_b, s)$ be a b-metric space with $s > 1$ and let $\mathcal{J}: \mathcal{E} \to \mathcal{P}_b(\mathcal{E})$ be a multivalued almost orbitally (F, δ_b)-contraction. Suppose that $(\mathcal{E}, d_b, s)$ is $\mathcal{J}$-orbitally complete (for the same $x_0 \in \mathcal{E}$). If F is continuous and $\mathcal{J}x$ is closed for all $x \in \overline{\mathcal{O}(x_0; \mathcal{J})}$, or $\mathcal{J}$ has $\mathcal{J}$-orbitally closed graph, then $\mathcal{J}$ has a fixed point in $\mathcal{E}$.*

Proof. Starting from the given point x_0, choose a sequence $\{x_n\}$ in $\mathcal{E}$ such that $x_{n+1} \in \mathcal{J}x_n$, for all $n \geq 0$. Now, if $x_{n_0} \in \mathcal{J}x_{n_0}$ for some n_0, then the proof is finished. Therefore, we assume $x_n \neq x_{n+1}$ for all $n \geq 0$. So $d_b(x_{n+1}, x_{n+2}) > 0$ and $\delta_b(\mathcal{J}x_n, \mathcal{J}x_{n+1}) > 0$ for all $n \geq 0$.

Using the condition (2) for elements $x = x_n$, $y = x_{n+1}$, for arbitrary $n \geq 0$ we have

$$\tau + F(sd_b(x_{n+1}, x_{n+2})) \leq \tau + F(s\delta_b(\mathcal{J}x_n, \mathcal{J}x_{n+1})) \leq F(\Theta_1(x_n, x_{n+1}) + \lambda\Theta_2(x_n, x_{n+1}))$$

where

$$\begin{aligned}\Theta_1(x_n, x_{n+1}) &= \max\left\{\begin{array}{c} d_b(x_n, x_{n+1}), \mathcal{D}_b(x_n, \mathcal{J}x_n), \mathcal{D}_b(x_{n+1}, \mathcal{J}x_{n+1}), \\ \frac{1}{2s}[\mathcal{D}_b(x_n, \mathcal{J}x_{n+1}) + \mathcal{D}_b(x_{n+1}, \mathcal{J}x_n)] \end{array}\right\} \\ &\leq \max\left\{d_b(x_n, x_{n+1}), d_b(x_n, x_{n+1}), d_b(x_{n+1}, x_{n+2}), \frac{1}{2s}d_b(x_n, x_{n+2}))\right\} \\ &= \max\left\{d_b(x_n, x_{n+1}), d_b(x_{n+1}, x_{n+2}), \frac{1}{2s}d_b(x_n, x_{n+2})\right\}\end{aligned}$$

and

$$\Theta_2(x_n, x_{n+1}) = \min\{\mathcal{D}_b(x_n, \mathcal{J}x_n), \mathcal{D}_b(x_{n+1}, \mathcal{J}x_{n+1}), \mathcal{D}_b(x_n, \mathcal{J}x_{n+1}), \mathcal{D}_b(x_{n+1}, \mathcal{J}x_n)\} = 0.$$

As $\frac{1}{2s}d_b(x_n, x_{n+2}) \leq \max\{d_b(x_n, x_{n+1}), d_b(x_{n+1}, x_{n+2})\}$, it follows that

$$\tau + F(sd_b(x_{n+1}, x_{n+2})) \leq F(\max\{d_b(x_n, x_{n+1}), d_b(x_{n+1}, x_{n+2})\}). \tag{4}$$

Suppose that $d_b(x_n, x_{n+1}) \leq d_b(x_{n+1}, x_{n+2})$, for some positive integer n. Then from (4), we have

$$\tau + F(sd_b(x_{n+1}, x_{n+2})) \leq F(d_b(x_{n+1}, x_{n+2})),$$

a contradiction with (F1). Hence,

$$\max\{d_b(x_n,x_{n+1}),d_b(x_{n+1},x_{n+2})\}=d_b(x_n,x_{n+1}),$$

and consequently

$$\tau+F(sd_b(x_{n+1},x_{n+2}))\leq F(d_b(x_n,x_{n+1})) \text{ for all } n\in\mathbb{N}\cup\{0\}. \tag{5}$$

It follows by (5) and the property (F4) that

$$\tau+F(s^n d_b(x_n,x_{n+1}))\leq F(s^{n-1}d_b(x_{n-1},x_n)) \text{ for all } n\in\mathbb{N}\cup\{0\}. \tag{6}$$

Denote $\varrho_n=d(x_n,x_{n+1})$ for $n=0,1,2,\ldots$. Then, $\varrho_n>0$ for all n and, using (6), the following holds:

$$F(s^n\varrho_n)\leq F(s^{n-1}\varrho_{n-1})-\tau\leq F(s^{n-2}\varrho_{n-2})-2\tau\leq\cdots\leq F(\varrho_0)-n\tau \tag{7}$$

for all $n\in\mathbb{N}$. From (7), we get $F(s^n\varrho_n)\to-\infty$ as $n\to\infty$. Thus, from (F2), we have

$$s^n\varrho_n\to 0 \quad \text{as } n\to\infty. \tag{8}$$

Now, by the property (F3) there exists $k\in(0,1)$ such that

$$\lim_{n\to\infty}(s^n\varrho_n)^k F(s^n\varrho_n)=0. \tag{9}$$

By (7), the following holds for all $n\in\mathbb{N}$:

$$(s^n\varrho_n)^k F(s^n\varrho_n)-(s^n\varrho_n)^k F(\varrho_0)\leq(s^n\varrho_n)^k(-n\tau)\leq 0. \tag{10}$$

Passing to the limit as $n\to\infty$ in (10) and using (8) and (9), we obtain

$$\lim_{n\to\infty} n(s^n\varrho_n)^k=0$$

and hence $\lim_{n\to\infty} n^{1/k}s^n\varrho_n=0$. Now, the last limit implies that the series $\Sigma_{n=1}^{\infty}s^n\varrho_n$ is convergent and hence $\{x_n\}$ is a Cauchy sequence in $\mathcal{O}(x_0;\mathcal{J})$. Since $\mathcal{E}$ is $\mathcal{J}$-orbitally complete, there exists a $z\in\mathcal{E}$ such that

$$x_n\to z \ \text{ as } \ n\to\infty.$$

Suppose that $\mathcal{J}z$ is closed.

We observe that if there exists an increasing sequence $\{n_k\}\subset\mathbb{N}$ such that $x_{n_k}\in\mathcal{J}z$ for all $k\in\mathbb{N}$, since $\mathcal{J}z$ is closed and $\lim_{k\to\infty}x_{n_k}=z$, we deduce that $z\in\mathcal{J}z$ and hence the proof is completed. Then we assume that there exists $n_0\in\mathbb{N}$ such that $x_n\notin\mathcal{J}z$ for all $n\in\mathbb{N}$ with $n\geq n_0$. It follows that $\delta_b(\mathcal{J}x_n,\mathcal{J}z)>0$ for all $n\geq n_0$. Using the condition (2) for $x=x_n$, $y=z$, we have

$$\tau+F(s\mathcal{D}_b(x_{n+1},\mathcal{J}z))\leq\tau+F(s\delta_b(\mathcal{J}x_n,\mathcal{J}z))\leq F(\Theta_1(x_n,z)+\lambda\Theta_2(x_n,z)) \tag{11}$$

where

$$\begin{aligned}\Theta_1(x_n,z)&=\max\left\{d_b(x_n,z),\mathcal{D}_b(x_n,\mathcal{J}x_n),\mathcal{D}_b(z,\mathcal{J}z),\tfrac{\mathcal{D}_b(x_n,\mathcal{J}z)+\mathcal{D}_b(z,\mathcal{J}x_n)}{2s}\right\}\\&\leq\max\left\{d_b(x_n,z),d_b(x_n,x_{n+1}),\mathcal{D}_b(z,\mathcal{J}z),\tfrac{\mathcal{D}_b(x_n,\mathcal{J}z)+d_b(z,x_{n+1})}{2s}\right\}\\&\to\mathcal{D}_b(z,\mathcal{J}z),\quad\text{as } n\to\infty,\end{aligned}$$

and

$$\begin{aligned}\Theta_2(x_n,z)&=\min\left\{\mathcal{D}_b(x_n,\mathcal{J}x_n),\mathcal{D}_b(z,\mathcal{J}z),\mathcal{D}_b(x_n,\mathcal{J}z),\mathcal{D}_b(z,\mathcal{J}x_n)\right\}\\&\leq\min\left\{d_b(x_n,x_{n+1}),\mathcal{D}_b(z,\mathcal{J}z),\mathcal{D}_b(x_n,\mathcal{J}z),d_b(z,x_{n+1})\right\}\\&\to 0,\quad\text{as } n\to\infty.\end{aligned}$$

Since F and $\mathcal{D}_b$ are continuous, if $\mathcal{D}_b(z, \mathcal{J}z) > 0$, passing to the limit as $n \to \infty$ in (11), we obtain

$$\tau + F(s\mathcal{D}_b(z, \mathcal{J}z)) \leq F(\mathcal{D}_b(z, \mathcal{J}z)),$$

which is impossible since $\tau > 0$, $s \geq 1$ and F is strictly increasing. Hence, $\mathcal{D}_b(z, \mathcal{J}z) = 0$ and, since $\mathcal{J}z$ is closed, we have $z \in \mathcal{J}z$. Thus, z is a fixed point of $\mathcal{J}$.

Suppose that $G(\mathcal{J})$ is $\mathcal{J}$-orbitally closed.

Since $(x_n, x_{n+1}) \in G(\mathcal{J})$ for all $n \in \mathbb{N} \cup \{0\}$ and $\lim_{n\to\infty} x_n = z$, we have $(z, z) \in G(\mathcal{J})$ by the $\mathcal{J}$-orbitally closedness. Hence, $z \in \mathcal{J}z$.

It is proved that z is a fixed point of $\mathcal{J}$. □

The following corollaries follow from Theorem 3.2 by taking $F(\alpha) = \ln \alpha$, resp. $F(\alpha) = \alpha + \ln \alpha$ in (2).

Corollary 3.3. *Let $(\mathcal{E}, d_b, s)$ be a b-metric space with $s > 1$ and let $\mathcal{J}: \mathcal{E} \to \mathcal{P}_b(\mathcal{E})$ be a multivalued mapping satisfying, for some $\tau > 0$, $x_0 \in \mathcal{E}$, $\lambda \geq 0$, the condition*

$$s\delta_b(\mathcal{J}x, \mathcal{J}y) \leq e^{-\tau}\{\Theta_1(x, y) + \lambda\Theta_2(x, y)\}$$

for all $x, y \in \overline{\mathcal{O}(x_0; \mathcal{J})}$ with $\min\{\delta_b(\mathcal{J}x, \mathcal{J}y), d_b(x, y)\} > 0$, where Θ_1, Θ_2 are given by (3). Suppose that $(\mathcal{E}, d_b, s)$ is $\mathcal{J}$-orbitally complete (for the same $x_0 \in \mathcal{E}$). If $\mathcal{J}x$ is closed for all $x \in \overline{\mathcal{O}(x_0; \mathcal{J})}$, or $\mathcal{J}$ has $\mathcal{J}$-orbitally closed graph, then $\mathcal{J}$ has a fixed point in $\mathcal{E}$.

Corollary 3.4. *Let $(\mathcal{E}, d_b, s)$ be a b-metric space with $s > 1$ and let $\mathcal{J}: \mathcal{E} \to \mathcal{P}_b(\mathcal{E})$ be a multivalued mapping satisfying, for some $\tau > 0$, $x_0 \in \mathcal{E}$, $\lambda \geq 0$, the condition*

$$\frac{s\delta_b(\mathcal{J}x, \mathcal{J}y)}{\Theta_1(x, y) + \lambda\Theta_2(x, y)} e^{s\delta_b(\mathcal{J}x, \mathcal{J}y) - (\Theta_1(x,y) + \lambda\Theta_2(x,y))} \leq e^{-\tau}$$

for all $x, y \in \overline{\mathcal{O}(x_0; \mathcal{J})}$ with $\min\{\delta_b(\mathcal{J}x, \mathcal{J}y), d_b(x, y)\} > 0$, where Θ_1, Θ_2 are given by (3). Suppose that $(\mathcal{E}, d_b, s)$ is $\mathcal{J}$-orbitally complete (for the same $x_0 \in \mathcal{E}$). If $\mathcal{J}x$ is closed for all $x \in \overline{\mathcal{O}(x_0; \mathcal{J})}$, or $\mathcal{J}$ has $\mathcal{J}$-orbitally closed graph, then $\mathcal{J}$ has a fixed point in $\mathcal{E}$.

Example 3.5. *This example is inspired by [19, Example 2.3].*

Let $\mathcal{E} = [0, 1]$ be equipped with b-metric $d_b(x, y) = (x - y)^2$ (with $s = 2$). Consider the mapping $\mathcal{J} : \mathcal{E} \to \mathcal{P}_{cb}(\mathcal{E})$ given by

$$\mathcal{J}x = \begin{cases} \{\frac{1}{2}\}, & 0 \leq x < 1, \\ [0, \frac{1}{4}], & x = 1. \end{cases}$$

If $x, y \in [0, 1)$ then $\delta_b(\mathcal{J}x, \mathcal{J}y) = 0$. Let $x \in [0, 1)$ and $y = 1$. Then $\mathcal{J}x = \{\frac{1}{2}\}$, $\mathcal{J}y = [0, \frac{1}{4}]$, $\delta_b(\mathcal{J}x, \mathcal{J}y) = \frac{1}{4}$,

$$\Theta_1(x, y) = \max\{(1 - x)^2, (\tfrac{1}{2} - x)^2, (\tfrac{3}{4})^2, \tfrac{1}{4}[\mathcal{D}(x, \mathcal{J}y) + \tfrac{1}{4}]\} \geq \tfrac{9}{16},$$
$$\Theta_2(x, y) = \min\{(\tfrac{1}{2} - x)^2, (\tfrac{3}{4})^2, \mathcal{D}(x, \mathcal{J}y), \tfrac{1}{4}]\} \geq 0.$$

Take $\tau = \frac{1}{16} + \ln \frac{9}{8} > 0$, $F(\alpha) = \alpha + \ln \alpha$ and $\lambda \geq 0$. Then

$$\begin{aligned} \tau + F(s\delta_b(\mathcal{J}x, \mathcal{J}y)) &= \frac{1}{16} + \ln \frac{9}{8} + 2 \cdot \frac{1}{4} + \ln(2 \cdot \frac{1}{4}) \\ &= \frac{9}{16} + \ln \frac{9}{16} = F(\frac{9}{16}) \leq F(\Theta_1(x, y) + \lambda\Theta_2(x, y)). \end{aligned}$$

Hence, the conditions of Theorem 3.2 (more precisely, Corollary 3.4) are fulfilled and $\mathcal{J}$ has a fixed point (which is $z = \frac{1}{2}$).

However, in the case $x \in [0, 1)$, $y = 1$, it is $\mathcal{H}_b(\mathcal{J}x, \mathcal{J}y) = \frac{1}{4}$ and hence

$$s\mathcal{H}_b(\mathcal{J}x, \mathcal{J}y) = \frac{1}{2} < (1 - x)^2 = d_b(x, y)$$

for $x < 1 - \frac{1}{\sqrt{2}}$. Thus, no number τ and function $F \in \mathfrak{F}_s$ exist such that the condition (1) is satisfied. So, Theorem 2.4 cannot be used to obtain the desired conclusion.

The following corollary is a special case of Theorem 3.2 when $\mathcal{J}$ is a single-valued mapping.

Corollary 3.6. *Let $(\mathcal{E}, d_b, s)$ be a b-metric space with $s > 1$ and let $\mathcal{J} : \mathcal{E} \to \mathcal{E}$ be a self-mapping such that $\mathcal{E}$ is $\mathcal{J}$-orbitally complete (at some x_0). Suppose that $F \in \mathfrak{F}_s$ and there exist $\tau > 0$, $\lambda \geq 0$ such that*

$$\tau + F(sd_b(\mathcal{J}x, \mathcal{J}y)) \leq F(\Theta_1'(x, y) + \lambda\Theta_2'(x, y)), \tag{12}$$

for all $x, y \in \overline{\mathcal{O}(x_0; \mathcal{J})}$ with $\min\{d_b(\mathcal{J}x, \mathcal{J}y), d_b(x, y)\} > 0$, where

$$\Theta_1'(x, y) = \max\left\{d_b(x, y), d_b(x, \mathcal{J}x), d_b(y, \mathcal{J}y), \frac{d_b(x, \mathcal{J}y) + d_b(y, \mathcal{J}x)}{2}\right\}$$

and

$$\Theta_2'(x, y) = \min\{d_b(x, \mathcal{J}x), d_b(y, \mathcal{J}y), d_b(x, \mathcal{J}y), d_b(y, \mathcal{J}x)\}.$$

If F is continuous, then $\mathcal{J}$ has a fixed point in $\mathcal{E}$.

4 Multivalued almost (F, δ_b)-contraction pair and relevance to common fixed point results

In this section, we prove a common fixed point theorem for a pair of multivalued mappings satisfying certain conditions.

First we introduce the notion of multivalued almost (F, δ_b)-contraction pair in b-metric spaces.

Definition 4.1. *Let $(\mathcal{E}, d_b, s)$ be a b-metric space with $s > 1$. Two multivalued mappings $\mathcal{J}_1, \mathcal{J}_2 : \mathcal{E} \to \mathcal{P}_b(\mathcal{E})$ are said to form a multivalued almost (F, δ_b)-contraction pair, if $F \in \mathfrak{F}_s$ and there exist $\tau > 0$, $\lambda \geq 0$ such that*

$$\tau + F(s\delta_b(\mathcal{J}_1x, \mathcal{J}_2y)) \leq F(\Delta_1(x, y) + \lambda\Delta_2(x, y)), \tag{13}$$

for all $x, y \in \mathcal{E}$ with $\min\{\delta_b(\mathcal{J}_1x, \mathcal{J}_2y), d_b(x, y)\} > 0$, where

$$\Delta_1(x, y) = \max\left\{d_b(x, y), \mathcal{D}_b(x, \mathcal{J}_1x), \mathcal{D}_b(y, \mathcal{J}_2y), \frac{1}{2s}[\mathcal{D}_b(x, \mathcal{J}_2y) + \mathcal{D}_b(y, \mathcal{J}_1x)]\right\}$$

and

$$\Delta_2(x, y) = \min\{\mathcal{D}_b(x, \mathcal{J}_1x), \mathcal{D}_b(y, \mathcal{J}_2y), \mathcal{D}_b(x, \mathcal{J}_2y), \mathcal{D}_b(y, \mathcal{J}_1x)\}.$$

If (13) is satisfied just for $x, y \in \overline{\mathcal{O}(x_0; \mathcal{J}_1, \mathcal{J}_2)}$ (for some $x_0 \in \mathcal{E}$), we say that $(\mathcal{J}_1, \mathcal{J}_2)$ is a multivalued almost orbitally (F, δ_b)-contraction pair.

The main result of this section is the following theorem.

Theorem 4.2. *Let $(\mathcal{E}, d_b, s)$ be a b-metric space with $s > 1$ and let $\mathcal{J}_1, \mathcal{J}_2 : \mathcal{E} \to \mathcal{P}_b(\mathcal{E})$ form a multivalued almost orbitally (F, δ_b)-contraction pair (for some x_0). Assume that $\mathcal{E}$ is $(\mathcal{J}_1, \mathcal{J}_2)$-orbitally complete at x_0. If F is continuous and $\mathcal{J}_1$ and $\mathcal{J}_2$ are orbitally continuous at x_0, then $\mathcal{J}_1$ and $\mathcal{J}_2$ have a common fixed point in $\mathcal{E}$.*

Proof. Starting with the given point x_0, choose a sequence $\{x_n\} \subset \mathcal{E}$ satisfying

$$x_{2n+1} \in \mathcal{J}_2x_{2n}, \quad x_{2n+2} \in \mathcal{J}_1x_{2n+1}, \quad \text{for } n \in \{0, 1, \dots\}$$

and let $a_n = d_b(x_n, x_{n+1})$. If $x_{n_0} \in \mathcal{J}_2x_{n_0}$ or $x_{n_0} \in \mathcal{J}_1x_{n_0}$ for some n_0, then the proof is finished. So assume $x_n \neq x_{n+1}$ for all $n \geq 0$. We claim that

$$\lim_{n\to\infty} s^n a_n = 0.$$

Suppose that n is an odd number. Substituting $x = x_n$ and $y = x_{n+1}$ in (13), we obtain

$$\tau + F(sd_b(x_n, x_{n+1})) \leq \tau + F(s\delta_b(\mathcal{J}_1x_n, \mathcal{J}_2x_{n+1})) \leq F(\Delta_1(x_n, x_{n+1}) + \lambda\Delta_2(x_n, x_{n+1})), \tag{14}$$

where

$$\Delta_1(x_n, x_{n+1}) = \max\left\{\begin{matrix} d_b(x_n, x_{n+1}), \mathcal{D}_b(x_n, \mathcal{J}_1 x_n), \mathcal{D}_b(x_{n+1}, \mathcal{J}_2 x_{n+1}), \\ \frac{1}{2s}\mathcal{D}_b(x_n, \mathcal{J}_2 x_{n+1}) + \mathcal{D}_b(x_{n+1}, \mathcal{J}_1 x_n) \end{matrix}\right\}$$
$$\leq \max\left\{d_b(x_n, x_{n+1}),\ d_b(x_n, x_{n+1}),\ d_b(x_{n+1}, x_{n+2}), \frac{1}{2s} d_b(x_n, x_{n+2})\right\}$$
$$\leq \max\{d_b(x_n, x_{n+1}),\ d_b(x_{n+1}, x_{n+2})\}$$

as $\frac{1}{2s}\delta_b(x_n, x_{n+2}) \leq \max\{\delta_b(x_n, x_{n+1}), d_b(x_{n+1}, x_{n+2})\}$ and

$$\Delta_2(x_n, x_{n+1}) = \min\{\mathcal{D}_b(x_n, \mathcal{J}_1 x_n), \mathcal{D}_b(x_{n+1}, \mathcal{J}_2 x_{n+1}), \mathcal{D}_b(x_n, \mathcal{J}_2 x_{n+1}), \mathcal{D}_b(x_{n+1}, \mathcal{J}_1 x_n)\} = 0.$$

Therefore it follows from (14) that

$$\tau + F(d_b(x_n,\ x_{n+1})) \leq F(\max\{d_b(x_{n-1}, x_n),\ d_b(x_n, x_{n+1})\}). \tag{15}$$

Suppose that $d_b(x_{n-1}, x_n) \leq d_b(x_n, x_{n+1})$. Then from (15), we have

$$\tau + F(s d_b(x_n, x_{n+1})) \leq F(d_b(x_n, x_{n+1})),$$

a contradiction, which means that

$$\max\{d_b(x_n, x_{n+1}), d_b(x_{n+1}, x_{n+2})\} = d_b(x_n, x_{n+1}).$$

Consequently, $\tau + F(s d_b(x_n, x_{n+1})) \leq F(d_b(x_{n-1}, x_n))$, that is

$$\tau + F(s a_n) \leq F(a_{n-1}). \tag{16}$$

In a similar way, we can establish inequality (16) when n is an even number.

It follows by (16) and property (F4) that

$$\tau + F(s^n a_n) \leq F(s^{n-1} a_{n-1}) \text{ for all } n \in \mathbb{N} \cup \{0\}.$$

Similarly as in Theorem 3.2, we can prove that the sequence $\{x_n\}$ is a b-Cauchy sequence in $\mathcal{O}(x_0; \mathcal{J}_1, \mathcal{J}_2)$. Since $\mathcal{E}$ is $(\mathcal{J}_2, \mathcal{J}_1)$-multivalued orbitally complete at x_0, there exists a $z \in \mathcal{E}$ such that

$$x_n \to z \text{ as } n \to \infty.$$

If $\mathcal{J}_2$ and $\mathcal{J}_1$ are orbitally continuous, then clearly $\mathcal{J}_2 z = \mathcal{J}_1 z = z$. □

Consequences similar to Corollaries 3.3 and 3.4 can be formulated in an obvious way.

If in Theorem 4.2, $\mathcal{J}_1$ and $\mathcal{J}_2$ are single-valued mappings, we deduce the following result.

Corollary 4.3. *Let $(\mathcal{E}, d_b, s)$ be a b-metric space with $s > 1$ and let $\mathcal{J}_1, \mathcal{J}_2 : \mathcal{E} \to \mathcal{E}$ be self-mappings such that $\mathcal{E}$ is $(\mathcal{J}_1, \mathcal{J}_2)$-orbitally complete (at some x_0). Suppose that $F \in \mathfrak{F}_s$ and there exist $\tau > 0$, $\lambda \geq 0$ such that*

$$\tau + F(s d_b(\mathcal{J}_1 x, \mathcal{J}_2 y)) \leq F(\Delta_1(x, y) + \lambda \Delta_2(x, y)),$$

for all $x, y \in \overline{\mathcal{O}(x_0; \mathcal{J}_1, \mathcal{J}_2)}$ (for the same x_0) with $\min\{d_b(\mathcal{J}x, \mathcal{J}y), d_b(x, y)\} > 0$, where

$$\Delta_1(x, y) = \max\left\{d_b(x, y), d_b(x, \mathcal{J}_1 x), d_b(y, \mathcal{J}_2 y), \frac{d_b(x, \mathcal{J}_2 y) + d_b(y, \mathcal{J}_1 x)}{2s}\right\}$$

and

$$\Delta_2(x, y) = \min\{d_b(x, \mathcal{J}_1 x), d_b(y, \mathcal{J}_2 y), d_b(x, \mathcal{J}_2 y), d_b(y, \mathcal{J}_1 x)\}.$$

If F is continuous and $\mathcal{J}_1$ and $\mathcal{J}_2$ are $(\mathcal{J}_1, \mathcal{J}_2)$-orbitally continuous at x_0, then $\mathcal{J}_1$ and $\mathcal{J}_2$ have a common fixed point.

We illustrate the preceding result with the following example (inspired by [20, Example 2.10]).

Example 4.4. *Let the set $\mathcal{E} = [0, +\infty)$ be equipped with b-metric $d_b(x, y) = (x - y)^2$ ($s = 2$) and define $\mathcal{J}_1, \mathcal{J}_2 : \mathcal{E} \to \mathcal{E}$ by*

$$\mathcal{J}_1 x = \begin{cases} \frac{1}{2}x, & 0 \le x \le \frac{1}{2}, \\ x, & x > \frac{1}{2}, \end{cases} \qquad \mathcal{J}_2 x = \begin{cases} \frac{1}{3}x, & 0 \le x \le \frac{1}{3}, \\ x, & x > \frac{1}{3}. \end{cases}$$

Take $x_0 = \frac{1}{2}$. Then it is easy to show that

$$\mathcal{O}(x_0; \mathcal{J}_1, \mathcal{J}_2) \subset \left\{ \frac{1}{2^k \cdot 3^l} : k, l \in \mathbb{N} \right\}, \qquad \overline{\mathcal{O}(x_0; \mathcal{J}_1, \mathcal{J}_2)} = \mathcal{O}(x_0; \mathcal{J}_1, \mathcal{J}_2) \cup \{0\}.$$

We will check that the contractive condition of Corollary 4.3 is fulfilled for $x, y \in \mathcal{O}(x_0; \mathcal{J}_1, \mathcal{J}_2)$ with $\tau = \ln \frac{9}{8}$ and $F(\alpha) = \ln \alpha$. Indeed, it takes the form

$$2\left(\frac{x}{2} - \frac{y}{3}\right)^2 \le \frac{8}{9} \max \left\{ (x - y)^2, \frac{1}{4}x^2, \frac{4}{9}y^2, \frac{1}{2}\left[\left(x - \frac{y}{3}\right)^2 + \left(y - \frac{x}{2}\right)^2 \right] \right\},$$

which, after the substitution $y = tx$, $t \ge 0$ reduces to

$$2\left(\frac{1}{2} - \frac{1}{3}t\right)^2 \le \frac{8}{9} \max \left\{ (1 - t)^2, \frac{1}{4}, \frac{4}{9}t^2, \frac{1}{2}\left[\left(1 - \frac{t}{3}\right)^2 + \left(t - \frac{1}{2}\right)^2 \right] \right\}.$$

The last inequality can be easily checked by considering possible values of the parameter $t \ge 0$.

All other conditions are also fulfilled, and hence, by Corollary 4.3, we conclude that $\mathcal{J}_1$ and $\mathcal{J}_2$ have a common fixed point (which is $z = 0$).

5 Application to Fredholm integral inclusions

In this section we apply the obtained results to achieve the existence of solutions for a certain Fredholm-type integral inclusion. The application is inspired by [3, 21].

Consider the following integral inclusion of Fredholm type.

$$x(t) \in f(t) + \int_a^b \mathcal{K}(t, s, x(s))\, ds, \qquad t \in [a, b]. \tag{17}$$

Here, $f \in C[a, b]$ is a given real function and $\mathcal{K} : [a, b] \times [a, b] \times \mathbb{R} \to \mathcal{P}_{cb}(\mathbb{R})$ a given set-valued operator; $x \in C[a, b]$ is the unknown function.

Now, for $p \ge 1$, consider the b-metric d_b on $C[a, b]$ defined by

$$d_b(x, y) = (\max_{t \in [a,b]} |x(t) - y(t)|)^p = \max_{t \in [a,b]} |x(t) - y(t)|^p \tag{18}$$

for all $x, y \in C[a, b]$. Then $(C[a, b], d_b, 2^{p-1})$ is a complete b-metric space. Let $\mathcal{D}_b$ and δ_b have the respective meanings.

We will assume the following:

(**I**) For each $x \in C[a, b]$, the operator $\mathcal{K}_x(t, s) := \mathcal{K}(t, s, x(s))$, $(t, s) \in [a, b] \times [a, b]$ is continuous.

(**II**) there exists a continuous function $\Upsilon : [a, b]^2 \to [0, +\infty)$ such that

$$|k_u(t, s) - k_v(t, s)|^p \le \Upsilon(t, s) \cdot \left(\begin{array}{c} \max \left\{ \begin{array}{c} d_b(u(s), v(s)), \mathcal{D}_b(u(s), \mathcal{K}(t, s, u(s))), \\ \mathcal{D}_b(v(s), \mathcal{K}(t, s, v(s))), \\ \frac{\mathcal{D}_b(u(s), \mathcal{K}(t, s, v(s))) + \mathcal{D}_b(v(s), \mathcal{K}(t, s, u(s)))}{2^p} \end{array} \right\} \\ + \lambda \min \left\{ \begin{array}{c} \mathcal{D}_b(u(s), \mathcal{K}(t, s, u(s))), \mathcal{D}_b(v(s), \mathcal{K}(t, s, v(s))), \\ \mathcal{D}_b(u(s), \mathcal{K}(t, s, v(s))), \mathcal{D}_b(v(s), \mathcal{K}(t, s, u(s))) \end{array} \right\} \end{array} \right)$$

for all $t,s \in [a,b]$, all $u,v \in C[a,b]$ and all $k_u(t,s) \in \mathcal{K}_u(t,s)$, $k_v(t,s) \in \mathcal{K}_v(t,s)$, where $\lambda \geq 0$, $p > 1$;

(**III**) there exists $\tau \in [1,+\infty)$ such that

$$\sup_{t\in[a,b]} \int_a^b \Upsilon(t,\tau)\,d\tau \leq \frac{e^{-\tau}}{2^{p-1}}.$$

Theorem 5.1. *Under the conditions* (**I**)–(**III**), *the integral inclusion* (17) *has a solution in* $C[a,b]$.

Proof. Let $\mathcal{E} = C[a,b]$ (with b-metric d_b as defined in (18)) and consider the set-valued operator $\mathcal{J} : \mathcal{E} \to \mathcal{P}_{cb}(\mathcal{E})$ defined by

$$\mathcal{J}x = \left\{ y \in \mathcal{E} : y(t) \in f(t) + \int_a^b \mathcal{K}(t,s,x(s))\,ds,\ t \in [a,b] \right\}.$$

It is clear that the set of solutions of the integral inclusion (17) coincides with the set of fixed points of the operator $\mathcal{J}$. Hence, we have to prove that under the given conditions, $\mathcal{J}$ has at least one fixed point in $\mathcal{E}$. For this, we shall check that the conditions of Theorem 3.2 hold true.

Let $x \in \mathcal{E}$ be arbitrary. For the set-valued operator $\mathcal{K}_x(t,s) : [a,b]\times[a,b] \to \mathcal{P}_{cb}(\mathbb{R})$, it follows from the Michael's selection theorem that there exists a continuous operator $k_x : [a,b]\times[a,b] \to \mathbb{R}$ such that $k_x(t,s) \in \mathcal{K}_x(t,s)$ for each $(t,s) \in [a,b]\times[a,b]$. It follows that $f(t) + \int_a^b k_x(t,s)\,ds \in \mathcal{J}x$. Hence, $\mathcal{J}x \neq \emptyset$. Since f and $\mathcal{K}_x$ are continuous on $[a,b]$, resp. $[a,b]^2$, their ranges are bounded and hence $\mathcal{J}x$ is bounded, i.e., $\mathcal{J} : \mathcal{E} \to \mathcal{P}_{cb}(\mathcal{E})$.

We will check that the contractive condition (2) holds for $\mathcal{J}$ in $\mathcal{E}$ with some $\tau > 0$, $\lambda \geq 0$ and $F \in \mathfrak{F}_s$, i.e.,

$$\tau + F(s\delta_b(\mathcal{J}x_1,\mathcal{J}x_2)) \leq F\left(\begin{array}{c} \max\left\{d_b(x_1,x_2), \mathcal{D}_b(x_1,\mathcal{J}x_1), \mathcal{D}_b(x_2,\mathcal{J}x_2), \frac{\mathcal{D}_b(x_1,\mathcal{J}x_2)+\mathcal{D}_b(x_2,\mathcal{J}x_1)}{2^p}\right\} \\ +\lambda\min\{\mathcal{D}_b(x_1,\mathcal{J}x_1), \mathcal{D}_b(x_2,\mathcal{J}x_2), \mathcal{D}_b(x_1,\mathcal{J}x_2), \mathcal{D}_b(x_2,\mathcal{J}x_1)\} \end{array}\right) \tag{19}$$

for elements $x_1, x_2 \in \mathcal{E}$. Let $y_1 \in \mathcal{J}x_1$ be arbitrary, i.e.,

$$y_1(t) \in f(t) + \int_a^b \mathcal{K}(t,s,x_1(s))\,ds, \quad t \in [a,b]$$

holds true. This means that for all $t,s \in [a,b]$ there exists $k_{x_1}(t,s) \in \mathcal{K}_{x_1}(t,s) = \mathcal{K}(t,s,x_1(s))$ such that $y_1(t) = f(t) + \int_a^b k_{x_1}(t,s)\,ds$ for $t \in [a,b]$.

For all $x_1, x_2 \in \mathcal{E}$, it follows from (**II**) that

$$|k_{x_1}(t,s) - k_{x_2}(t,s)|^p \leq \Upsilon(t,s)\left(\begin{array}{c} \max\left\{\begin{array}{c} d_b(x_1(s),x_2(s)), \mathcal{D}_b(x_1(s),\mathcal{K}(t,s,x_1(s))), \\ \mathcal{D}_b(x_2(s),\mathcal{K}(t,s,x_2(s))), \\ \frac{\mathcal{D}_b(x_1(s),\mathcal{K}(t,s,x_2(s)))+\mathcal{D}_b(x_2(s),\mathcal{K}(t,s,x_1(s)))}{2^p} \end{array}\right\} \\ +\lambda\min\left\{\begin{array}{c} \mathcal{D}_b(x_1(s),\mathcal{K}(t,s,x_1(s))), \mathcal{D}_b(x_2(s),\mathcal{K}(t,s,x_2(s))), \\ \mathcal{D}_b(x_1(s),\mathcal{K}(t,s,x_2(s))), \mathcal{D}_b(x_2(s),\mathcal{K}(t,s,x_1(s))) \end{array}\right\} \end{array}\right).$$

It means that there exists $z(t,s) \in \mathcal{K}_{x_2}(t,s)$ such that

$$|k_{x_1}(t,s) - z(t,s)|^p \leq \Upsilon(t,s)\left(\begin{array}{c} \max\left\{\begin{array}{c} d_b(x_1(s),x_2(s)), \mathcal{D}_b(x_1(s),\mathcal{K}_{x_1}(t,s)), d_b(x_2(s),z(t,s)), \\ \frac{d_b(x_1(s),z(t,s))+\mathcal{D}_b(x_2(s),\mathcal{K}_{x_1}(t,s))}{2^p} \end{array}\right\} \\ +\lambda\min\left\{\begin{array}{c} \mathcal{D}_b(x_1(s),\mathcal{K}_{x_1}(t,s)), d_b(x_2(s),z(t,s)), \\ d_b(x_1(s),z(t,s)), \mathcal{D}_b(x_2(s),\mathcal{K}_{x_1}(t,s)) \end{array}\right\} \end{array}\right) =: R(t,s),$$

for all $t,s \in [a,b]$.

Denote by $\mathcal{U}(t,s) : [a,b]\times[a,b] \to \mathcal{P}_{cb}(\mathbb{R})$ the operator defined by

$$\mathcal{U}(t,s) = \mathcal{K}_{x_2}(t,s) \cap \{u \in \mathbb{R} : d_b(k_{x_1}(t,s),u)) \leq R(t,s)\}.$$

Since, by (**I**), $\mathcal{U}$ is lower semicontinuous, it follows that there exists a continuous operator $k_{x_2} : [a,b] \times [a,b] \to \mathbb{R}$ such that $k_{x_2}(t,s) \in \mathcal{U}(t,s)$ for $t, s \in [a,b]$. Then $y_2(t) := f(t) + \int_a^b k_{x_2}(t,s)\, ds$ satisfies that

$$y_2(t) \in f(t) + \int_a^b \mathcal{K}(t,s,x_2(s))\, ds, \quad t \in [a,b],$$

i.e., $y_2 \in \mathcal{J}x_2$ and

$$\begin{aligned}
d_b(y_1, y_2) &\le \max_{t \in [a,b]} \int_a^b |k_{x_1}(t,s) - k_{x_2}(t,s)|^p\, ds \\
&\le \max_{t \in [a,b]} \int_a^b \Upsilon(t,s) \left(\begin{array}{c} \max \left\{ \begin{array}{c} d_b(x_1(s), x_2(s)), d_b(x_1(s), k_{x_1}(t,s), d_b(x_2(s), k_{x_2}(t,s))), \\ \frac{d_b(x_1(s), k_{x_2}(t,s)) + d_b(x_2(s), k_{x_1}(t,s))}{2^p} \end{array} \right\} \\ +\lambda \min \left\{ \begin{array}{c} d_b(x_1(s), k_{x_1}(t,s)), d_b(x_2(s), k_{x_2}(t,s)), \\ d_b(x_1(s), k_{x_2}(t,s)), d_b(x_2(s), k_{x_1}(t,s)) \end{array} \right\} \end{array} \right) ds \\
&\le \frac{e^{-\tau}}{2^{p-1}} \left(\begin{array}{c} \max \left\{ d_b(x_1, x_2), \mathcal{D}_b(x_1, \mathcal{J}x_1), \mathcal{D}_b(x_2, \mathcal{J}x_2), \frac{\mathcal{D}_b(x_1, \mathcal{J}x_2)) + \mathcal{D}_b(x_2, \mathcal{J}x_1)}{2^p} \right\} \\ +\lambda \min\{\mathcal{D}_b(x_1, \mathcal{J}x_1), \mathcal{D}_b(x_2, \mathcal{J}x_2), \mathcal{D}_b(x_1, \mathcal{J}x_2), \mathcal{D}_b(x_2, \mathcal{J}x_1)\} \end{array} \right).
\end{aligned}$$

for all $t, s \in [a,b]$.

Thus, we obtain that

$$\delta_b(\mathcal{J}x_1, \mathcal{J}x_2) \le \frac{e^{-\tau}}{2^{p-1}} \left(\begin{array}{c} \max \left\{ d_b(x_1, x_2), \mathcal{D}_b(x_1, \mathcal{J}x_1), \mathcal{D}_b(x_2, \mathcal{J}x_2), \frac{\mathcal{D}_b(x_1, \mathcal{J}x_2) + \mathcal{D}_b(x_2, \mathcal{J}x_1)}{2^p} \right\} \\ +\lambda \min\{\mathcal{D}_b(x_1, \mathcal{J}x_1), \mathcal{D}_b(x_2, \mathcal{J}x_2), \mathcal{D}_b(x_1, \mathcal{J}x_2), \mathcal{D}_b(x_2, \mathcal{J}x_1)\} \end{array} \right).$$

(This shows again that the sets $\mathcal{J}x_1$ and $\mathcal{J}x_2$ are bounded.) By passing to logarithms, we write

$$\ln(s\delta_b(\mathcal{J}x_1, \mathcal{J}x_2)) \le \ln e^{-\tau} \left(\begin{array}{c} \max \left\{ d_b(x_1, x_2), \mathcal{D}_b(x_1, \mathcal{J}x_1), \mathcal{D}_b(x_2, \mathcal{J}x_2), \frac{\mathcal{D}_b(x_1, \mathcal{J}x_2) + \mathcal{D}_b(x_2, \mathcal{J}x_1)}{2^p} \right\} \\ +\lambda \min\{\mathcal{D}_b(x_1, \mathcal{J}x_1), \mathcal{D}_b(x_2, \mathcal{J}x_2), \mathcal{D}_b(x_1, \mathcal{J}x_2), \mathcal{D}_b(x_2, \mathcal{J}x_1)\} \end{array} \right).$$

Taking the function $F \in \mathfrak{F}_s$ defined by $F(\alpha) = \ln \alpha$, we obtain that the condition (19) is fulfilled.

Using Theorem 3.2, we conclude that the given integral inclusion has a solution. □

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