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Solutions for a class of nonlinear Volterra integral and integro-differential equation using cyclic (φ, ψ, θ)-contraction

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Abstract
We establish the existence and uniqueness of solutions for a class of nonlinear Volterra integral and integro-differential equations using fixed-point theorems for a new variant of cyclic (φ, ψ, θ)-contractive mappings. Nontrivial examples are given to support the usability of our results.
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1 Introduction and preliminaries

Integral equation methods are very useful for solving many problems in several applied fields like mathematical economics and optimal control theory, because these problems are often reduced to integral equations. Since these equations usually cannot be solved explicitly, so it is necessary to get different numerical techniques. There are numerous advanced and efficient methods, which have been focusing on the solution of integral equations.

Integral equations appear in many forms. Two distinct ways that depend on the limits of integration are used to characterize integral equations, namely:

1. If the limits of integration are fixed, the integral equation is called a Fredholm integral equation given in the form:

$$u(x) = f(x) + \int_a^b K(x,t)u(t)\,dt, \tag{1.1}$$

where a and b are constants.

2. If at least one limit is a variable, the equation is called a Volterra integral equation given in the form:

$$u(x) = f(x) + \int_a^x K(x,t)u(t)\,dt. \tag{1.2}$$

It is interesting to point out that any equation that includes both integrals and derivatives of the unknown function $u(x)$ is called an integro-differential equation. The Volterra integro-differential equation is of the form:

$$u^{(n)}(x) = f(x) + \int_a^x K(x,t)u(t)\,dt, \quad \text{where } u^{(n)} = \frac{d^n u}{dx^n}. \tag{1.3}$$

Nonlinear analysis is a remarkable confluence of topology, analysis and applied mathematics. Indeed, the fixed-point theory is one of the most rapidly growing topic of nonlinear functional analysis. It is a vast and interdisciplinary subject whose study belongs to several mathematical domains such as: classical analysis, functional analysis, operator theory, topology and algebraic topology, *etc.* This topic has grown very rapidly perhaps due to its interesting applications in various fields within and out side the mathematics such as: integral equations, initial and boundary value problems for ordinary and partial differential equations, game theory, optimal control, nonlinear oscillations, variational inequalities, complementary problems, economics and others.

The celebrated Banach contraction principle is a fundamental piece both in several branches of functional analysis and in many applications. This important fixed-point theorem can be stated as follows.

Theorem 1.1 [1] *Let* $(\mathcal{X}, d)$ *be a complete metric space and* $\mathcal{T}$ *be a self-map of* $\mathcal{X}$ *satisfying*:

$$d(\mathcal{T}x, \mathcal{T}y) \leq kd(x,y) \quad \forall x,y \in \mathcal{X}, \tag{1.4}$$

where k *is a constant in* $(0,1)$. *Then,* $\mathcal{T}$ *has a unique fixed point* $\xi \in \mathcal{X}$.

Due to its relevance, generalizations of Banach's fixed-point theorem have been studied by many authors (see, *e.g.*, [2] and references cited therein). Many works have been done for getting the solution of linear and nonlinear Volterra integral and integro-differential equations using Banach's fixed-point theorem (see Pachpatte [3, 4] and references cited therein).

A very important fact that condition (1.4) implies continuity of $\mathcal{T}$, suggests in a natural way the question of obtaining fixed-point results for metric spaces where the involved self-map is not necessarily continuous. This question is answered by Kirk *et al.* [5] and turned the area of investigation of fixed point by introducing cyclic representations and cyclic contractions, which can be stated as follows:

A mapping $\mathcal{T} : \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ is called cyclic if $\mathcal{T}(\mathcal{A}) \subseteq \mathcal{B}$ and $\mathcal{T}(\mathcal{B}) \subseteq \mathcal{A}$, where $\mathcal{A}, \mathcal{B}$ are nonempty subsets of a metric space $(\mathcal{X}, d)$. Moreover, $\mathcal{T}$ is called cyclic contraction if there exists $k \in (0,1)$ such that $d(\mathcal{T}x, \mathcal{T}y) \leq kd(x,y)$ for all $x \in \mathcal{A}$ and $y \in \mathcal{B}$. Notice that although a contraction is continuous, cyclic contraction need not to be. This is one of the important gains of this theorem which motivates, in a natural way, the following notion.

Definition 1.1 (See [5, 6]) Let $(\mathcal{X}, d)$ be a complete metric space. Let p be a positive integer, $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_p$ be nonempty subsets of $\mathcal{X}$, $\mathcal{Y} = \bigcup_{i=1}^{p} \mathcal{A}_i$ and $\mathcal{T} : \mathcal{Y} \to \mathcal{Y}$. Then $\mathcal{Y} = \bigcup_{i=1}^{p} \mathcal{A}_i$ is said to be a cyclic representation of $\mathcal{Y}$ with respect to $\mathcal{T}$ if

(i) $\mathcal{A}_i$, $i = 1, 2, \ldots, p$, are nonempty closed sets, and

(ii) $\mathcal{T}(\mathcal{A}_1) \subseteq \mathcal{A}_2, \ldots, \mathcal{T}(\mathcal{A}_{p-1}) \subseteq \mathcal{A}_p, \mathcal{T}(\mathcal{A}_p) \subseteq \mathcal{A}_1$.

Following [5], a number of fixed-point theorems on cyclic representation of $\mathcal{Y}$ with respect to a self-mapping $\mathcal{T}$ have appeared (see, *e.g.*, [6–14]).

To continue the investigation specified in [5], a new variant of cyclic contractive mappings satisfying generalized altering distance function, which is followed by the existence and uniqueness of fixed points for such mappings is discussed here. The obtained result generalizes and improves many existing theorems in the literature. Some examples are given in the support of our results. Finally, applications to the solutions for a class of nonlinear Volterra integral and integro-differential equation using cyclic (φ,ψ,θ)-contraction is presented.

2 Main results

In the sequel, we designate $\mathbb{R}$ the set of all real numbers, the set of all real nonnegative numbers by $\mathbb{R}^+$ and the set of all natural numbers by $\mathbb{N}$.

To introduce a new variant of cyclic contraction we need the notion of different type of altering distance function.

Definition 2.1 [15] A function $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ is called an altering distance function if the following properties are satisfied:

(a) φ is continuous and nondecreasing, and
(b) $\varphi(t)=0 \Leftrightarrow t=0$.

Let Φ denote the set of all functions $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ satisfying
(a) φ is continuous nondecreasing;
(b) $\varphi^{-1}(\{0\})=\{0\}$.
Let Ψ denote the set of all functions $\psi:\mathbb{R}^+\to\mathbb{R}^+$ satisfying
(c) $\lim_{t\to r^+}\psi(t)>0$ (and finite) for all $r>0$;
(d) $\lim_{t\to 0^+}\psi(t)=0$.
Let Θ denote the set of all functions $\theta:{\mathbb{R}^+}^4\to\mathbb{R}^+$ satisfying
(e) θ is continuous;
(f) $\theta(t_1,t_2,t_3,t_4)=0$ if and only if $t_1t_2t_3t_4=0$.
The examples of function ψ are given in [14]; see also [5, 16].
Now, we give some examples of functions θ satisfying (e) and (f).

Example 2.1 The following functions belong to Θ:

(1) $\theta(t_1,t_2,t_3,t_4)=\hbar\min\{t_1,t_2,t_3,t_4\}$;
(2) $\theta(t_1,t_2,t_3,t_4)=t_1t_2t_3t_4$;
(3) $\theta(t_1,t_2,t_3,t_4)=\ln(1+t_1t_2t_3t_4)$;
(4) $\theta(t_1,t_2,t_3,t_4)=e^{t_1t_2t_3t_4}-1$.

Now we can state the notion of cyclic (φ,ψ,θ)-contraction as the following.

Definition 2.2 Let $(\mathcal{X},d)$ be a metric space. Let p be a positive integer, $\mathcal{A}_1,\mathcal{A}_2,\dots,\mathcal{A}_p$ be nonempty subsets of $\mathcal{X}$ and $\mathcal{Y}=\bigcup_{i=1}^p\mathcal{A}_i$. An operator $\mathcal{T}:\mathcal{Y}\to\mathcal{Y}$ is called cyclic (φ,ψ,θ)-contractive, if

(I) $\mathcal{Y}=\bigcup_{i=1}^p\mathcal{A}_i$ is a cyclic representation of $\mathcal{Y}$ with respect to $\mathcal{T}$;
(II) for any $(x,y)\in\mathcal{A}_i\times\mathcal{A}_{i+1}$, $i=1,2,\dots,p$ (with $\mathcal{A}_{p+1}=\mathcal{A}_1$),

$$\varphi\bigl(d(\mathcal{T}x,\mathcal{T}y)\bigr)\le\varphi\bigl(d(x,y)\bigr)-\psi\bigl(d(x,y)\bigr)+\theta\bigl(d(x,\mathcal{T}x),d(y,\mathcal{T}y),d(x,\mathcal{T}y),d(y,\mathcal{T}x)\bigr),$$

where $\varphi\in\Phi$, $\psi\in\Psi$ and $\theta\in\Theta$.

Example 2.2 Let $\mathcal{X} = [0,1]$ with the usual metric. Suppose $\mathcal{A}_1 = [0, \frac{1}{8}]$ and $\mathcal{A}_2 = [\frac{1}{8}, 1]$ and $\mathcal{X} = \bigcup_{i=1}^{2} \mathcal{A}_i$. Define $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ such that

$$\mathcal{T}x = \begin{cases} \frac{1}{8}, & x \in [0,1), \\ 0, & x = 1. \end{cases} \tag{2.1}$$

Clearly, $\mathcal{A}_1$ and $\mathcal{A}_2$ are closed subsets of $\mathcal{X}$. Moreover $\mathcal{T}(\mathcal{A}_i) \subset \mathcal{A}_{i+1}$ for $i = 1, 2$, so that $\bigcup_{i=1}^{2} \mathcal{A}_i$ is a cyclic representation of $\mathcal{X}$ with respect to $\mathcal{T}$. Furthermore, let $\varphi, \psi, \theta : \mathbb{R}^+ \to \mathbb{R}^+$ defined by

$$\varphi(t) = \frac{t}{2}, \qquad \psi(t) = \frac{t}{8} \quad \text{and} \quad \theta(t_1, t_2, t_3, t_4) = \hbar \min\{t_1, t_2, t_3, t_4\} \quad \forall t, t_1, t_2, t_3, t_4 \in \mathbb{R}^+.$$

Now we show that $\mathcal{T}$ satisfies cyclic (φ, ψ, θ)-contractive.

For $x \in \mathcal{A}_1$, $y \in \mathcal{A}_2$ (or $x \in \mathcal{A}_2$, $y \in \mathcal{A}_1$).

- When $x \in [0, \frac{1}{8}]$ and $y \in [\frac{1}{8}, 1)$, we deduce $d(\mathcal{T}x, \mathcal{T}y) = 0$ and equation (II) is trivially satisfied.
- When $x \in [0, \frac{1}{8}]$ and $y = 1$, we deduce $d(\mathcal{T}x, \mathcal{T}y) = \frac{1}{8}$, $\theta(d(x, \mathcal{T}x), d(y, \mathcal{T}y), d(x, \mathcal{T}y), d(y, \mathcal{T}x)) = 0$, and then equation (II) holds as it reduces to

$$\begin{aligned} \varphi\big(d(\mathcal{T}x, \mathcal{T}y)\big) &= \frac{1}{16} \leq \frac{3}{8}|x - 1| \\ &= \varphi\big(|x-1|\big) - \psi\big(|x-1|\big) + \theta\big(d(x, \mathcal{T}x), d(y, \mathcal{T}y), d(x, \mathcal{T}y), d(y, \mathcal{T}x)\big). \end{aligned}$$

Example 2.3 Let $\mathcal{X} = \mathbb{R}$ with the usual metric. Suppose $\mathcal{A}_1 = [-1, 0] = \mathcal{A}_3$ and $\mathcal{A}_2 = [0, 1] = \mathcal{A}_4$ and $Y = \bigcup_{i=1}^{4} A_i$. Define $\mathcal{T} : \mathcal{Y} \to \mathcal{Y}$ such that $\mathcal{T}x = \frac{-x}{3}$ for all $x \in \mathcal{Y}$. It is clear that $\bigcup_{i=1}^{4} \mathcal{A}_i$ is a cyclic representation of $\mathcal{Y}$ with respect to $\mathcal{T}$.

Define $\varphi, \psi, \theta : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$\varphi(t) = \frac{t}{4}, \qquad \psi(t) = \frac{t}{16} \quad \text{and} \quad \theta(t_1, t_2, t_3, t_4) = t_1 t_2 t_3 t_4 \quad \forall t, t_1, t_2, t_3, t_4 \geq 0.$$

It can be easily shown that the map $\mathcal{T}$ is cyclic (φ, ψ, θ)-contractive. Indeed, let $L = \varphi(d(\mathcal{T}x, \mathcal{T}y)) = \frac{|x-y|}{12}$ and

$$R = \frac{3}{16}|x - y| = \varphi\big(d(x,y)\big) - \psi\big(d(x,y)\big) + \theta\big(d(x, \mathcal{T}x), d(y, \mathcal{T}y), d(x, \mathcal{T}y), d(y, \mathcal{T}x)\big).$$

- When $x = -1$ and $y = 0$, we deduce $L = \frac{1}{12}$ and $R = \frac{3}{16}$.
- When $x = 0$ and $y = 1$, we deduce $L = \frac{1}{12}$ and $R = \frac{3}{16}$.
- When $x = -1$ and $y = 1$, we deduce $L = \frac{1}{6}$ and $R = \frac{499}{648}$.

Similarly other cases can be discussed. Hence, $\mathcal{T}$ is a cyclic contractive (φ, ψ, θ)-condition.

Our main result is the following.

Theorem 2.1 *Let $(\mathcal{X}, d)$ be a complete metric space, $p \in \mathbb{N}$, $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_p$ nonempty closed subsets of $\mathcal{X}$ and $\mathcal{Y} = \bigcup_{i=1}^{p} \mathcal{A}_i$. Suppose the mapping $\mathcal{T} : \mathcal{Y} \to \mathcal{Y}$ is cyclic (φ, ψ, θ)-contractive, for some $\varphi \in \Phi$, $\psi \in \Psi$ and $\theta \in \Theta$. Then $\mathcal{T}$ has a unique fixed point. Moreover, the fixed point of $\mathcal{T}$ belongs to $\bigcap_{i=1}^{p} \mathcal{A}_i$.*

Proof Let $x_0 \in \mathcal{A}_1$ (such a point exists since $\mathcal{A}_1 \neq \emptyset$). Define a sequence $\{x_n\}$ in $\mathcal{X}$ by:

$$x_{n+1} = \mathcal{T} x_n, \quad n = 0, 1, 2, \ldots.$$

If there is $k \in \mathbb{N} \cup \{0\}$ such that $x_k = x_{k+1}$, then $x_k = x_n$ for all $n \geq k$, so x_k is a fixed point of $\mathcal{T}$ and $x_k \in \bigcap_{i=1}^{p} \mathcal{A}_i$.

Then we assume that

$$x_n \neq x_{n+1} \quad \forall n \in \mathbb{N} \cup \{0\}. \tag{2.2}$$

We shall prove that the sequence $\{d(x_n, x_{n+1})\}$ is nonincreasing with

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.3}$$

Indeed, suppose that, for some $n \in \mathbb{N}$,

$$d(x_{n+1}, x_{n+2}) > d(x_n, x_{n+1}).$$

Using this together with the properties of functions ψ, φ, θ, we get

$$\begin{aligned} \varphi\big(d(x_{n+1}, x_{n+2})\big) &\leq \varphi\big(d(x_n, x_{n+1})\big) - \psi\big(d(x_n, x_{n+1})\big) \\ &\quad + \theta\big(d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), d(x_n, x_{n+2}), d(x_{n+1}, x_{n+1})\big) \\ &\leq \varphi\big(d(x_n, x_{n+1})\big) - \psi\big(d(x_n, x_{n+1})\big), \end{aligned}$$

which, in view of the fact that $\psi \geq 0$, yields

$$\varphi\big(d(x_{n+1}, x_{n+2})\big) \leq \varphi\big(d(x_n, x_{n+1})\big),$$

which, in turn, by condition (a) we deduce that

$$d(x_{n+1}, x_{n+2}) \leq d(x_n, x_{n+1}) \quad \forall n \in \mathbb{N}. \tag{2.4}$$

Then, $\{d(x_{n+1}, x_{n+2})\}$ is a nonincreasing sequence of positive real numbers. This implies that there exists $r \geq 0$ such that

$$\lim_{n\to\infty} d(x_{n+1}, x_{n+2}) = r. \tag{2.5}$$

Since

$$\begin{aligned} \varphi\big(d(x_{n+1}, x_{n+2})\big) &\leq \varphi\big(d(x_n, x_{n+1})\big) - \psi\big(d(x_n, x_{n+1})\big) \\ &\quad + \theta\big(d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), d(x_n, x_{n+2}), d(x_{n+1}, x_{n+1})\big) \\ &= \varphi\big(d(x_n, x_{n+1})\big) - \psi\big(d(x_n, x_{n+1})\big) \end{aligned} \tag{2.6}$$

we deduce, passing to the limit as $n \to \infty$ in (2.6) and using continuity of φ, that

$$\varphi(r) \leq \varphi(r) - \lim_{n\to\infty} \psi\big(d(x_n, x_{n+1})\big).$$

From condition (c) and using (2.5), we have

$$\lim_{n\to\infty} \psi\big(d(x_n, x_{n+1})\big) = \lim_{t\to r^+} \psi(t) > 0.$$

We get

$$\varphi(r) \le \varphi(r) - \lim_{t\to r^+} \psi(t) < \varphi(r),$$

a contradiction, and thus $r = 0$. Hence, (2.3) is proved.

Now, we shall prove that $\{x_n\}$ is a Cauchy sequence in $(\mathcal{X}, d)$. Suppose to the contrary that $\{x_n\}$ is not a Cauchy sequence, then there exists $\varepsilon > 0$ for which we can find two sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ such that for all positive integers k,

$$n(k) > m(k) > k, \quad d(x_{m(k)}, x_{n(k)}) \ge \varepsilon, \qquad d(x_{m(k)}, x_{n(k)-1}) < \varepsilon. \tag{2.7}$$

Using (2.7) and the triangle inequality, we get

$$\begin{aligned}\varepsilon &\le d(x_{n(k)}, x_{m(k)}) \\ &\le d(x_{m(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{n(k)}) \\ &< \varepsilon + d(x_{n(k)}, x_{n(k)-1}).\end{aligned}$$

Passing to the limit as $k \to \infty$ in the above inequality and using (2.3), we obtain

$$\lim_{k\to\infty} d(x_{n(k)}, x_{m(k)}) = \varepsilon. \tag{2.8}$$

On the other hand, for all k, there exists $j(k) \in \{1, \dots, p\}$ such that $n(k) - m(k) + j(k) \equiv 1[p]$. Then $x_{m(k)-j(k)}$ (for k large enough, $m(k) > j(k)$) and $x_{n(k)}$ lie in different adjacently labeled sets $\mathcal{A}_i$ and $\mathcal{A}_{i+1}$ for certain $i \in \{1, \dots, p\}$.

Using the triangle inequality, we get

$$\begin{aligned}&\left|d(x_{m(k)-j(k)}, x_{n(k)}) - d(x_{n(k)}, x_{m(k)})\right| \\ &\quad \le d(x_{m(k)-j(k)}, x_{m(k)}) \\ &\quad \le \sum_{l=0}^{j(k)-1} d(x_{m(k)-j(k)+l}, x_{m(k)-j(k)+l+1}) \\ &\quad \le \sum_{l=0}^{p-1} d(x_{m(k)-j(k)+l}, x_{m(k)-j(k)+l+1}) \to 0 \quad \text{as } k \to \infty \text{ (from (2.3))},\end{aligned}$$

which, by (2.8), implies that

$$\lim_{k\to\infty} d(x_{m(k)-j(k)}, x_{n(k)}) = \varepsilon. \tag{2.9}$$

Using (2.3), we have

$$\lim_{k\to\infty} d(x_{m(k)-j(k)+1}, x_{m(k)-j(k)}) = 0 \tag{2.10}$$

and

$$\lim_{k\to\infty} d(x_{n(k)+1}, x_{n(k)}) = 0. \tag{2.11}$$

Again, using the triangle inequality, we get

$$\left|d(x_{m(k)-j(k)}, x_{n(k)+1}) - d(x_{m(k)-j(k)}, x_{n(k)})\right| \le d(x_{n(k)}, x_{n(k)+1}).$$

Passing to the limit as $k \to \infty$ in the above inequality, and using (2.11) and (2.9), we get

$$\lim_{k\to\infty} d(x_{m(k)-j(k)}, x_{n(k)+1}) = \varepsilon. \tag{2.12}$$

Therefore, from the inequality

$$\left|d(x_{n(k)+1}, x_{m(k)-j(k)+1}) - d(x_{m(k)-j(k)}, x_{n(k)+1})\right| \le d(x_{m(k)-j(k)}, x_{m(k)-j(k)+1})$$

we deduce, passing to the limit as $k \to \infty$, and using (2.3) and (2.12), that

$$\lim_{k\to\infty} d(x_{m(k)-j(k)+1}, x_{n(k)+1}) = \varepsilon. \tag{2.13}$$

Hence, by the continuity of φ and (2.13), we get

$$\varphi(\varepsilon) = \lim_{k\to+\infty} \varphi\bigl(d(\mathcal{T}x_{m(k)-j(k)}, \mathcal{T}x_{n(k)})\bigr). \tag{2.14}$$

Using (II), we obtain

$$\begin{aligned} &\varphi\bigl(d(x_{m(k)-j(k)+1}, x_{n(k)+1})\bigr) \\ &\quad\le \varphi\bigl(d(x_{m(k)-j(k)}, x_{n(k)})\bigr) - \psi\bigl(d(x_{m(k)-j(k)}, x_{n(k)})\bigr) - \theta\bigl(d(x_{m(k)-j(k)+1}, x_{m(k)-j(k)}), \\ &\qquad d(x_{n(k)+1}, x_{n(k)}), d(x_{m(k)-j(k)+1}, x_{n(k)}), d(x_{n(k)+1}, x_{m(k)-j(k)})\bigr) \end{aligned} \tag{2.15}$$

for all k. Now, it follows from (2.9), (2.10), (2.11) and the properties of Φ, Ψ, Θ we get

$$\begin{aligned} \lim_{k\to\infty} \varphi\bigl(d(\mathcal{T}x_{m(k)-j(k)}, \mathcal{T}x_{n(k)})\bigr) &\le \varphi(\varepsilon) - \lim_{t\to r^+} \psi(t) + \theta(\varepsilon, 0, 0, \varepsilon) \\ &= \varphi(\varepsilon) - \lim_{t\to r^+} \psi(t) < \varphi(\varepsilon). \end{aligned} \tag{2.16}$$

Now, combining (2.14) with the above inequality, we get

$$\varphi(\varepsilon) \le \varphi(\varepsilon), \tag{2.17}$$

a contradiction since $\varepsilon > 0$. Thus, we proved that $\{x_n\}$ is a Cauchy sequence in $(\mathcal{X}, d)$.

Since $(\mathcal{X}, d)$ is complete, there exists $\xi \in \mathcal{X}$ such that

$$\lim_{n\to\infty} x_n = \xi. \tag{2.18}$$

We shall prove that

$$\xi \in \bigcap_{i=1}^{p} \mathcal{A}_i. \tag{2.19}$$

From condition (I), and since $x_0 \in \mathcal{A}_1$, we have $\{x_{np}\}_{n\geq 0} \subseteq \mathcal{A}_1$. Since $\mathcal{A}_1$ is closed, from (2.18), we get that $\xi \in \mathcal{A}_1$. Again, from the condition (I), we have $\{x_{np+1}\}_{n\geq 0} \subseteq \mathcal{A}_2$. Since $\mathcal{A}_2$ is closed, from (2.18), we get that $\xi \in \mathcal{A}_2$. Continuing this process, we obtain (2.19).

Now, we shall prove that ξ is a fixed point of $\mathcal{T}$. Indeed, from (2.19), since for all n, there exists $i(n) \in \{1, 2, \dots, p\}$ such that $x_n \in \mathcal{A}_{i(n)}$, applying (II) with $x = \xi$ and $y = x_n$, we obtain

$$\begin{aligned}
\varphi\bigl(d(\mathcal{T}\xi, x_{n+1})\bigr) &= \varphi\bigl(d(\mathcal{T}\xi, \mathcal{T}x_n)\bigr) \\
&\leq \varphi\bigl(d(\xi, x_n)\bigr) - \psi\bigl(d(\xi, x_n)\bigr) \\
&\quad + \theta\bigl(d(\xi, \mathcal{T}\xi), d(x_n, x_{n+1}), d(\xi, x_{n+1}), d(x_n, \mathcal{T}\xi)\bigr) \\
&\leq \varphi\bigl(d(\xi, x_n)\bigr) + \theta\bigl(d(\xi, \mathcal{T}\xi), d(x_n, x_{n+1}), d(\xi, x_{n+1}), d(x_n, \mathcal{T}\xi)\bigr)
\end{aligned} \tag{2.20}$$

for all n. Passing to the limit as $n \to \infty$ in (2.20), and using (2.18), we get

$$\begin{aligned}
\varphi\bigl(d(\xi, \mathcal{T}\xi)\bigr) &\leq \varphi(0) + \theta\bigl(d(\xi, \mathcal{T}\xi), 0, 0, d(\xi, \mathcal{T}\xi)\bigr) \\
&= 0,
\end{aligned}$$

which holds unless $\varphi(d(\xi, \mathcal{T}\xi)) = 0$, so

$$\xi = \mathcal{T}\xi \tag{2.21}$$

that is, ξ is a fixed point of $\mathcal{T}$.

Finally, we prove that ξ is the unique fixed point of $\mathcal{T}$. Assume that ζ is another fixed point of $\mathcal{T}$, that is, $\mathcal{T}\zeta = \zeta$. By the condition (I), this implies that $\zeta \in \bigcap_{i=1}^{p} \mathcal{A}_i$. Then we can apply (II) for $x = \xi$ and $y = \zeta$. We obtain

$$\begin{aligned}
\varphi\bigl(d(\xi, \zeta)\bigr) &= \varphi\bigl(d(\mathcal{T}\xi, \mathcal{T}\zeta)\bigr) \\
&\leq \varphi\bigl(d(\xi, \zeta)\bigr) - \psi\bigl(d(\xi, \zeta)\bigr) + \theta\bigl(d(\xi, \mathcal{T}\xi), d(\zeta, \mathcal{T}\zeta), d(\xi, \mathcal{T}\zeta), d(\zeta, \mathcal{T}\xi)\bigr) \\
&= \varphi\bigl(d(\xi, \zeta)\bigr) - \psi\bigl(d(\xi, \zeta)\bigr),
\end{aligned}$$

which, by the fact that $\psi \geq 0$, implies

$$\varphi\bigl(d(\xi, \zeta)\bigr) \leq \varphi\bigl(d(\xi, \zeta)\bigr),$$

a contradiction, and thus $d(\xi, \zeta) = 0$, that is, $\xi = \zeta$. Thus, we proved the uniqueness of the fixed point. □

In the following, we deduce some fixed-point theorems from our main result given by Theorem 2.1.

If we take $p = 1$ and $\mathcal{A}_1 = \mathcal{X}$ in Theorem 2.1, then we get immediately the following fixed-point theorem.

Corollary 2.1 *Let $(\mathcal{X}, d)$ be a complete metric space and let $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ satisfying the following condition: there exist $\varphi \in \Phi$, $\psi \in \Psi$ and $\theta \in \Theta$ in Theorem 2.1 such that*

$$\varphi\big(d(\mathcal{T}x, \mathcal{T}y)\big) \le \varphi\big(d(x,y)\big) - \psi\big(d(x,y)\big) + \theta\big(d(x,\mathcal{T}x), d(y,\mathcal{T}y), d(x,\mathcal{T}y), d(y,\mathcal{T}x)\big)$$

for all $x, y \in \mathcal{X}$. Then $\mathcal{T}$ has a unique fixed point.

Remark 2.1 Corollary 2.1 extends and generalizes many existing fixed-point theorems in the literature [1, 9, 10, 15–24].

Now, it is easy to state a corollary of Theorem 2.1 involving a contraction of integral type.

Corollary 2.2 *Let $\mathcal{T}$ satisfy the conditions of Theorem 2.1, except that condition (II) is replaced by the following: there exists a positive Lebesgue integrable function u on $\mathbb{R}_+$ such that $\int_0^{\varepsilon} u(t)\,dt > 0$ for each $\varepsilon > 0$ and that*

$$\begin{aligned}\int_0^{\varphi(d(\mathcal{T}x,\mathcal{T}y))} u(t)\,dt \le{}& \int_0^{\varphi(d(x,y))} u(t)\,dt - \int_0^{\psi(d(x,y))} u(t)\,dt \\ &+ \int_0^{\theta(d(x,y),d(x,\mathcal{T}x),d(y,\mathcal{T}y),d(x,\mathcal{T}y),d(y,\mathcal{T}x))} u(t)\,dt.\end{aligned}$$

Then $\mathcal{T}$ has a unique fixed point. Moreover, the fixed point of $\mathcal{T}$ belongs to $\bigcap_{i=1}^{p} \mathcal{A}_i$.

A number of fixed-point results may be obtained by assuming different forms for the functions ψ, φ and θ. In particular, fixed-point results under various contractive conditions may be derived from the above theorems.

For example, if we consider $\varphi(t) = t$, $\psi(t) = (1-k)t$, we obtain the following results.

Corollary 2.3 *Let $(\mathcal{X}, d)$ be a complete metric space, $p \in \mathbb{N}$, $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_p$ nonempty closed subsets of $\mathcal{X}$, $\mathcal{Y} = \bigcup_{i=1}^{p} \mathcal{A}_i$ and $\mathcal{T} : \mathcal{Y} \to \mathcal{Y}$ such that*

(I)′ *$\mathcal{Y} = \bigcup_{i=1}^{p} \mathcal{A}_i$ is a cyclic representation of $\mathcal{Y}$ with respect to $\mathcal{T}$;*
(II)′ *for any $(x, y) \in \mathcal{A}_i \times \mathcal{A}_{i+1}$, $i = 1, 2, \dots, p$ (with $\mathcal{A}_{p+1} = \mathcal{A}_1$),*

$$d(\mathcal{T}x, \mathcal{T}y) \le kd(x,y) + \theta\big(d(x,\mathcal{T}x), d(y,\mathcal{T}y), d(x,\mathcal{T}y), d(y,\mathcal{T}x)\big), \tag{2.22}$$

where $k \in [0, 1)$ and $\theta \in \Theta$.

Then $\mathcal{T}$ has a unique fixed point. Moreover, the fixed point of $\mathcal{T}$ belongs to $\bigcap_{i=1}^{p} \mathcal{A}_i$.

Taking $\theta \equiv 0$ in (2.22), we obtain the following.

Corollary 2.4 *Let $(\mathcal{X}, d)$ be a complete metric space, $p \in \mathbb{N}$, $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_p$ nonempty closed subsets of $\mathcal{X}$, $\mathcal{Y} = \bigcup_{i=1}^{p} \mathcal{A}_i$ and $\mathcal{T} : \mathcal{Y} \to \mathcal{Y}$ such that*

(I)′ *$\mathcal{Y} = \bigcup_{i=1}^{p} \mathcal{A}_i$ is a cyclic representation of $\mathcal{Y}$ with respect to $\mathcal{T}$;*

(II)′ *for any* $(x,y) \in \mathcal{A}_i \times \mathcal{A}_{i+1}$, $i = 1,2,\ldots,p$ (*with* $\mathcal{A}_{p+1} = \mathcal{A}_1$),

$$d(\mathcal{T}x,\mathcal{T}y) \leq kd(x,y),$$

where $k \in [0,1)$.

Then $\mathcal{T}$ *has a unique fixed point. Moreover, the fixed point of* $\mathcal{T}$ *belongs to* $\bigcap_{i=1}^{p} \mathcal{A}_i$.

Next, we present some examples showing how our Theorem 2.1 can be used.

Example 2.4 Considering Example 2.3, $\mathcal{T}$ satisfy all conditions of Theorem 2.1 ($p = 4$), and $\mathcal{T}$ has a unique fixed point (which is $x^* = 0 \in \bigcap_{i=1}^{4} A_i$).

Example 2.5 Let $\mathcal{X} = \mathbb{R}$ endowed with the usual metric. Assume $\mathcal{A}_1 = \cdots = \mathcal{A}_m = \{0, \frac{1}{3}, 1\}$ so that $Y = \bigcup_{i=1}^{m} \mathcal{A}_i = \{0, \frac{1}{3}, 1\}$. Define $\mathcal{T} : \mathcal{Y} \to \mathcal{Y}$ such that $\mathcal{T}(0) = \mathcal{T}(1/3) = 0$ and $\mathcal{T}(1) = 1/3$. It is clear that $(\mathcal{X},d)$ is a complete metric space and $\mathcal{Y} = \bigcup_{i=1}^{m} \mathcal{A}_i$ is a cyclic representation of $\mathcal{Y}$ with respect to $\mathcal{T}$. Define also $\psi,\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ by $\varphi(t) = \frac{t}{2}$, $\psi(t) = \frac{t}{4}$ and $\psi,\varphi : {\mathbb{R}^+}^4 \to \mathbb{R}^+$ by $\theta(t_1,t_2,t_3,t_4) = e^{t_1t_2t_3t_4} - 1$ for all $t, t_1, t_2, t_3, t_4 \geq 0$.

It is easy to see that $\mathcal{T}$ satisfy condition of cyclic (φ,ψ,θ)-contraction and so all the hypotheses of Theorem 2.1 ($p = 4$) are satisfied and $\mathcal{T}$ has a unique fixed point (which is $x^* = 0 \in \bigcap_{i=1}^{4} A_i$).

3 An application to integro-differential equations

In this section we present an application of Corollary 2.3 to study the existence and uniqueness of solutions to certain Volterra and Fredholm type integro-differential equations. The examples are inspired by [3].

Consider the nonlinear Volterra type integro-differential equation of the form

$$x(t) = g(t) + \int_a^t f\big(t,s,x(s),x'(s)\big)\,ds \tag{3.1}$$

for $-\infty < a \leq t \leq b < \infty$, where x, g, f are real functions. We shall denote $J = [a,b]$. The functions g ($t \in J$) and f ($a \leq s \leq t \leq b$, $u,v \in \mathbb{R}$) are supposed to be continuous and continuously differentiable with respect to t.

For a real-valued function x, $t \in J$, continuous together with its first derivative x' for $t \in J$, we denote $|x(t)|_1 = |x(t)| + |x'(t)|$. Denote by $\mathcal{E}$ the space of functions which fulfill the condition

$$\big|x(t)\big|_1 = \mathcal{O}\big(\exp(\lambda t)\big), \quad t \in J, \tag{3.2}$$

where λ is a positive constant. Define the norm $\|\cdot\|_{\mathcal{E}}$ in the space $\mathcal{E}$ as

$$|x|_{\mathcal{E}} = \max_{t\in J}\big\{\big|x(t)\big|_1 \exp(-\lambda t)\big\}. \tag{3.3}$$

It is easy to see that $\mathcal{E}$ with the norm defined in (3.3) is a Banach space. We note that the condition (3.2) implies that there is a constant $N \geq 0$ such that $|x(t)|_1 \leq N\exp(\lambda t)$, $t \in J$. Using this fact in (3.3), we observe that

$$|x|_{\mathcal{E}} \leq N. \tag{3.4}$$

Define a mapping $\mathcal{T}: \mathcal{E} \to \mathcal{E}$ by

$$(\mathcal{T}x)(t) = g(t) + \int_a^t f\bigl(t, s, x(s), x'(s)\bigr)\,ds \tag{3.5}$$

for $x \in \mathcal{E}$. Note that, if $u^* \in \mathcal{E}$ is a fixed point of $\mathcal{T}$, then u^* is a solution of the problem (3.1). We shall prove the existence of a fixed point of $\mathcal{T}$ under the following conditions.

(I) There exist $(\alpha, \beta) \in \mathcal{E}^2$, $(\alpha_0, \beta_0) \in \mathbb{R}^2$ such that

$$\begin{aligned} &\alpha_0 \le \alpha(t) \le \beta(t) \le \beta_0, \\ &\alpha_0 \le \alpha'(t) \le \beta'(t) \le \beta_0, \quad t \in J, \end{aligned}$$

and for all $t \in J$, we have

$$\begin{aligned} &\alpha(t) \le g(t) + \int_a^t f\bigl(t, s, \beta(s), \beta'(s)\bigr)\,ds, \\ &\alpha'(t) \le g'(t) + f\bigl(t, t, \beta(t), \beta'(t)\bigr) + \int_a^t \frac{\partial}{\partial t} f\bigl(t, s, \beta(s), \beta'(s)\bigr)\,ds, \quad t \in J \end{aligned}$$

and

$$\begin{aligned} &\beta(t) \ge g(t) + \int_a^t f\bigl(t, s, \alpha(s), \alpha'(s)\bigr)\,ds, \\ &\beta'(t) \ge g'(t) + f\bigl(t, t, \alpha(t), \alpha'(t)\bigr) + \int_a^t \frac{\partial}{\partial t} f\bigl(t, s, \alpha(s), \alpha'(s)\bigr)\,ds, \quad t \in J. \end{aligned}$$

(II) $f: J \times J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies for $u, v \in \mathcal{E}$,

$$\begin{aligned} &u(t) \ge v(t) \quad \text{and} \quad u'(t) \ge v'(t) \quad \text{for } t \in J \\ &\quad \Rightarrow \quad f\bigl(t, s, u(s), u'(s)\bigr) \le f\bigl(t, s, v(s), v'(s)\bigr) \quad \text{and} \\ &\frac{\partial}{\partial t} f\bigl(t, s, u(s), u'(s)\bigr) \le \frac{\partial}{\partial t} f\bigl(t, s, v(s), v'(s)\bigr), \quad a \le s \le t \le b. \end{aligned}$$

(III) The function f and its derivative satisfy the conditions

$$\begin{aligned} &\bigl|f(t, s, u, v) - f(t, s, \overline{u}, \overline{v})\bigr| \le h_1(t, s)\bigl[|u - \overline{u}| + |v - \overline{v}|\bigr], \\ &\left|\frac{\partial}{\partial t} f(t, s, u, v) - \frac{\partial}{\partial t} f(t, s, \overline{u}, \overline{v})\right| \le h_2(t, s)\bigl[|u - \overline{u}| + |v - \overline{v}|\bigr] \end{aligned}$$

for $a \le s \le t \le b$, $u, v, \overline{u}, \overline{v} \in \mathcal{E}$, where $h_i \in C(J^2, \mathbb{R}^+)$ for $i = 1, 2$.

(IV) There exist nonnegative constants γ_1, γ_2 such that $\gamma_1 + \gamma_2 < 1$ and

$$\begin{aligned} &\int_a^t h_1(t, s) \exp(\lambda s)\,ds \le \gamma_1 \exp(\lambda t), \\ &h_1(t, t) \exp(\lambda t) + \int_a^t h_2(t, s) \exp(\lambda s)\,ds \le \gamma_2 \exp(\lambda t) \end{aligned}$$

for $t \in J$, where λ is given in (3.2).

(V) There exist nonnegative constants δ_1, δ_2 such that

$$\left|g(t)\right| + \int_a^t \left|f(t,s,0,0)\right| ds \le \delta_1 \exp(\lambda t),$$
$$\left|g'(t)\right| + \left|f(t,s,0,0)\right| + \int_a^t \left|\frac{\partial}{\partial t} f(t,s,0,0)\right| ds \le \delta_2 \exp(\lambda t)$$

for $a \le s \le t \le b$, where λ is given in (3.2).

We have the following result for the set

$$\mathcal{P} = \left\{u \in \mathcal{E} : \alpha(t) \le u(t) \le \beta(t), \alpha'(t) \le u'(t) \le \beta'(t), t \in J\right\}.$$

Theorem 3.1 *Under the assumptions* (I) *to* (V), *the integro-differential problem* (3.1) *has a unique solution in the set* $\mathcal{P}$.

Proof The proof of the theorem is divided into three parts.

(A): First, we show that $\mathcal{T}$ maps $\mathcal{E}$ into itself.

Differentiating both sides of (3.5) with respect to t, we get

$$(\mathcal{T}x)'(t) = g'(t) + f\big(t,t,x(t),x'(t)\big) + \int_a^t \frac{\partial}{\partial t} f\big(t,s,x(s),x'(s)\big)\, ds. \tag{3.6}$$

Evidently, $\mathcal{T}x$, $(\mathcal{T}x)'$ are continuous on J. We verify that (3.2) is fulfilled. From (3.3), (3.6) and using conditions (IV), (V) and (3.4), we have

$$\begin{aligned}
\left|(\mathcal{T}x)(t)\right| &\le \left|g(t)\right| + \int_a^t \left|f\big(t,s,x(s),x'(s)\big) - f(t,s,0,0) + f(t,s,0,0)\right| ds \\
&\le \left|g(t)\right| + \int_a^t \left|f(t,s,0,0)\right| ds + \int_a^t h_1(t,s)\left|x(s)\right|_1 ds \\
&\le \delta_1 \exp(\lambda t) + |x|_{\mathcal{E}} \int_a^t h_1(t,s)\exp(\lambda s)\, ds \\
&\le [\delta_1 + N\gamma_1]\exp(\lambda t)
\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}
\left|(\mathcal{T}x)'(t)\right| &\le \left|g'(t)\right| + \left|f\big(t,t,x(t),x'(t)\big) - f(t,t,0,0) + f(t,t,0,0)\right| \\
&\quad + \int_a^t \left|\frac{\partial}{\partial t} f\big(t,s,x(s),x'(s)\big) - \frac{\partial}{\partial t} f(t,s,0,0) + \frac{\partial}{\partial t} f(t,s,0,0)\right| ds \\
&\le \left|g'(t)\right| + \left|f(t,t,0,0)\right| + \int_a^t \left|\frac{\partial}{\partial t} f(t,s,0,0)\right| ds + h_1(t,t)\left|x(t)\right|_1 \\
&\quad + \int_a^t h_2(t,s)\left|x(s)\right|_1 ds \\
&\le \delta_2 \exp(\lambda t) + |x|_{\mathcal{E}} h_1(t,t)\exp(\lambda t) + |x|_{\mathcal{E}} \int_a^t h_2(t,s)\exp(\lambda s)\, ds \\
&\le [\delta_2 + N\gamma_2]\exp(\lambda t).
\end{aligned} \tag{3.8}$$

Combining (3.7) and (3.8), we get

$$\bigl|(\mathcal{T}x)(t)\bigr|_1 \le \bigl[\delta_1 + \delta_2 + N(\gamma_1 + \gamma_2)\bigr]\exp(\lambda t). \tag{3.9}$$

It follows from (3.9) that $\mathcal{T}x \in \mathcal{E}$. This proves that $\mathcal{T}$ maps $\mathcal{E}$ into itself.

(B): Define closed subsets of $\mathcal{E}$, $\mathcal{A}_1$ and $\mathcal{A}_2$ by

$$\mathcal{A}_1 = \bigl\{u \in \mathcal{E} : u(t) \le \beta(t), u'(t) \le \beta'(t) \text{ for } t \in J\bigr\}$$

and

$$\mathcal{A}_2 = \bigl\{u \in \mathcal{E} : u(t) \ge \alpha(t), u'(t) \ge \alpha'(t) \text{ for } t \in J\bigr\}.$$

We shall prove that

$$\mathcal{T}(\mathcal{A}_1) \subseteq \mathcal{A}_2 \quad \text{and} \quad \mathcal{T}(\mathcal{A}_2) \subseteq \mathcal{A}_1. \tag{3.10}$$

Let $u \in \mathcal{A}_1$, that is,

$$u(t) \le \beta(t) \quad \text{and} \quad u'(t) \le \beta'(t) \quad \text{for all } t \in J.$$

Using condition (II), we obtain that

$$f\bigl(t,s,u(s),u'(s)\bigr) \ge f\bigl(t,s,\beta(s),\beta'(s)\bigr) \quad \text{and} \tag{3.11}$$

$$\frac{\partial}{\partial t} f\bigl(t,s,u(s),u'(s)\bigr) \le \frac{\partial}{\partial t} f\bigl(t,s,\beta(s),\beta'(s)\bigr) \quad \text{for } a \le s \le t \le b. \tag{3.12}$$

The inequality (3.11) with condition (I) imply that

$$(\mathcal{T}u)(t) = g(t) + \int_a^t f\bigl(t,s,u(s),u'(s)\bigr)\,ds \ge g(t) + \int_a^t f\bigl(t,s,\beta(s),\beta'(s)\bigr)\,ds \ge \alpha(t)$$

for all $t \in J$. The inequality (3.12) with condition (I) imply that

$$\begin{aligned}(\mathcal{T}u)'(t) &= g'(t) + f\bigl(t,t,u(t),u'(t)\bigr) + \int_a^t \frac{\partial}{\partial t} f\bigl(t,s,u(s),u'(s)\bigr)\,ds \\ &\ge g'(t) + f\bigl(t,t,\beta(t),\beta'(t)\bigr) + \int_a^t \frac{\partial}{\partial t} f\bigl(t,s,\beta(s),\beta'(s)\bigr)\,ds \ge \alpha'(t)\end{aligned}$$

for all $t \in J$. Hence, we have $\mathcal{T}u \in \mathcal{A}_2$.

Similarly, if $u \in \mathcal{A}_2$, it can be proved that $\mathcal{T}u \in \mathcal{A}_1$ holds. Thus, (3.10) is fulfilled.

(C): We verify that the operator $\mathcal{T}$ is a cyclic (φ,ψ,θ)-contraction map.

Let $(u,v) \in \mathcal{A}_1 \times \mathcal{A}_2$, that is, for all $t \in J$,

$$\begin{aligned} &u(t) \le \beta(t) \le \beta_0, \qquad u'(t) \le \beta'(t) \le \beta_0, \\ &v(t) \ge \alpha(t) \ge \alpha_0, \qquad v'(t) \ge \alpha'(t) \ge \alpha_0. \end{aligned}$$

Using the properties (3.5) and (3.6) of $\mathcal{T}$ and conditions (III), (IV) and (V), we conclude that

$$\begin{aligned}
\left|(\mathcal{T}u)(t) - (\mathcal{T}v)(t)\right| &\le \int_a^t \left|f\left(t,s,u(s),u'(s)\right) - f\left(t,s,v(s),v'(s)\right)\right| ds \\
&\le \int_a^t h_1(t,s)\left|u(s) - v(s)\right|_1 ds \\
&\le |u-v|_{\mathcal{E}} \int_a^t h_1(t,s)\exp(\lambda s)\, ds \\
&\le |u-v|_{\mathcal{E}}\gamma_1 \exp(\lambda t)
\end{aligned} \tag{3.13}$$

and

$$\begin{aligned}
\left|(\mathcal{T}u)'(t) - (\mathcal{T}v)'(t)\right| &\le \left|f\left(t,t,u(t),u'(t)\right) - f\left(t,t,v(t),v'(t)\right)\right| \\
&\quad + \int_a^t \left|\frac{\partial}{\partial t} f\left(t,s,u(s),u'(s)\right) - \frac{\partial}{\partial t} f\left(t,s,v(s),v'(s)\right)\right| ds \\
&\le h_1(t,t)\left|u(t) - v(t)\right|_1 + \int_a^t h_2(t,s)\left|u(s) - v(s)\right|_1 ds \\
&\le |u-v|_{\mathcal{E}} h_1(t,t)\exp(\lambda t) + |u-v|_{\mathcal{E}} \int_a^t h_2(t,s)\exp(\lambda s)\, ds \\
&\le |u-v|_{\mathcal{E}}\gamma_2 \exp(\lambda t)
\end{aligned} \tag{3.14}$$

for $t \in J$. Combining (3.13) and (3.14), we get

$$\left|(\mathcal{T}u)(t) - (\mathcal{T}v)(t)\right|_1 \le |u-v|_{\mathcal{E}}(\gamma_1 + \gamma_2)\exp(\lambda t). \tag{3.15}$$

From (3.15) we obtain (with $k = \gamma_1 + \gamma_2 < 1$)

$$\begin{aligned}
|\mathcal{T}u - \mathcal{T}v|_{\mathcal{E}} &\le k|u-v|_{\mathcal{E}} \\
&\le k|u-v|_{\mathcal{E}} + \theta\left(|u-\mathcal{T}u|_{\mathcal{E}}, |v-\mathcal{T}v|_{\mathcal{E}}, |u-\mathcal{T}v|_{\mathcal{E}}, |v-\mathcal{T}u|_{\mathcal{E}}\right).
\end{aligned} \tag{3.16}$$

Using the same technique, we can show that the above inequality also holds if we take $(u,v) \in \mathcal{A}_2 \times \mathcal{A}_1$. All other conditions of Corollary 2.3 are fulfilled for the complete metric space $(\mathcal{A}_1 \cup \mathcal{A}_2, |\cdot|_{\mathcal{E}})$ and $\mathcal{T}$ restricted to $\mathcal{A}_1 \cup \mathcal{A}_2$ (with $p = 2$).

We conclude that the operator $\mathcal{T}$ has a unique fixed point u^*, and hence the integro-differential equation (3.1) has a unique solution in the set $\mathcal{P}$. □

4 An application to nonlinear Volterra integral equations in two variables

In this section, we present application of Theorem 2.1 to study the existence and uniqueness of solutions to certain nonlinear Volterra integral equations.

Consider the nonlinear Volterra integral equation in two variables of the form [4]:

$$u(x,y) = f(x,y) + \int_0^x g\left(x,y,\xi,u(\xi,y)\right) d\xi + \int_0^x \int_0^y h\left(x,y,\sigma,\tau,u(\sigma,\tau)\right) d\tau\, d\sigma, \tag{4.1}$$

where f, g, h are given functions and u is the unknown function to be found.

Let $C(S_1, S_2)$ the class of continuous functions from the set S_1 to the set S_2. We denote by $E = \mathbb{R}^+ \times \mathbb{R}^+$, $E_1 = \{f(x,y,s) : 0 \le s \le x < \infty, y \in \mathbb{R}^+\}$; and $E_2 = \{f(x,y,s,t) : 0 \le s \le x < \infty, 0 \le t \le y < \infty\}$.

Throughout, we assume that $f \in C(E, \mathbb{R})$, $g \in C(E_1 \times \mathbb{R}, \mathbb{R})$, $h \in C(E_2 \times \mathbb{R}, \mathbb{R})$.

Denote by $\mathcal{S}$ the space of functions $z \in C(E, \mathbb{R})$ which fulfill the condition

$$\bigl|z(x,t)\bigr| = \mathcal{O}\bigl(\exp\bigl(\lambda(x+y)\bigr)\bigr), \tag{4.2}$$

where λ is a positive constant. Define the norm in the space $\mathcal{S}$ as

$$|z|_{\mathcal{S}} = \sup_{(x,y)\in\mathcal{S}} \bigl[\bigl|z(x,t)\bigr| \exp\bigl(-\lambda(x+y)\bigr)\bigr]. \tag{4.3}$$

It is easy to see that $\mathcal{S}$ with the norm defined in (4.3) is a Banach space. We note that the condition (4.2) implies that there is a constant $M_0 \ge 0$ such that $|z(x,t)| \le M_0 \exp(\lambda(x+y))$. Using this fact in (4.3), we observe that

$$|z|_S \le M_0. \tag{4.4}$$

Define a mapping $\mathcal{T} : \mathcal{S} \to \mathcal{S}$ by

$$\begin{aligned}(\mathcal{T}u)(x,y) ={}& f(x,y) + \int_0^x g\bigl(x,y,\xi,u(\xi,y)\bigr)\,d\xi \\ &+ \int_0^x \int_0^y h\bigl(x,y,\sigma,\tau,u(\sigma,\tau)\bigr)\,d\tau\,d\sigma \end{aligned} \tag{4.5}$$

for $u \in \mathcal{S}$. Note that, if $u^* \in \mathcal{S}$ is a fixed point of $\mathcal{T}$, then u^* is a solution of the problem (4.1).

We shall prove the existence of a fixed point of $\mathcal{T}$ under the following conditions.

(I) There exist $(\alpha,\beta) \in \mathcal{S}^2$, $(\alpha_0,\beta_0) \in \mathbb{R}^2$ such that

$$\alpha_0 \le \alpha(x,t) \le \beta(x,t) \le \beta_0(x,t) \in \mathbb{R}^+ \times \mathbb{R}^+$$

and for all $(x,t) \in \mathbb{R}^+ \times \mathbb{R}^+$, we have

$$\alpha(x,t) \le f(x,t) + \int_0^x g\bigl(t,s,\xi,\beta(\xi,s)\bigr)\,d\xi + \int_0^x \int_0^y h\bigl(t,s,\sigma,\tau,\beta(\sigma,\tau)\bigr)\,d\tau\,d\sigma$$

and

$$\beta(x,t) \ge f(x,t) + \int_0^x g\bigl(t,s,\xi,\alpha(\xi,s)\bigr)\,d\xi + \int_0^x \int_0^y h\bigl(t,s,\sigma,\tau,\alpha(\sigma,\tau)\bigr)\,d\tau\,d\sigma.$$

(II) The functions g, h in equation (4.1) satisfy the conditions

$$\begin{aligned}&\bigl|g(x,y,\xi,u) - g(x,y,\xi,\overline{u})\bigr| \le h_1(x,y,\xi)|u-\overline{u}|, \\ &\bigl|h(x,y,\sigma,\tau,u) - h(x,y,\sigma,\tau,\overline{u})\bigr| \le h_2(x,y,\sigma,\tau)|u-\overline{u}|,\end{aligned}$$

where $h_1 \in C(E_1, \mathbb{R}^+)$, $h_2 \in C(E_2, \mathbb{R}^+)$.

(III) There exist nonnegative constants $\delta_1 < 1$, δ_2 such that

$$\int_0^x h_1(x,y,\xi)\exp\bigl(\lambda(x+y)\bigr)\,d\xi + \int_0^x\int_0^y h_2(x,y,\sigma,\tau)\exp\bigl(\lambda(\sigma+\tau)\bigr)\,d\tau\,d\sigma \\ \leq \delta_1\exp\bigl(\lambda(x+y)\bigr)$$

and

$$\left|f(x,y) + \int_0^x g(x,y,\xi,0)\,d\xi + \int_0^x\int_0^y h(x,y,\sigma,\tau,0)\,d\tau\,d\sigma\right| \leq \delta_2\exp\bigl(\lambda(x+y)\bigr),$$

where λ is as given in (4.2).

(IV) There exist $(\alpha,\beta) \in \mathcal{S}^2$ such that $\alpha(t) \leq \beta(t)$ for $t \in \mathbb{R}^+$ and that

$$(\mathcal{T}\alpha)(x,t) \leq \beta(x,t) \quad \text{and} \quad (\mathcal{T}\beta)(x,t) \geq \alpha(x,t) \quad \text{for } (x,t) \in \mathbb{R}^+ \times \mathbb{R}^+.$$

We have the following result for the set

$$\mathcal{W} = \bigl\{u \in \mathcal{S} : \alpha(x,y) \leq u(x,y) \leq \beta(x,y), (x,y) \in \mathbb{R}^+ \times \mathbb{R}^+\bigr\}.$$

Theorem 4.1 *Under the assumptions* (I) *to* (IV), *the integral problem* (4.1) *has a unique solution in the set* $\mathcal{W}$.

Proof We proof of the theorem in three steps.

Step 1: First we show that $\mathcal{T}$ maps $\mathcal{S}$ into itself.

Evidently, $\mathcal{T}u$ is continuous on $\mathcal{S}$ and $\mathcal{T}u \in \mathbb{R}$. We verify that (4.2) is fulfilled. From (4.3), and using conditions (II), (III) and (4.4), we have

$$\begin{aligned}
\bigl|(\mathcal{T}u)(x,y)\bigr| &\leq \left|f(x,y) + \int_0^x g(x,y,\xi,0)\,d\xi + \int_0^x\int_0^y h(x,y,\sigma,\tau,0)\,d\tau\,d\sigma\right| \\
&\quad + \int_0^x \left|g\bigl(x,y,\xi,u(\xi,y)\bigr) - \int_0^x g(x,y,\xi,0)\right| d\xi \\
&\quad + \int_0^x\int_0^y \bigl|h\bigl(x,y,\sigma,\tau,u(\sigma,\tau)\bigr) - h(x,y,\sigma,\tau,0)\bigr|\,d\tau\,d\sigma \\
&\leq \delta_2\exp\bigl(\lambda(x+y)\bigr) + \int_0^x h_1(x,y,\xi)\bigl|u(\xi,y)\bigr|\,d\xi \\
&\quad + \int_0^x\int_0^y h_2(x,y,\sigma,\tau)\bigl|u(\sigma,\tau)\bigr|\,d\tau\,d\sigma \\
&\leq \delta_2\exp\bigl(\lambda(x+y)\bigr) + |u|_S\left[\int_0^x h_1(x,y,\xi)\exp\bigl(\lambda(x+y)\bigr)\,d\xi\right. \\
&\quad \left. + \int_0^x\int_0^y h_2(x,y,\sigma,\tau)\exp\bigl(\lambda(\sigma,\tau)\bigr)\,d\tau\,d\sigma\right] \\
&\leq [\delta_2 + M_0\delta_1]\exp\bigl(\lambda(x+y)\bigr).
\end{aligned} \tag{4.6}$$

It follows from (4.6) that $\mathcal{T}u \in \mathcal{S}$. This proves that $\mathcal{T}$ maps $\mathcal{S}$ into itself.

Step 2: Define closed subsets of $\mathcal{S}$, $\mathcal{A}_1$ and $\mathcal{A}_2$ by

$$\mathcal{A}_1 = \bigl\{u \in \mathcal{S} : u(x,t) \leq \beta(x,t) \text{ for } (x,t) \in \mathbb{R}^+ \times \mathbb{R}^+\bigr\}$$

and

$$\mathcal{A}_2 = \{u \in \mathcal{S} : u(x,t) \geq \alpha(x,t) \text{ for } (x,t) \in \mathbb{R}^+ \times \mathbb{R}^+\}.$$

We shall prove that

$$\mathcal{T}(\mathcal{A}_1) \subseteq \mathcal{A}_2 \quad \text{and} \quad \mathcal{T}(\mathcal{A}_2) \subseteq \mathcal{A}_1. \tag{4.7}$$

Let $u \in \mathcal{A}_1$, that is,

$$u(x,t) \leq \beta(x,t) \quad \text{for all } (x,t) \in \mathbb{R}^+ \times \mathbb{R}^+.$$

Using condition (II), we obtain that

$$g(x,t,\xi,u(\xi,t)) \geq g(x,t,\xi,\beta(\xi,t)) \tag{4.8}$$

and

$$h(x,t,\sigma,\tau,u(\sigma,\tau)) \geq h(x,t,\sigma,\tau,\beta(\sigma,\tau)) \tag{4.9}$$

for $0 \leq \xi \leq x < \infty$, $0 \leq \tau \leq t < \infty$.

The inequalities (4.8) and (4.9) with conditions (I) and (IV) imply that

$$\begin{aligned}(\mathcal{T}u)(x,t) &= f(x,t) + \int_0^x g(x,t,\xi,u(\xi,t))\,d\xi \\ &\quad + \int_0^x \int_0^y h(x,t,\sigma,\tau,u(\sigma,\tau))\,d\tau\,d\sigma \\ &\geq f(x,t) + \int_0^x g(x,t,\xi,\beta(\xi,t))\,d\xi \\ &\quad + \int_0^x \int_0^y h(x,t,\sigma,\tau,\beta(\sigma,\tau))\,d\tau\,d\sigma \geq \alpha(x,t)\end{aligned}$$

for all $(x,t) \in \mathbb{R}^+ \times \mathbb{R}^+$. Hence, we have $\mathcal{T}u \in \mathcal{A}_2$.

Similarly, if $u \in \mathcal{A}_2$, it can be proved that $\mathcal{T}u \in \mathcal{A}_1$ holds. Thus, (4.7) is fulfilled.

Step 3: We verify that the operator $\mathcal{T}$ is a cyclic generalized ψ-weakly contractive mapping.

Let $(u,v) \in \mathcal{A}_1 \times \mathcal{A}_2$, that is, for all $t \in J$,

$$u(x,t) \leq \beta(x,t) \leq \beta_0, \qquad v(x,t) \geq \alpha(x,t) \geq \alpha_0.$$

Using the properties (4.5) of $\mathcal{T}$ and conditions (II) and (III), we conclude that

$$\begin{aligned}&\left|(\mathcal{T}u)(x,y) - (\mathcal{T}v)(x,y)\right| \\ &\quad \leq \int_0^x \left|g(x,y,\xi,u(\xi,y)) - g(x,y,\xi,v(\xi,y))\right|\,d\xi \\ &\qquad + \int_0^x \int_0^y \left|h(x,y,\sigma,\tau,u(\sigma,\tau)) - h(x,y,\sigma,\tau,v(\sigma,\tau))\right|\,d\tau\,d\sigma\end{aligned}$$

$$\begin{aligned}
&\le \int_0^x h_1(x,y,\xi)\bigl|u(\xi,y)-v(\xi,y)\bigr|\,d\xi \\
&\quad + \int_0^x \int_0^y h_2(x,y,\sigma,\tau)\bigl|u(\sigma,\tau)-v(\sigma,\tau)\bigr|\,d\tau\,d\sigma \\
&\le |u-v|_{\mathcal{S}}\biggl[\int_0^x h_1(x,y,\xi)\exp\bigl(\lambda(x+y)\bigr)\,d\xi \\
&\quad + \int_0^x \int_0^y h_2(x,y,\sigma,\tau)\exp\bigl(\lambda(\sigma,\tau)\bigr)\,d\tau\,d\sigma\biggr] \\
&\le \delta_1|u-v|_{\mathcal{S}}\exp\bigl(\lambda(x+y)\bigr).
\end{aligned} \tag{4.10}$$

From (4.10), we obtain (with $k = \delta_1 < 1$)

$$|\mathcal{T}u-\mathcal{T}v|_{\mathcal{S}} \le k|u-v|_{\mathcal{S}}. \tag{4.11}$$

Considering the functions $\psi,\varphi,\theta:\mathbb{R}^+\to\mathbb{R}^+$ defined by

$$\varphi(t)=t \quad \text{and} \quad \psi(t)=(1-k)t,$$

we get

$$\begin{aligned}
\varphi\bigl(|\mathcal{T}u-\mathcal{T}v|_{\mathcal{S}}\bigr) &\le \varphi\bigl(|u-v|_{\mathcal{S}}\bigr)-\psi\bigl(|u-v|_{\mathcal{S}}\bigr) \\
&\quad + \theta\bigl(|u-\mathcal{T}u|_{\mathcal{S}},|v-\mathcal{T}v|_{\mathcal{S}},|u-\mathcal{T}v|_{\mathcal{S}},|v-\mathcal{T}u|_{\mathcal{S}}\bigr).
\end{aligned}$$

Using the same technique, we can show that the above inequality also holds if we take $(u,v)\in\mathcal{A}_2\times\mathcal{A}_1$. All other conditions of Theorem 2.1 are fulfilled for the complete metric space $(\mathcal{A}_1\cup\mathcal{A}_2,|\cdot|_{\mathcal{S}})$ and $\mathcal{T}$ restricted to $\mathcal{A}_1\cup\mathcal{A}_2$ (with $p=2$).

We conclude that the operator $\mathcal{T}$ has a unique fixed point u^* and, hence, the integro-differential equation (4.1) has a unique solution in the set $\mathcal{W}$. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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