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Research Article

Some New Fixed Point Theorems in Partial Metric Spaces with Applications

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Recently, a number of fixed point theorems for contraction type mappings in partial metric spaces have been obtained by various authors. Most of these theorems can be obtained from the corresponding results in metric spaces. The purpose of this paper is to present certain fixed point results for single and multivalued mappings in partial metric spaces which cannot be obtained from the corresponding results in metric spaces. Besides discussing some useful examples, an application to Volterra type system of integral equations is also discussed.

Dedicated to Professor Shyam Lal Singh on his 75th birthday

1. Introduction and Preliminaries

Throughout this paper $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{R}^+$ denote the set of all natural numbers, the set of all real numbers, and the set of all nonnegative real numbers, respectively.

The well-known Banach contraction theorem (BCT) has been generalized and extended by many authors in various ways. In 1974, Ćirić [1] introduced the notion of *quasi-contraction* and obtained a forceful generalization of Banach contraction theorem.

Definition 1. A self-mapping T of a metric space X is a quasi-contraction if there exists a number $r \in [0,1)$ such that, for all $x, y \in X$,

$$d(Tx, Ty) \leq rm(x, y), \qquad \textbf{(C)}$$

where $m(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}$.

Theorem 2 (see [1]). *A quasi-contraction on a complete metric space has a unique fixed point.*

We remark that a quasi-contraction for a self-mapping on a metric space is considered as the most general among contractions listed by Rhoades [2].

In 2006, Proinov [3] established an equivalence between two types of generalizations of the BCT. The first type involves Meir-Keeler [4] type contraction conditions and the second type involves Boyd and Wong [5] type contraction conditions. Further, generalizing certain results of Jachymski [6] and Matkowski [7] he obtained the following general fixed point theorem, which extends Ćirić's quasi-contraction.

Theorem 3 (see [3], Th. 4.1). *Let T be a continuous and asymptotically regular self-mapping on a complete metric space (X,d) satisfying the following conditions:*

(P1) $d(Tx,Ty) \leq \varphi(D(x,y))$ *for all* $x, y \in X$,

(P2) $d(Tx,Ty) < D(x,y)$, *whenever* $D(x,y) \neq 0$, *where* $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$, *a function satisfying the following: for any* $\varepsilon > 0$ *there exists* $\delta > \varepsilon$ *such that* $\varepsilon < t < \delta$ *implies* $\varphi(t) \leq \varepsilon$, $\gamma \geq 0$, *and*

$$D(x, y) = d(x, y) + \gamma [d(x, Tx) + d(y, Ty)]. \quad (1)$$

Then T *has a unique fixed point.*

Moreover if $\gamma = 1$ *and* φ *is continuous with* $\varphi(t) < t$ *for all* $t > 0$, *then the continuity of* T *can be dropped.*

A mapping T satisfying the conditions (P1) and (P2) is called a Proinov contraction. The following example shows the generality of Proinov contraction over quasi-contraction.

Example 4 (see [8]). Let $X = \{1, 2, 3\}$ with the usual metric d and $T : X \to X$ such that

$$T1 = 1,\quad T2 = 3,\quad T3 = 1. \quad (2)$$

The mapping T does not satisfy the condition (**C**). However, T satisfies the conditions (P1) and (P2) with $\varphi(t) = 2t/(1+\gamma)$, where $\gamma > 1$.

On the other hand, in 1994, Matthews [9] introduced the notion of partial metric spaces to study the denotational semantics of dataflow networks. It is widely recognized that partial metric spaces play an important role in constructing models in the theory of computation (see [10] and references thereof). Matthews also obtained the partial metric version of Banach contraction theorem. Subsequently, many authors studied partial metric spaces and their topological properties and obtained a number of fixed point theorems for single and multivalued mappings (cf. [9–27] and many others).

In [28], Haghi et al. pointed out that some fixed point generalizations to partial metric spaces can be obtained from the corresponding results in metric spaces. To demonstrate facts they considered certain cases. Motivated by Proinov's results, in this paper, we present some fixed point theorems in partial metric spaces which cannot be obtained from the corresponding results in metric spaces. Indeed, we obtain some fixed and common fixed point theorems for single and multivalued mappings in the setting of partial metric spaces. Our results complement, extend, and generalize a number of fixed point theorems including some recent results in [10, 11, 14, 16, 23] and others. Besides discussing some useful examples, an application to Volterra type system of integral equations is also given. Finally, we show that fixed point problems discussed herein are well-posed and have limit shadowing property.

For the sake of completeness, we recall the following definitions and results from [9, 10, 14].

Definition 5. A partial metric on a nonempty set X is a function $p : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$

(p_1) $p(x, x) = p(y, y) = p(x, y)$ if and only if $x = y$;

(p_2) $p(x, x) \leq p(x, y)$;

(p_3) $p(x, y) = p(y, x)$;

(p_4) $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$.

The pair (X, p) is called a partial metric space.

A partial metric p on X generates a T_0-topology τ_p on X with a base of the family of open p-balls $\{B_p(x, r) : x \in X, r > 0\}$, where

$$B_p(x, r) = \{y \in X : p(x, y) < p(x, x) + r\}. \quad (3)$$

If p is a partial metric on X, then the function $p^s : X \times X \to \mathbb{R}^+$ given by

$$p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y) \quad (4)$$

for all $x, y \in X$ is a metric on X.

Example 6 (see [10, 14]). Let $X = \mathbb{R}^+$ and $p : X \times X \to \mathbb{R}^+$ given by $p(x, y) = \max\{x, y\}$ for all $x, y \in X$. Then (X, p) is a partial metric space.

Example 7 (see [9, 14]). Let $X = \{[a, b] : a, b \in \mathbb{R},\ a \leq b\}$. Then $p([a, b], [c, d]) = \max\{b, d\} - \min\{a, c\}$ defines a partial metric p on X.

Definition 8. Let (X, p) be a partial metric space. Then one has the following:

(1) A sequence $\{x_n\}$ in X converges to a point $x \in X$ if and only if $\lim_{n\to\infty} p(x, x_n) = p(x, x)$.

(2) A sequence $\{x_n\}$ in X is Cauchy if $\lim_{n,m\to\infty} p(x_n, x_m)$ exists and is finite.

(3) X is complete if every Cauchy sequence $\{x_n\}$ in X converges to a point $x \in X$, that is, $p(x, x) = \lim_{n,m\to\infty} p(x_n, x_m)$.

Lemma 9 (see [9]). *Let* (X, p) *be a partial metric space. Then* X *is complete if and only if the metric space* (X, p^s) *is complete. Furthermore,* $\lim_{n\to\infty} p^s(x_n, z) = 0$ *if and only if*

$$p(z, z) = \lim_{n\to\infty} p(x_n, z) = \lim_{n,m\to\infty} p(x_n, x_m). \quad (5)$$

In [25], Romaguera introduced the following notions of 0-Cauchy sequence and 0-complete partial metric spaces. He obtained a characterization of completeness for partial metric space using the notion of 0-completeness.

Definition 10. A sequence $\{x_n\}$ in a partial metric space (X, p) is 0-Cauchy if $\lim_{n,m\to\infty} p(x_n, x_m) = 0$. The partial metric space (X, p) is 0-complete if each 0-Cauchy sequence in X converges to a point $x \in X$ such that $p(x, x) = 0$.

Notice that every 0-Cauchy sequence in (X, p) is Cauchy in (X, p^s) and every complete partial metric space is 0-complete. However, a 0-complete partial metric space need not be complete (cf. [29] and [25]).

A subset A of X is closed (resp., compact) in (X, p) if it is closed (resp., compact) with respect to the topology τ_p induced by p on X. The subset A is bounded in (X, p) if there exist $x_0 \in X$ and $M > 0$ such that $a \in B_p(x_0, M)$ for all $a \in A$; that is,

$$p(x_0, a) < p(a, a) + M \quad \forall a \in A. \tag{6}$$

Let $CB^p(X)$ be the collection of all nonempty, closed, and bounded subsets of X with respect to the partial metric p. For $A \in CB^p(X)$, one defines

$$p(x, A) = \inf_{y \in A} p(x, y). \tag{7}$$

For $A, B \in CB^p(X)$,

$$\begin{aligned} \delta_p(A, B) &= \sup_{a \in A} p(a, B), \\ \delta_p(B, A) &= \sup_{b \in B} p(b, A), \\ H_p(A, B) &= \max\{\delta_p(A, B), \delta_p(B, A)\}. \end{aligned} \tag{8}$$

Proposition 11 (see [14]). *Let (X, p) be a partial metric space. For any $A, B, C \in CB^p(X)$, one has*

(δ_1) $\delta_p(A, A) = \sup\{p(a, a) : a \in A\}$;

(δ_2) $\delta_p(A, A) \le \delta_p(A, B)$;

(δ_3) $\delta_p(A, B) = 0$ *implies that* $A \subseteq B$;

(δ_4) $\delta_p(A, B) \le \delta_p(A, C) + \delta_p(C, B) - \inf_{c \in C} p(c, c)$.

Proposition 12 (see [14]). *Let (X, p) be a partial metric space. For any $A, B, C \in CB^p(X)$, one has*

(H1) $H_p(A, A) \le H_p(A, B)$;

(H2) $H_p(A, B) = H_p(B, A)$;

(H3) $H_p(A, B) \le H_p(A, C) + H_p(C, B) - \inf_{c \in C} p(c, c)$;

(H4) $H_p(A, B) = 0 \Rightarrow A = B$. *But the converse is not true.*

In view of Propositions 11 and 12, H_p is a partial Hausdorff metric induced by the partial metric p.

2. Auxiliary Results

Hitzler and Seda [19] obtained the following result to establish a relation between a partial metric and the corresponding metric on a nonempty set X.

Proposition 13 (see [19, 28]). *Let (X, p) be a partial metric space. Then the function $d : X \times X \to \mathbb{R}^+$ defined by $d(x, y) = 0$ whenever $x = y$ and $d(x, y) = p(x, y)$ whenever $x \ne y$ is a metric on X such that $\tau_{p^s} \subseteq \tau_d$. Moreover, (X, d) is complete if and only if (X, p) is 0-complete.*

The following lemma is the key result in [28].

Lemma 14 (see [28]). *Let (X, p) be a partial metric space, $T : X \to X$ a self-mapping, d the metric constructed in Proposition 13, and $x, y \in X$. Define*

$$\begin{aligned} M_d(x, y) = \max\Big\{&d(x, y), d(x, Tx), d(y, Ty), \\ &\frac{1}{2}[d(x, Ty) + d(y, Tx)]\Big\}, \\ M_p(x, y) = \max\Big\{&p(x, y), p(x, Tx), p(y, Ty), \\ &\frac{1}{2}[p(x, Ty) + p(y, Tx)]\Big\}. \end{aligned} \tag{9}$$

Then $M_d(x, y) = M_p(x, y)$ for all $x, y \in X$ with $x \ne y$.

Using Proposition 13 and Lemma 14 above, we obtain the following result.

Lemma 15. *Let (X, p) be a partial metric space and $T : X \to X$ a self-mapping. Suppose $d : X \times X \to \mathbb{R}^+$ is the constructed metric in Proposition 13 and $x, y \in X$. Define*

$$\begin{aligned} \mu_d(x, y) &= d(x, y) + d(x, Tx) + d(y, Ty), \\ \mu_p(x, y) &= p(x, y) + p(x, Tx) + p(y, Ty). \end{aligned} \tag{10}$$

Then

(a) $\mu_d(x, y) \le \mu_p(x, y)$ *for all* $x, y \in X$;

(b) $M_p(x, y) \le \mu_p(x, y)$ *for all* $x, y \in X$.

Proof. To prove (a), we shall consider three cases and the rest of the cases will follow in the same manner.

Case 1 ($x = y$). One has

$$\begin{aligned} \mu_d(x, y) &= 0 + d(x, Tx) + d(y, Ty) \\ &\le p(x, x) + p(x, Tx) + p(y, Ty) \\ &= \mu_p(x, y). \end{aligned} \tag{11}$$

Case 2 ($x = Tx$). One has

$$\begin{aligned} \mu_d(x, y) &= d(x, y) + 0 + d(y, Ty) \\ &\le p(x, y) + p(x, x) + p(y, Ty) \\ &= \mu_p(x, y). \end{aligned} \tag{12}$$

Case 3 ($y = Ty$). One has

$$\begin{aligned} \mu_d(x, y) &= d(x, y) + d(x, Tx) + 0 \\ &\le p(x, y) + p(x, Tx) + p(y, y) \\ &= \mu_p(x, y). \end{aligned} \tag{13}$$

The proof of (b) follows easily form [8, page 3300]. □

3. Single Valued Mappings

For the sake of brevity, in this section, we shall use the following denotations:

(1) Φ the class of functions $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that ϕ is continuous nondeceasing function satisfying $\phi(t) < t$ and the series $\sum_{n\geq 1}^{\infty} \phi^n(t)$ converges for all $t > 0$;

(2) Ψ the class of functions $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that ψ is upper semicontinuous from the right satisfying $\psi(t) < t$ for all $t > 0$.

Let $A, B, S, T : X \to X$ be mappings.

(3) $M(x, y) = \max\{p(Sx, Ty), p(Ax, Sx), p(By, Ty), (1/2)[p(Sx, By) + p(Ax, Ty)]\}$;

(4) $\alpha(x, y) = p(Sx, Ty) + p(Ax, Sx) + p(By, Ty)$.

Remark 16. It can be easily seen that

(i) $M(x, y) \leq \alpha(x, y)$ for all $x, y \in X$ (see [8]);

(ii) $\Phi \subset \Psi$.

Browder and Petryshyn [30] introduced the notion of *asymptotic regularity* for a single valued mapping in a metric space (see also [3], page 547).

Definition 17. A self-mapping T of a metric space (X, d) is *asymptotically regular* at a point $x \in X$ if

$$\lim_{n\to\infty} d\left(T^n x, T^{n+1} x\right) = 0. \tag{14}$$

If T is asymptotically regular at each point of X then one says that T is asymptotically regular on X.

Sastry et al. [31] and Singh et al. [32] extended the above definition to three mappings as follows.

Definition 18. Let S, T, and f be self-mappings of a metric space (X, d). The pair (S, T) is asymptotically regular with respect to f at a point $x_0 \in X$ if there exists a sequence $\{x_n\}$ in X such that

$$\begin{aligned} fx_{2n+1} &= Sx_{2n}, \\ fx_{2n+2} &= Tx_{2n+1} \end{aligned} \tag{15}$$

for all $n \in \mathbb{N} \cup \{0\}$ and

$$\lim_{n\to\infty} d\left(fx_n, fx_{n+1}\right) = 0. \tag{16}$$

If $S = T$ then one gets the definition of asymptotic regularity of T with respect to f (see, for instance, Rhoades et al. [33]). Further, if f is the identity mapping on X, then one gets the usual definition of asymptotic regularity for a mapping T.

We extend the above notion to four self-mappings on a partial metric space as follows.

Definition 19. Let A, B, S, and T be self-mappings of a partial metric space (X, p). The mappings A, B, S, and T will be called asymptotically regular at $x_0 \in X$ if there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\begin{aligned} y_{2n} &= Tx_{2n+1} = Ax_{2n}; \\ y_{2n+1} &= Sx_{2n+2} = Bx_{2n+1}. \end{aligned} \tag{17}$$

$n \in \mathbb{N} \cup \{0\}$ and

$$\lim_{n\to\infty} p\left(y_n, y_{n+1}\right) = 0. \tag{18}$$

The following theorem is the main result in [10].

Theorem 20. *Suppose A, B, S, and T are self-mappings of a complete partial metric space (X, p) such that $AX \subseteq TX$, $BX \subseteq SX$, and*

$$p\left(Ax, By\right) \leq \phi\left(M\left(x, y\right)\right), \tag{19}$$

for all $x, y \in X$, where $\phi \in \Phi$.

If one of AX, BX, TX, and SX is a closed subset of (X, p), then

(i) *A and S have a coincidence point;*

(ii) *B and T have a coincidence point.*

Moreover, if the pairs (A, S) and (B, T) are weakly compatible, then A, B, S, and T have a unique common fixed point.

Now we present a more general result than Theorem 20.

Theorem 21. *Let A, B, S, and T be self-mappings of a partial metric space (X, p) such that*

(A) *$AX \subseteq TX$ and $BX \subseteq SX$;*

(B) *A, B, S, and T are asymptotically regular at $x_0 \in X$;*

$$p\left(Ax, By\right) \leq \psi\left(\alpha\left(x, y\right)\right), \tag{20}$$

for all $x, y \in X$, where $\psi \in \Psi$.

If one of AX, BX, TX, and SX is a 0-complete subspace of X, then

(a) *A and S have a coincidence point;*

(b) *B and T have a coincidence point;*

(c) *A and S have a common fixed point provided that the pair (A, S) is commuting at one of their coincidence points;*

(d) *B and T have a common fixed point provided that the pair (B, T) is commuting at one of their coincidence points.*

Moreover, the mappings A, B, S, and T have a common fixed point provided that (c) and (d) are true.

Proof. Let $x_0 \in X$ be such that A, B, S, and T are asymptotically regular at x_0. Since $AX \subseteq TX$, there exists $x_1 \in X$ such that $Tx_1 = Ax_0$. Also since $BX \subseteq SX$, there exists $x_2 \in X$

such that $Sx_2 = Bx_1$. Continuing this process, we construct sequences $\{x_n\}$ and $\{y_n\}$ in X defined by

$$\begin{aligned} y_{2n} &= Tx_{2n+1} = Ax_{2n}; \\ y_{2n+1} &= Sx_{2n+2} = Bx_{2n+1}, \end{aligned} \tag{21}$$

where $n \in \mathbb{N} \cup \{0\}$. Since A, B, S, and T are asymptotically regular at x_0, we have

$$\lim_{n\to\infty} p(y_n, y_{n+1}) = 0. \tag{22}$$

We claim that $\{y_n\}$ is a 0-Cauchy sequence. Suppose $\{y_n\}$ is not 0-Cauchy. Then there exist $\beta > 0$ and increasing sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that

$$\begin{aligned} p\left(y_{2m_k}, y_{2n_k+1}\right) &\geq \beta, \\ p\left(y_{2m_k}, y_{2n_k}\right) &< \beta, \end{aligned} \tag{23}$$

for all $n \leq 2m_k < 2n_k + 1$. By the triangle inequality, we have

$$\begin{aligned} p\left(y_{2m_k}, y_{2n_k+1}\right) &\leq p\left(y_{2m_k}, y_{2n_k}\right) + p\left(y_{2n_k}, y_{2n_k+1}\right) \\ &\quad - p\left(y_{2n_k}, y_{2n_k}\right) \\ &\leq p\left(y_{2m_k}, y_{2n_k}\right) + p\left(y_{2n_k}, y_{2n_k+1}\right). \end{aligned} \tag{24}$$

Thus $\lim_{k\to\infty} p(y_{2m_k}, y_{2n_k+1}) = \beta$. Now, by (20), we have

$$\begin{aligned} p\left(y_{2m_k}, y_{2n_k+1}\right) &= p\left(Ax_{2m_k}, Bx_{2n_k+1}\right) \\ &\leq \psi\left(\alpha\left(x_{2m_k}, x_{2n_k+1}\right)\right) = \psi\left(p\left(Sx_{2m_k}, Tx_{2n_k+1}\right)\right. \\ &\quad \left. + p\left(Ax_{2m_k}, Sx_{2m_k}\right) + p\left(Bx_{2n_k+1}, Tx_{2n_k+1}\right)\right) \\ &= \psi\left(p\left(y_{2m_k-1}, y_{2n_k}\right) + p\left(y_{2m_k}, y_{2m_k-1}\right)\right. \\ &\quad \left. + p\left(y_{2n_k+1}, y_{2n_k}\right)\right). \end{aligned} \tag{25}$$

Since ψ is upper semicontinuous from the right, we deduce that

$$\beta \leq \limsup_{k\to\infty} \psi\left(\alpha\left(x_{2m_k}, x_{2n_k+1}\right)\right) \leq \psi(\beta), \tag{26}$$

a contradiction. Therefore $\lim_{m,n\to\infty} p(y_n, y_m) = 0$.

Suppose that SX is a 0-complete subspace of X. Then the subsequence $\{y_{2n}\}$ being contained in SX has a limit in SX. Call it z. Let $u \in S^{-1}z$. Then $Su = z$. Note that the subsequence $\{y_{2n+1}\}$ also converges to z. By (20), we have

$$\begin{aligned} p\left(Au, Bx_{2n+1}\right) &\leq \psi\left(\alpha\left(u, x_{2n+1}\right)\right) \\ &= \psi\left(p\left(Su, Tx_{2n+1}\right) + p(Au, Su)\right. \\ &\quad \left. + p\left(Bx_{2n+1}, Tx_{2n+1}\right)\right) = \psi\left(p\left(Su, y_{2n}\right)\right. \\ &\quad \left. + p(Au, Su) + p\left(y_{2n+1}, y_{2n}\right)\right). \end{aligned} \tag{27}$$

Since ψ is upper semicontinuous from the right, making $k \to \infty$ implies $p(Au, Su) \leq \psi(p(Au, Su)) < p(Au, Su)$, a contradiction, unless $p(Au, Su) = 0$. Therefore $Au = Su = z$ and u is a coincidence point of A and S.

Since $AX \subseteq TX$, $z = Au \in TX$. Hence there exists $v \in X$ such that $Au = Tv$. Again, by (20), we have

$$\begin{aligned} p(Au, Bv) &\leq \psi(\alpha(u, v)) \\ &= \psi(p(Su, Tv) + p(Au, Su) + p(Bv, Tv)) \\ &= \psi(p(Bv, Au)). \end{aligned} \tag{28}$$

Thus $p(Au, Bv) \leq \psi(p(Au, Bv)) < p(Au, Bv)$, a contradiction, unless $p(Au, Bv) = 0$. Therefore $Tv = Au = Bv$ and v is a coincidence point of B and T.

If the pairs (A, S) and (B, T) are commuting at u and v, respectively, then

$$\begin{aligned} &ASu = SAu, \\ &AAu = ASu = SAu = SSu; \\ &BTv = TBv, \\ &TTv = TBv = BTv = BBv. \end{aligned} \tag{29}$$

Now, in view of (20), it follows that

$$\begin{aligned} p(AAu, Au) &= p(AAu, Bv) \leq \psi(\alpha(Au, v)) \\ &= \psi(p(SAu, Tv) + p(AAu, SAu) + p(Bv, Tv)) \\ &= \psi(p(AAu, Au)), \end{aligned} \tag{30}$$

a contradiction. Therefore $AAu = Au = SAu$ and Au is a common fixed point of A and S. Similarly, Bv is a common fixed point of B and T. Since $Au = Bv$, we conclude that Au is a common fixed point of A, B, S, and T. The proof is similar when TX is a complete subspace of X. The cases in which AX or BX is a complete subspace of X are also similar since $AX \subseteq TX$ and $BX \subseteq SX$. □

When $A = B$ and $S = T = \mathrm{id}$ (the identity mapping) in Theorem 21, we get the following result which extends a result of Romaguera [23].

Corollary 22. *Let A be an asymptotically regular self-mapping of a partial metric space (X, p) such that*

$$p(Ax, Ay) \leq \psi\left(\mu_p(x, y)\right), \tag{31}$$

for all $x, y \in X$, where $\psi \in \Psi$. Then A has a fixed point.

The following example shows the generality of our results.

Example 23. Let $X = \{0, 1, 2\}$ endowed with the partial metric $p(x, y) = \max\{x, y\}$ for all $x, y \in X$. Then (X, p)

is a 0-complete partial metric space. Define the mappings $A, B, S, T : X \to X$ by

$$\begin{aligned} A &= B, \\ S &= T, \\ A0 &= A1 = 0, \\ A2 &= 2; \\ T0 &= 0, \\ T1 &= 1, \\ T2 &= 2. \end{aligned} \tag{32}$$

Define sequences $\{x_n\}$ and $\{y_n\}$ by

$$\begin{aligned} x_0 &= x_n = 0; \\ y_{2n} &= Tx_{2n+1} = Ax_{2n}; \\ y_{2n+1} &= Sx_{2n+2} = Bx_{2n+1}, \end{aligned} \tag{33}$$

where $n \in \mathbb{N} \cup \{0\}$. Then the mappings A, B, S, and T are asymptotically regular at 0.

Further,

$$p(Ax, By) \le \psi(\alpha(x, y)), \tag{34}$$

for all $x, y \in X$, where $\alpha(x, y) = p(Sx, Ty) + p(Ax, Sx) + p(By, Sy)$ and $\psi(t) = 3t/4$. So the assumptions of Theorem 21 are fulfilled and A, B, S, and T have fixed points 0 and 2.

On the other hand, $p(A1, B2) > \phi(M(1, 2))$ for any ϕ. Therefore the mappings A, B, S, and T do not satisfy the requirements of Theorem 20.

Example 24. Let $X = [0, 1]$ be endowed with the partial metric $p(x, y) = \max\{x, y\}$ for all $x, y \in X$. Then (X, p) is a 0-complete partial metric space. Define the mappings $A, B, S, T : X \to X$ by

$$\begin{aligned} Ax &= \frac{1}{2}x^2, \\ Bx &= \frac{1}{8}x^2, \\ Sx &= \frac{1}{6}x^2, \\ Tx &= x^2. \end{aligned} \tag{35}$$

Then $AX \subseteq TX$ and $BX \subseteq SX$. Now define the function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ by $\psi(t) = (3/4)t$. We now show that

$$p(Ax, By) \le \psi(\alpha(x, y)), \tag{36}$$

for all $x, y \in X$, where $\alpha(x, y) = p(Sx, Ty) + p(Ax, Sx) + p(By, Sy)$. For this, let $x, y \in X$ with $y \le x$. Then

$$\begin{aligned} p(Ax, By) &= p\left(\frac{1}{2}x^2, \frac{1}{8}y^2\right) = \frac{1}{2}x^2, \\ \alpha(x, y) &= p(Sx, Ty) + p(Ax, Sx) + p(By, Sy) \\ &= p\left(\frac{1}{6}x^2, y^2\right) + p\left(\frac{1}{2}x^2, \frac{1}{6}x^2\right) \\ &\quad + p\left(\frac{1}{8}y^2, \frac{1}{6}y^2\right) \ge \frac{1}{6}x^2 + \frac{1}{2}x^2 = \frac{2}{3}x^2. \end{aligned} \tag{37}$$

Then

$$\psi(\alpha(x, y)) \ge \psi\left(\frac{2}{3}x^2\right) = \frac{1}{2}x^2 = p(Ax, By). \tag{38}$$

Now we show that the mappings A, B, S, and T are asymptotically regular. For this, by Definition 19, we have for all $n \in \mathbb{N} \cup \{0\}$

$$\begin{aligned} y_{2n} &= Tx_{2n+1} = Ax_{2n} = (x_{2n+1})^2 = \frac{1}{2}(x_{2n})^2, \\ y_{2n+1} &= Sx_{2n+2} = Bx_{2n+1} = \frac{1}{6}(x_{2n+2})^2 = \frac{1}{8}(x_{2n+1})^2. \end{aligned} \tag{39}$$

Solving the above two equations we get

$$\begin{aligned} x_{2n+1} &= \frac{1}{\sqrt{2}}x_{2n}, \\ x_{2n+2} &= \sqrt{\frac{3}{8}}x_{2n}. \end{aligned} \tag{40}$$

So for given $x_0 \in X$ we can define a sequence $\{x_n\}$ by using (40) and then we can easily define sequence $\{y_n\}$ by (39). It can be easily seen that $\lim_{n\to\infty} p(y_n, y_{n+1}) = 0$.

On the other hand, for $y = 0$ and $x > 0$, we have

$$\begin{aligned} p(Ax, B0) &= p\left(\frac{1}{2}x^2, 0\right) = \frac{1}{2}x^2, \\ M(x, y) &= \max\Big\{p(Sx, T0), p(Ax, Sx), p(B0, T0), \\ &\quad \frac{1}{2}[p(Sx, B0) + p(Ax, T0)]\Big\} = \max\Big\{p\left(\frac{1}{6}x^2, 0\right), \\ &\quad p\left(\frac{1}{2}x^2, \frac{1}{6}x^2\right), p(0, 0), \\ &\quad \frac{1}{2}\left[p\left(\frac{1}{6}x^2, 0\right) + p\left(\frac{1}{2}x^2, 0\right)\right]\Big\} = \frac{1}{2}x^2. \end{aligned} \tag{41}$$

Therefore $p(Ax, B0) = (1/2)x^2 = M(x, y) > \phi(M(x, y))$ for any ϕ. Therefore the mappings A, B, S, and T do not satisfy the requirement of Theorem 20.

Now we give an example in which the mapping is not asymptotically regular at each point of interval but satisfies condition (4) for one mapping.

Example 25. Let $X = [0,1]$ be endowed with the partial metric $p(x,y) = \max\{x,y\}$ for all $x, y \in X$. Then (X,p) is a 0-complete partial metric space. Define the mapping by

$$Tx = \begin{cases} 0, & \text{if } x \in \left[0, \frac{1}{2}\right], \\ 1, & \text{otherwise } x \in \left(\frac{1}{2}, 1\right]. \end{cases} \quad (42)$$

Now define the function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ by $\varphi(t) = (3/4)t$. It can be easily seen that T is asymptotically regular at each point of interval $[0,1/2]$ and not asymptotically regular at any point of interval $(1/2,1]$. Now we show that

$$p(Ax, By) \leq \varphi(\alpha(x,y)), \quad (43)$$

for all $x, y \in X$, where $\alpha(x,y) = p(x,y) + p(x,Tx) + p(y,Ty)$. For this we distinguish the following cases.

Case 1. If $x, y \in [0,1/2]$ with $x \leq y$, we have

$$p(Tx, Ty) = 0 \leq \varphi(\alpha(x,y)). \quad (44)$$

Case 2. If $x \in [0,1/2]$ and $y \in (1/2,1]$, we have

$$\begin{aligned} \varphi(\alpha(x,y)) &= \varphi(p(x,y) + p(x,Tx) + p(y,Ty)) \\ &= \frac{3}{4}(y + x + 1) \geq \frac{3}{4}\left(\frac{1}{2} + 1\right) = \frac{9}{8} > 1 \\ &= p(Tx, Ty). \end{aligned} \quad (45)$$

Case 3. If $x, y \in (1/2,1]$ with $x \leq y$, we have

$$\begin{aligned} \varphi(\alpha(x,y)) &= \varphi(p(x,y) + p(x,Tx) + p(y,Ty)) \\ &= \frac{3}{4}(y + 1 + 1) \geq \frac{3}{2} > 1 = p(Tx, Ty). \end{aligned} \quad (46)$$

4. Multivalued Mappings

Rhoades et al. [33] and Singh et al. [32] extended the concept of *asymptotic regularity* from single valued to multivalued mappings in metric spaces. We extend it to partial metric spaces.

Definition 26. Let (X,p) be a partial metric space and $S : X \to CB^p(X)$. The mapping S is *asymptotically regular* at $x_0 \in X$ if, for any sequence $\{x_n\}$ in X and each sequence $\{y_n\}$ in X such that $y_n \in Sx_{n-1}$,

$$\lim_{n\to\infty} p(y_n, y_{n+1}) = 0 \quad (47)$$

for $n \in \mathbb{N} \cup \{0\}$.

Aydi et al. [14] obtained the following equivalent to the well-known multivalued contraction theorem due to Nadler Jr. [34].

Theorem 27. *Let (X,p) be a complete partial metric space. If $T : X \to CB^p(X)$ is a multivalued mapping such that, for all $x, y \in X$ and $k \in (0,1)$, one has*

$$H_p(Tx, Ty) \leq kp(x,y), \quad (48)$$

then T has a fixed point.

In [24], Romaguera pointed out that if $X = \mathbb{R}^+$ and $p : X \times X \to \mathbb{R}^+$ defined by $p(x,y) = \max\{x,y\}$ for all $x, y \in X$, then $CB^p(X) = \emptyset$ and the approach used in Theorem 27 and elsewhere has a disadvantage that the fixed point theorems for self-mappings may not be derived from it, when $CB^p(X) = \emptyset$. To overcome this problem he introduced the concept of mixed multivalued mappings and obtained a different version of Nadler Jr.'s theorem in a partial metric space.

Definition 28. Let (X,p) be a partial metric space. A mapping $T : X \to X \cup CB^p(X)$ is called a mixed multivalued mapping on X if T is a multivalued mapping on X such that for each $x \in X$ either $Tx \in X$ or $Tx \in CB^p(X)$.

A self-mapping $T : X \to X$ and a multivalued mapping $T : X \to CB^p(X)$ both are mixed multivalued mappings (see also [35]).

Motivated by Proinov's theorem and the above facts, we obtain the following result, which extends Theorem 27 above and Corollary 2.5 in [16].

Theorem 29. *Let (X,p) be a 0-complete partial metric space and $S : X \to X \cup C^p(X)$ a continuous mixed multivalued mapping such that*

(S1) *$H_p(Sx,Sy) \leq \varphi(\mu(x,y))$ for all $x, y \in X$;*

(S2) *$H_p(Sx,Sy) < \mu(x,y)$ whenever $\mu(x,y) \neq 0$,*

where φ is as in Theorem 3, $C^p(X)$ is a collection of all nonempty compact subsets of X, $\gamma \geq 0$, and

$$\mu(x,y) = p(x,y) + \gamma[p(x,Sx) + p(y,Sy)]. \quad (49)$$

If S is asymptotically regular at $x_0 \in X$, then S has a fixed point.

Moreover, if $\gamma = 1$ and φ is continuous and satisfies $\varphi(t) < t$ for all $t > 0$, then the continuity of S can be dropped.

Proof. We construct a sequence $\{x_n\}$ in X in the following way. Let $x_0 \in X$ such that S is asymptotically regular at x_0. Let x_1 be any element of Sx_0. If $x_0 = x_1$ or $x_1 \in Sx_1$, then x_1 is a fixed point of S and there is nothing to prove. Assume that $x_1 \notin Sx_1$ and Sx_1 is not singleton. Then $Sx_1 \in C^p(X)$ and by compactness of Sx_1 we can choose $x_2 \in Sx_1$ such that

$$p(x_1, x_2) \leq H_p(Sx_0, Sx_1). \quad (50)$$

If $Sx_1 = \{x_2\}$ is a singleton, then obviously

$$p(x_1, x_2) \leq H_p(Sx_0, Sx_1). \quad (51)$$

Therefore, in either case, we have

$$p(x_1, x_2) \leq H_p(Sx_0, Sx_1). \quad (52)$$

Again, since Sx_2 is compact, we choose a point $x_3 \in Sx_2$ such that

$$p(x_2, x_3) \leq H_p(Sx_1, Sx_2). \quad (53)$$

Continuing in the same manner we get

$$p(x_n, x_{n+1}) \leq H_p(Sx_{n-1}, Sx_n). \quad (54)$$

Following largely [3, 8], we show that the sequence $\{x_n\}$ is a 0-Cauchy. Fix $\varepsilon > 0$. Since φ is as in Theorem 3, there exists $\delta > \varepsilon$ such that, for any $t \in (0, \infty)$,

$$\varepsilon < t < \delta \Longrightarrow \varphi(t) \le \varepsilon. \tag{55}$$

Without loss of generality we may assume that $\delta \le 2\varepsilon$. Since S is asymptotically regular at x_0,

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = 0. \tag{56}$$

So, there exists an integer $N_1 \ge 1$ such that

$$p(x_n, x_{n+1}) \le H_p(Sx_{n-1}, Sx_n) < \frac{\delta - \varepsilon}{1 + 2\gamma}, \tag{57}$$

for all $n \ge N_1$. By induction we shall show that

$$p(x_n, x_m) \le H_p(Sx_{n-1}, Sx_{m-1}) < \frac{\delta + 2\gamma\varepsilon}{1 + 2\gamma}, \tag{58}$$

for all $m, n \in \mathbb{N}$ with $m \ge n \ge N_1$.

Let $n \ge N_1$ be fixed. Obviously, (58) holds for $m = n + 1$. Assuming (58) to hold for an integer $m \ge n$, we shall prove it for $m + 1$. By the triangle inequality, we get

$$\begin{aligned} p(x_n, x_{m+1}) &\le p(x_n, x_{n+1}) + p(x_{n+1}, x_{m+1}) \\ &\quad - p(x_{n+1}, x_{n+1}) \\ &\le p(x_n, x_{n+1}) + p(x_{n+1}, x_{m+1}) \\ &\le p(x_n, x_{n+1}) + H_p(Sx_n, Sx_m). \end{aligned} \tag{59}$$

We claim that

$$H_p(Sx_n, Sx_m) \le \varepsilon. \tag{60}$$

To prove the above claim, we consider two cases.

Case 1 ($\mu(x_n, x_m) \le \varepsilon$). By (S2) it follows that $H_p(Sx_n, Sx_m) \le \mu(x_n, x_m) \le \varepsilon$, and (60) holds.

Case 2 ($\mu(x_n, x_m) > \varepsilon$). By (S1), we have

$$H_p(Sx_n, Sx_m) \le \varphi(\mu(x_n, x_m)). \tag{61}$$

By the definition of $\mu(x, y)$, we obtain

$$\begin{aligned} \mu(x_n, x_m) &= p(x_n, x_m) \\ &\quad + \gamma[p(x_n, Sx_n) + p(x_m, Sx_m)] \\ &= p(x_n, x_m) \\ &\quad + \gamma[p(x_n, x_{n+1}) + p(x_m, x_{m+1})]. \end{aligned} \tag{62}$$

Using (57) and (58) in this inequality, we get

$$\mu(x_n, x_m) < \frac{\delta + 2\gamma\varepsilon}{1 + 2\gamma} + \gamma\frac{\delta - \varepsilon}{1 + 2\gamma} + \gamma\frac{\delta - \varepsilon}{1 + 2\gamma} = \delta. \tag{63}$$

Thus $\varepsilon < \mu(x_n, x_m) < \delta$. Hence (55) implies that $\varphi(\mu(x_n, x_m)) \le \varepsilon$. Now (61) implies (60). By (60), (59), and (57), it follows that

$$\begin{aligned} p(x_n, x_{m+1}) &\le p(x_n, x_{n+1}) + H_p(Sx_n, Sx_m) \\ &< \frac{\delta - \varepsilon}{1 + 2\gamma} + \varepsilon = \frac{\delta + 2\gamma\varepsilon}{1 + 2\gamma}. \end{aligned} \tag{64}$$

This proves (58). Since $\delta \le 2\varepsilon$, (58) implies that $p(x_n, x_m) < 2\varepsilon$ for all integers m and n with $m \ge n \ge N_1$ and hence $\{x_n\}$ is a 0-Cauchy sequence. Since X is 0-complete, $\{x_n\}$ has a limit. Call it z. We note that

$$p(z, z) = \lim_{m,n\to\infty} p(x_n, x_m) = 0. \tag{65}$$

If S is continuous, then obviously $z \in Sz$ is a fixed point of S.

Moreover, if $\gamma = 1$ and φ is continuous and satisfies $\varphi(t) < t$ for all $t > 0$, then it follows from (S2) that

$$\begin{aligned} p(x_{n+1}, Sz) &\le H_p(Sx_n, Sz) \\ &\le \varphi(p(x_n, z) + p(x_n, Sx_n) + d(z, Sz)) \\ &\le \varphi(p(x_n, z) + p(x_n, x_{n+1}) + d(z, Sz)). \end{aligned} \tag{66}$$

Making $n \to \infty$,

$$d(z, Sz) \le \varphi(d(z, Sz)), \tag{67}$$

a contradiction, unless $z \in Sz$. □

Now we present a slightly modified version of Theorem 29 to obtain a new result.

Theorem 30. *Let (X, p) be a partial metric space and $T : X \to X$ and $S : X \to C^p(X)$ such that $SX \subseteq TX$ and*

(i) $H_p(Sx, Sy) \le \varphi(h(x, y))$ *for all* $x, y \in X$,

where $h(x, y) = p(Tx, Ty) + p(Tx, Sx) + p(Ty, Sy)$, *and φ is as in Theorem 3 and is continuous;*

(ii) $H_p(Sx, Sy) < h(x, y)$ *whenever* $h(x, y) \ne 0$.

If S is asymptotically regular and either SX or TX is a complete subspace of X, then S and T have a coincidence point z.

Further, S and T have a common fixed point provided that $SSz = Sz$ and S and T are commuting at their coincidence point z.

Proof. It can be completed using the proofs of Theorem 29 above and Theorem 2.2 in [8]. □

The following example illustrates our results.

Example 31 (see [36]). Let $X = \{0, 1, 2\}$ and $p : X \times X \to \mathbb{R}^+$ defined by

$$\begin{aligned} p(0,0) &= 0, \\ p(1,1) &= p(2,2) = \frac{1}{4}, \\ p(1,0) &= \frac{1}{3}, \\ p(2,0) &= \frac{3}{5}, \\ p(2,1) &= \frac{2}{5}, \\ p(x,y) &= p(y,x) \quad \forall x, y \in X. \end{aligned} \tag{68}$$

Clearly, (X, p) is a 0-complete partial metric space. Now, define the mapping $S : X \to C^p(X)$ such that

$$Sx = \begin{cases} \{0\} & \text{if } x \neq 2, \\ \{0,1\} & \text{if } x = 2. \end{cases} \tag{69}$$

It can be easily seen that S is asymptotically regular at 0 and

$$H_p(Sx, Sy) \leq \varphi(\mu(x,y)) \tag{70}$$

with $\gamma \geq 1$ and $\varphi(t) = 3t/4$. Therefore all the assumptions of Theorem 29 are fulfilled and 0 is a fixed point of S.

The following example shows the generality of our results.

Example 32. Let $X = \{0, 1, 4\}$ be endowed with the partial metric $p : X \times X \to \mathbb{R}^+$ defined by

$$p(x,y) = \frac{1}{4}|x-y| + \frac{1}{2}\max\{x,y\} \quad \forall x, y \in X. \tag{71}$$

Clearly, (X, p) is a 0-complete partial metric space. Now, define the mapping $S : X \to C^p(X)$ such that

$$\begin{aligned} S0 &= S1 = \{0\}, \\ S4 &= \{0,4\}. \end{aligned} \tag{72}$$

It can be easily seen that S is continuous and asymptotically regular and for all $x, y \in X$,

$$H_p(Sx, Sy) \leq \varphi(\mu(x,y)) \tag{73}$$

with $\gamma \geq 1$ and $\varphi(t) = 4t/5$. Therefore the assumptions of Theorem 29 are fulfilled.

On the other hand,

$$\begin{aligned} H_p(S0, S4) &= H_p(\{0\}, \{0,4\}) \\ &= \max\{p(0,\{0,4\}), \max\{p(0,0), p(4,0)\}\} = 3 \\ &> 3k \end{aligned} \tag{74}$$

for all $k \in (0,1)$; Theorem 27 and Corollary 2.5 in [16] are not satisfied by the mapping S.

Example 33. Let $X = [0,2]$ be endowed with the partial metric $p : X \times X \to \mathbb{R}^+$ defined by

$$p(x,y) = \frac{1}{4}|x-y| + \frac{1}{2}\max\{x,y\} \quad \forall x, y \in X. \tag{75}$$

Now define the mapping $S : X \to C^p(X) \cup X$ such that

$$Sx = \begin{cases} \left[0, \frac{7}{8}x\right], & \text{if } x \in [0,1], \\ \{1\}, & \text{if } x \in (1,2]. \end{cases} \tag{76}$$

It can be easily seen that $p^s(x,y) = |x-y|$ and (X, p^s) is a complete metric space. Now, by Lemma 9, (X, p) is a complete partial metric space and hence 0-complete. Now we show that

$$H_p(Sx, Sy) \leq \varphi(p(x,y) + \gamma[p(x,Sx) + p(y,Sy)]) \tag{77}$$

for all $x, y \in X$ with $\gamma \geq 5/7$ and $\varphi(t) = (7/8)t$. For this we distinguish the following cases.

Case 1. If $x, y \in [0,1]$ with $x \leq y$, we have

$$\begin{aligned} H_p(Sx, Sy) &= \max\{\delta_p(Sx, Sy), \delta_p(Sy, Sx)\} \\ &= \max\left\{\frac{7}{16}x, \frac{21}{32}y - \frac{7}{32}x\right\} \\ &= \frac{21}{32}y - \frac{7}{32}x. \end{aligned} \tag{78}$$

Therefore, for all $\gamma \geq 0$ and $\varphi(t) = (7/8)t$, we have

$$\begin{aligned} \varphi(\mu(x,y)) &= \varphi(p(x,y) + \gamma[p(x,Sx) + p(y,Sy)]) \\ &\geq \varphi(p(x,y)) = \frac{7}{8}\left(\frac{3}{4}y - \frac{1}{4}x\right) \\ &= \frac{21}{32}y - \frac{7}{32}x = H_p(Sx, Sy). \end{aligned} \tag{79}$$

Similar conclusion can be done for $y < x$.

Case 2. If $x \in [0,1]$ and $y \in (1,2]$, then we have

$$\begin{aligned} H_p(Sx, Sy) &= \max\{\delta_p(Sx, Sy), \delta_p(Sy, Sx)\} \\ &= \max\left\{\frac{3}{4}, \frac{3}{4} - \frac{7}{32}x\right\} = \frac{3}{4}. \end{aligned} \tag{80}$$

Therefore, for all $\gamma \geq 5/7$ and $\varphi(t) = (7/8)t$, we have

$$\begin{aligned} \varphi(\mu(x,y)) &= \varphi(p(x,y) + \gamma[p(x,Sx) + p(y,Sy)]) \\ &= \frac{7}{8}\left(\frac{3}{4}y - \frac{1}{4}x + \gamma\left[\frac{17}{32}x + \frac{3}{4}y - \frac{1}{4}\right]\right) \\ &= \frac{7}{32}(2+2\gamma)y + \frac{7}{32}(y-x) \end{aligned}$$

$$+\frac{7}{8}\gamma\left[\frac{17}{32}x+\frac{1}{4}(y-1)\right]\geq\frac{3}{4}$$
$$=H_p(Sx,Sy). \tag{81}$$

Case 3. If $x, y \in (1,2]$ with $x \leq y$, then we have

$$H_p(Sx,Sy)=H_p(\{1\},\{1\})=\frac{1}{2}. \tag{82}$$

Hence, for all $\gamma \geq 5/7$ and $\varphi(t)=(7/8)t$, we have

$$\begin{aligned}\varphi(\mu(x,y)) &= \varphi(p(x,y)+\gamma[p(x,Sx)+p(y,Sy)])\\ &= \frac{7}{8}\left(\frac{3}{4}y-\frac{1}{4}x+\gamma\left[\frac{3}{4}x-\frac{1}{4}+\frac{3}{4}y-\frac{1}{4}\right]\right)\\ &= \frac{7}{32}(2+3\gamma)y+\frac{7}{32}(y-x)+\frac{7}{8}\gamma\frac{(3x-2)}{4}\geq\frac{1}{2}\\ &= H_p(Sx,Sy).\end{aligned} \tag{83}$$

Similar conclusion can be done for $y < x$.

On the other hand, at $x = 1$ and $y = 11/10$, we have

$$\begin{aligned}H_p(Sx,Sy) &= H_p\left(\left[0,\frac{7}{8}\right],\{1\}\right)\\ &= \max\left\{\delta_p\left(\left[0,\frac{7}{8}\right],\{1\}\right),\delta_p\left(\{1\},\left[0,\frac{7}{8}\right]\right)\right\}\\ &= \max\left\{\frac{3}{4},\frac{17}{32}\right\}=\frac{3}{4}.\end{aligned} \tag{84}$$

Now

$$\begin{aligned}H_p(Sx,Sy) &= H_p\left(\left[0,\frac{7}{8}\right],\{1\}\right)=\frac{3}{4}>k\cdot\frac{23}{40}\\ &= k\cdot p(x,y)\end{aligned} \tag{85}$$

for all $k \in [0,1)$ and

$$\begin{aligned}M(x,y) &= \max\left\{p(x,y),p(x,Sx),p(y,Sy),\right.\\ &\left.\frac{1}{2}[p(x,Sy)+p(y,Sx)]\right\}=\max\left\{\frac{23}{40},\frac{17}{32},\frac{23}{40},\frac{177}{320}\right\}\\ &= \frac{23}{40}.\end{aligned} \tag{86}$$

Hence

$$H_p(Sx,Sy)=\frac{3}{4}>k\cdot\frac{23}{40}=k\cdot M(x,y) \tag{87}$$

for all $k \in [0,1)$.

Remark 34. We remark that the mappings satisfying Theorems 2, 27, and 20 are also asymptotically regular. Therefore the extra assumption of asymptotical regularity in our results is not strong.

5. Existence of a Common Solution of Volterra Type Integral Equations

This section is inspired by the work given in paper [37] and the purpose of this section is to give an existence theorem for a solution of (88) given below.

Let $I = [0,K] \subset \mathbb{R}$ be a closed and bounded interval with $K > 0$. Consider the system of Volterra type integral equations:

$$\begin{aligned}u(t) &= g(t)+\int_0^t K_1(t,s,u(s))\,ds,\\ u(t) &= g(t)+\int_0^t K_2(t,s,u(s))\,ds,\\ u(t) &= g(t)+\int_0^t K_3(t,s,u(s))\,ds,\\ u(t) &= g(t)+\int_0^t K_4(t,s,u(s))\,ds,\end{aligned} \tag{88}$$

where $t \in [0,K]$ and $K_i : [0;K]\times[0;K]\times\mathbb{R} \to \mathbb{R}$ $(i \in \{1,2,3,4\})$ and $g : \mathbb{R} \to \mathbb{R}$ are continuous functions.

Let $C(I,\mathbb{R})$ be the set of real continuous functions defined on I and $T_i : C(I,\mathbb{R}) \to C(I,\mathbb{R})$ self-mappings defined by

$$\begin{gathered}T_iu(t)=g(t)+\int_0^t K_i(t,s,u(s))\,ds\\ \forall u\in C(I,\mathbb{R}),\ t\in I,\ i\in\{1,2,3,4\}.\end{gathered} \tag{89}$$

Clearly, u is a solution of (88) if and only if it is a common fixed point of T_i for $i \in \{1,2,3,4\}$.

We shall prove the existence of a common fixed point of T_i for $i \in \{1,2,3,4\}$ under certain conditions.

Theorem 35. *Suppose that the following hypotheses hold:*

(H1) *For any $u \in C(I,\mathbb{R})$, there exist $k_1, k_2 \in C(I,\mathbb{R})$ such that $T_1u = T_3k_1$, $T_2u = T_4k_2$.*

(H2) *For all $t \in I$, $u \in C(I,\mathbb{R})$,*

$$T_1T_4u(t)=T_4T_1u(t),\quad \textit{whenever } T_1u(t)=T_4u(t), \tag{90}$$

and for all $t \in I$, $u \in C(I,\mathbb{R})$,

$$T_2T_3u(t)=T_3T_2u(t),\quad \textit{whenever } T_2u(t)=T_3u(t). \tag{91}$$

(H3) *There exists a continuous function $\hbar : I\times I \to \mathbb{R}^+$ such that for all $t, s \in I$ and $u, v \in C(I,\mathbb{R})$*

$$\begin{aligned}&\left|K_1(t,s,u(s))-K_2(t,s,v(s))\right|\leq\hbar(t,s)\\ &\quad\cdot\left[\left|T_4u(s)-T_3v(s)\right|+\left|T_1u(s)-T_4u(s)\right|\right.\\ &\quad\left.+\left|T_2v(s)-T_3v(s)\right|\right].\end{aligned} \tag{92}$$

(**H4**) *For sequences* $\{h_n\}$ *and* $\{k_n\}$ *in* $C(I, \mathbb{R})$ *such that*

$$k_{2n} = T_3 h_{2n} = T_1 h_{2n}; \qquad k_{2n+1} = T_4 h_n = T_2 h_{2n+1}, \tag{93}$$

one has

$$\lim_{n\to\infty} |k_n(t) - k_{n+1}(t)| = 0 \quad \forall t \in I,\ n \in \mathbb{N} \cup \{0\}. \tag{94}$$

(**H5**) $\sup_{t\in I} \int_0^t \hbar(t,s)ds \le 3/4$.

Then system (88) of integral equations has a solution $u^* \in C(I, \mathbb{R})$.

Proof. For $x \in X = C(I, \mathbb{R})$ define $\|x\|_\tau = \max_{t\in[0,K]}\{|x(t)|e^{-\tau t}\}$, where $\tau \ge 1$ is taken arbitrary. Notice that $\|\cdot\|_\tau$ is a norm equivalent to the maximum norm and $(X, \|\cdot\|_\tau)$ is a Banach space (cf. [38, 39]). The metric induced by this norm is given by

$$d_\tau(x,y) = \max_{t\in[0,K]} \left\{|x(t) - y(t)| e^{-\tau t}\right\}, \tag{95}$$

for all $x, y \in X$.

Now, consider X endowed with the partial metric given by

$$p_\tau(x,y) = \begin{cases} d_\tau(x,y), & \text{if } \|x\|_\tau, \|y\|_\tau \le 1, \\ d_\tau(x,y) + \tau, & \text{otherwise.} \end{cases} \tag{96}$$

Obviously, (X, p_τ) is 0-complete but not complete. In fact, the associated metric $p^s_\tau(x,y) = 2p_\tau(x,y) - p_\tau(x,x) - p_\tau(y,y)$ given by

$$p^s_\tau(x,y) = \begin{cases} 2d_\tau(x,y), & \text{if } (\|x\|_\tau, \|y\|_\tau \le 1) \text{ or } (\|x\|_\tau, \|y\|_\tau > 1), \\ 2d_\tau(x,y) + \tau, & \text{otherwise,} \end{cases} \tag{97}$$

is not complete [37]. It is evident that $x^* \in X$ is a solution of (88) if and only if x^* is a common fixed point of $T_i's$. By condition (**H1**), it is clear that

$$T_1(C(I,\mathbb{R})) \subset T_3(C(I,\mathbb{R})), \qquad T_2(C(I,\mathbb{R})) \subset T_4(C(I,\mathbb{R})). \tag{98}$$

From condition (**H2**), the pairs (T_1, T_4) and (T_2, T_3) are commuting. Using condition (**H3**) and maximum norm, the mappings T_i $(i = 1,2,3,4)$ are asymptotically regular on (X, p_τ).

Now, we show that condition (20) holds. Observe that this condition needs not be checked for $u = v \in X$. Then for $u, v \in X$ such that $\|u\|_\tau, \|v\|_\tau \le 1$, by assertions (**H3**)–(**H5**), we have

$$\begin{aligned} |T_1u(t) - T_2v(t)| &\le \int_0^t |K_1(t,s,u(t)) - K_2(t,s,v(t))|\, ds \\ &\le \left(\int_0^t \frac{3\tau}{4} [|T_4u(s) - T_3v(s)| + |T_1u(s) - T_4u(s)| + |T_2v(s) - T_3v(s)|]\, ds\right) \\ &= \left(\int_0^t \frac{3\tau}{4} [|T_4u(s) - T_3v(s)| + |T_1u(s) - T_4u(s)| + |T_2v(s) - T_3v(s)|]\, e^{-\tau s} e^{\tau s} ds\right) = \frac{3\tau}{4}\left(\int_0^t e^{\tau s} ds\right) \\ &\cdot [|T_4u(s) - T_3v(s)| + |T_1u(s) - T_4u(s)| + |T_2v(s) - T_3v(s)|]\, e^{-\tau s} = \frac{3\tau}{4}\left(\frac{e^{\tau t}}{\tau} - \frac{1}{\tau}\right) \\ &\cdot [\|T_4u(s) - T_3v(s)\|_\tau + \|T_1u(s) - T_4u(s)\|_\tau + \|T_2v(s) - T_3v(s)\|_\tau] = \frac{3\tau}{4}\left(\frac{e^{\tau t}}{\tau}\right) [\|T_4u(s) \\ &- T_3v(s)\|_\tau + \|T_1u(s) - T_4u(s)\|_\tau + \|T_2v(s) - T_3v(s)\|_\tau]. \end{aligned} \tag{99}$$

Since $\|u\|_\tau, \|v\|_\tau \le 1$, it follows that, for all $u, v \in X$ at least for some $t \in [0, K]$, we have

$$|T_1x(t) - T_2y(t)| e^{-\tau t} \le \frac{3}{4} \times [\|T_4u(s) - T_3v(s)\|_\tau + \|T_1u(s) - T_4u(s)\|_\tau + \|T_2v(s) - T_3v(s)\|_\tau]. \tag{100}$$

Now, by considering the control functions $\psi : [0, +\infty) \to [0, +\infty)$ defined by $\psi(t) = 3t/4$, for all $t > 0$, we get

$$p_\tau(T_1u, T_2v) \le \psi(p_\tau(T_4u(s), T_3v(s)) + p_\tau(T_1u(s), T_4u(s)) + p_\tau(T_2v(s), T_3v(s))). \tag{101}$$

Putting $A = T_1$, $B = T_2$, $T = T_3$, and $S = T_4$, then all the hypotheses of Theorem 21 are satisfied. Therefore A, B, S, and T have a common fixed point $u^* \in C(I, \mathbb{R})$; that is, u^* is a solution of system (88). □

6. Conclusions

The authors are able to present some general fixed point results for a wider class of mappings in partial metric spaces with illustrative examples and an application. Results presented herein cannot be directly obtained from the corresponding metric space versions.

Conflicts of Interest

The authors declare that they do not have any conflicts of interest.

Authors' Contributions

First and second authors wrote results. Third author prepared application and fourth author worked out some examples. In this way, each author equally contributed to this paper and read and approved the final manuscript.

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