Some applications of fractional calculus

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Abstract. The present paper investigates a second-order differential equation with a fractional derivative in the lower term, in which the order of the fractional derivative is in the range from zero to two and is not known in advance. This model is used to describe oscillatory processes in a viscous medium. A fundamentally new method has been developed for the approximate solution of the first boundary-value problem for the equation of string vibration taking into account friction in a medium with fractal geometry.

1. Introduction

In recent years, fractional calculus is in the focus of attention of many researchers in the field of science and technology. In this regard, we should note the paper [1], as a unique comprehensive review of fractional calculus and its application with the authoritative contribution of leading world experts. First of all, we note that fractional derivatives in space can be used to model anomalous diffusions or dispersions, and fractional derivatives in time can be used to model some processes with a "memory". Particular attention should be paid to an equation of the form

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + C_0 D_{0t}^{\alpha} u + C_1 D_{0x}^{\beta} u + F$$

which, in particular, used to describe the vibration of a string taking into account friction in a medium with fractal geometry. In this work, this equation is used to model changes in the deformation-strength characteristics of polymer concrete under loading.

2. Main results

Now, in domain $D = \{0 < x < 1,\ 0 < t < 1\}$ we consider the first boundary value problem for the equation for for the equation of vibration of a string with a fractional derivative of order α with respect to the spatial variable

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$$\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x^2}+C_1 D_{0x}^{\alpha}u+C_0 D_{0t}^{\beta}u,\ \ 0<\alpha,\beta<2, \tag{1}$$

$$u(0,t)=u(1,t)=0, \tag{2}$$

$$u(x,0)=\varphi(x), \tag{3}$$

$$u'_t(x,0)=\psi(x). \tag{4}$$

Here, $0<\alpha<2$, c - constant, $D_{0x}^{\alpha}u$ - fractional derivative of Riemann-Liouville type of order α. Fractional derivative of order α for function $f(x)$ in a point x ($0\le m-1<\alpha<m,\ m\in N$) defined by the formula

$$D^{\alpha}f(x)=\frac{d^m}{dx^m}\left(\frac{1}{\Gamma(m-a)}\int_a^x\frac{f(\tau)d\tau}{(x-\tau)^{\alpha+1-m}}\right).$$

Obtained results are applied [1, 2, see the references therein] for modeling changes in the deformation-strength characteristics of polymer concrete under loading. The solution to problem (1) - (2) - (3) - (4) will be sought by the Fourier method

$$u(x,t)=X(x)T(t). \tag{5}$$

We substitute (5) into equation (1), then for an unknown function $X(x)$ we obtain the two-point Dirichlet problem

$$X''(x)+C_1 D_{0x}^{\alpha}X=\lambda X(x), \tag{6}$$

$$X(0)=X(1)=0. \tag{7}$$

The solution to problem (6) - (7) was written out in [1, 2, see the references therein]. In particular, there was shown that the number λ is an eigenvalue of problem (6) - (7), if and only if λ is the zero of the function

$$\omega(\lambda)=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{C_n^k\lambda^{n-k}(-C_1)^k}{\Gamma(2n-k\beta+2)}$$

and the corresponding eigenfunctions $X_j(x)$ have the form

$$X_j(x)=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{C_n^k\lambda_j^{n-k}(-C_1)^k}{\Gamma(2n-k\beta+2)}x^{2n+1-k\alpha},\ \ j=1,2,3,... \tag{8}$$

(here λ_j - j-th eigenfunction of the problem (6)-(7)). The system of the eigenfunctions (8) is complete [1, 2, see the references therein] but not orthogonal, thus we construct the system

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$$\widetilde{X}_j(x) = (1-x) - \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{C_n^k \lambda_j^{n-k}(-C_1)^k}{\Gamma(2n-k\beta+2)} x^{2n+1-k\alpha}, \ j = 1,2,3,... \qquad (9)$$

which biorthogonal to the system of eigenfunctions

$$X_j(x) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{C_n^k \lambda_j^{n-k}(-C_1)^k}{\Gamma(2n-k\beta+2)} x^{2n+1-k\alpha}, \ j = 1,2,3,...$$

Next, we find the general solution to the equation

$$T(t) + c_0 D_{0t}^{\beta} T(t) = \lambda T(t).$$

As in the case of equation (6), we have

$$T_m(t) = A_m\left(t + \sum_{n=1}^{\infty}\sum_{k=0}^{n} \frac{C_n^k \lambda_m^{n-k}(-c_0)^k}{\Gamma(2n-k\alpha+2)} x^{2n+1-k\alpha}\right) + B_m\left(1 + \sum_{n=1}^{\infty}\sum_{k=0}^{n} \frac{C_n^k \lambda_m^{n-k}(-c_0)^k}{\Gamma(2n-k\alpha+1)} x^{2n-k\alpha}\right).$$

Let's designate

$$Z_m(t) = \left(t + \sum_{n=1}^{\infty}\sum_{k=0}^{n} \frac{C_n^k \lambda_m^{n-k}(-c_0)^k}{\Gamma(2n-k\alpha+2)} x^{2n+1-k\alpha}\right), \quad \widetilde{Z}_m(t) = \left(1 + \sum_{n=1}^{\infty}\sum_{k=0}^{n} \frac{C_n^k \lambda_m^{n-k}(-c_0)^k}{\Gamma(2n-k\alpha+1)} x^{2n-k\alpha}\right).$$

Then the solution to problem (1) - (2) - (3) - (4) is written out in the standard way

$$u(x,t) = \sum_{m=1}^{\infty} T_m(t) X_m(x) = \sum_{m=1}^{\infty} [A_m Z_m(t) + B_m \widetilde{Z}_m(t)] X_m, \qquad (10)$$

putting in the last expression $t = 0$, we have

$$\varphi(x) = \sum_{n=1}^{\infty} B_m \widetilde{Z}(0) Z(0) X_m(x), \qquad (11)$$

so

$$B_m = \frac{1}{\widetilde{Z}(0)(\varphi(x), \widetilde{X}_m(x))(X_m(x), \widetilde{X}_m(x))}.$$

To find A_m differentiate both parts (12) by t and let $t = 0$ we obtain, $\sum_{m=1}^{\infty}\left[A_m Z'_m(0) + B_m \widetilde{Z}'_m(0)\right] X_m(x) = \psi(x)$

from here

$$\left[A_m Z'_m(0) + B_m \widetilde{Z}'_m(0)\right]\left(X_m(x), \widetilde{X}_m(x)\right) = \left(\psi(x), \widetilde{X}_m(x)\right).$$

And finally we have,

$$A_m = \frac{1}{Z'_m(0)}\left[\frac{1}{(X_m(x), \widetilde{X}_m(x))} - B_m \widetilde{Z}'_m(0)\right]$$

which allows us to write a solution to problem (1) - (2) - (3) - (4) in the form (10).

In the papers of one of the authors, the equation (6) was used to model the deformation-strength characteristics of polymer concrete. The main problem using models based on fractional derivatives is the problem of identifying the parameters of this model, especially the order of the fractional derivative. In order to test the obtained results, we used the experimental data presented in [2]. Comparing the experimental data [2] with the calculated (for $\alpha = 1.47$) it was concluded that the constructed model is adequate. In this paper, only transverse vibrations are considered, all movements occur in one plane and the granule moves perpendicular to the axis. Then, to simulate changes in the deformation-strength characteristics of polymer concrete under loading, we have the following first boundary-value problem (here $u(x,t)$ - granule displacement in moment t)

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + C_0 D_{0x}^{\beta} u + C_1 D_{0t}^{1,47} u, \quad 0 < \alpha, \beta < 2, \tag{12}$$

$$u(0,t) = u(1,t) = 0, \tag{13}$$

$$u(x,0) = \varphi(x), \tag{14}$$

$$u'_t(x,0) = \psi(x), \tag{15}$$

whose solution according to formula (10) has the form

$$u(x,t) = \sum_{m=1}^{\infty} [A_m Z_m(t) + B_m \widetilde{Z}_m(t)],$$

where

$$\left\{X_j(x)\right\}_{j=1}^{\infty} = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{C_n^k \lambda_j^{n-k} (-C_1)^k}{\Gamma(2n - k\beta + 2)} x^{2n+1-1,47k},$$

$$\left\{\widetilde{X}_j(x)\right\}_{j=1}^{\infty} = (1-x) - \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{C_n^k \lambda_j^{n-k} (-C_1)^k}{\Gamma(2n - k\beta + 2)} (1-x)^{2n+1-1,47k}.$$

Let's find eigenvalues λ_j numerically using the high-level language of technical calculations MATLAB taking $\alpha = 1{,}47$, $C_1 = 1{,}8$ (according by [2]). Eigenvalues are presented in table 1:

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Table 1. Numerical results for eigenvalues of the problem (1) - (2) - (3) - (4)

λ_j	λ_j	λ_j	λ_j	λ_j
16.6	59.4	125.0	213.4	323.4

Then, an approximate solution to problem (12) - (13) - (14) - (15) will take the form

$$u(x,t) = \sum_{m=1}^{5} [A_m Z_m(t) + B_m \widetilde{Z}_m(t)]. \quad (16)$$

Formula (16) allows us to write a solution to the problem (15) - (16) - (17) - (18) if the functions $\varphi(x)$ and $\psi(x)$ continuously differentiable. Finally, it remains to determine the parameter β. This parameter can again be determined by the technique developed in [1, 2, see the literature there] since the parameter α is already defined.

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