Subclasses of Bi-Univalent Functions of Complex Order Based On Subordination Conditions Involving Wright Hypergeometric Functions

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Abstract. In this paper, we introduce and investigate a new subclass of bi-univalent functions $\sum$ of complex order defined in the open unit disk, which are associated with hypergeometric functions and satisfy subordinate conditions. Furthermore, we find estimates on the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions in the new subclass. Several (known or new) consequences of the results are also pointed out.

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1 Introduction

Let **A** denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

Further, by **S** we shall denote the class of all functions in **A** which are univalent in **U**. Some of the important and well-investigated subclasses of the univalent function class **S** include (for example) the class $\mathbf{S}^*(\alpha)$ of starlike functions of order α in **U** and the class **K** (α) of convex functions of order α in **U**. It is well known that every function $f \in \mathbf{S}$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z \qquad (z \in \mathbb{U})$$

and

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$$f(f^{-1}(w)) = w \qquad \left(|w| < r_0(f); r_0(f) \geq \frac{1}{4}\right),$$

where

$$g(w) = f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots. \quad (2)$$

A function $f \in \mathbf{A}$ is said to be bi-univalent in $\mathbf{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathbf{U}$. Let Σ denote the class of bi-univalent functions in $\mathbf{U}$ given by Eq. (1).

An analytic function f is subordinate to an analytic function $\boldsymbol{g}$, written $f(z) \prec g(z)$, provided there is an analytic function $\boldsymbol{w}$ defined on $\mathbf{U}$ with $w(0) = 0$ and $|w(z)| < 1$ satisfying $f(z) = g(w(z))$. Ma and Minda [1] unified various subclasses of starlike and convex functions for which either of the quantities $\frac{z f'(z)}{f(z)}$ or $1 + \frac{z f''(z)}{f'(z)}$ is subordinate to a more general superordinate function. For this purpose, they considered an analytic function φ with a positive real part in the unit disk $U, \varphi(0) = 1, \varphi'(0) > 0,$ and φ maps $\mathbf{U}$ onto a region starlike with respect to 1 and symmetric with respect to the real axis. The class of Ma-Minda starlike functions consists of functions $f \in \mathrm{A}$ satisfying the subordination $\frac{z f'(z)}{f(z)} \prec \varphi(z)$. Similarly, the class of Ma-Minda convex functions of functions $f \in A$ satisfying the subordination

$$1 + \frac{z f''(z)}{f'(z)} \prec \varphi(z)$$

A function f is bi-starlike of Ma-Minda type or bi-convex of Ma-Minda type if both f and f^{-1} are respectively Ma-Minda starlike or convex. These classes are denoted respectively by $\mathrm{S}_\Sigma^*(\varphi)$ and $\mathrm{K}_\Sigma(\varphi)$. In the sequel, it is assumed that φ is an analytic function with positive real part in the unit disk $\mathbf{U}$, satisfying $\varphi(0) = 1, \varphi'(0) > 0$, and $\varphi(\mathrm{U})$ is symmetric with respect to the real axis. Such a function has a series expansion of the form

$$\varphi(z) = 1 + B_1z + B_2z^2 + B_3z^3 + \cdots, \quad (B_1 > 0) \qquad (3)$$

For some intriguing examples of functions and characterization of the class Σ, one could refer to Srivastava, *et al.*, [2] and the references stated therein (see also, Hayami & Owa [3]). Recently there has been growing interest to study the bi-univalent functions class Σ (see [2-7]) and obtain non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$. The coefficient estimate problem for each of the following Taylor-Maclaurin coefficients $|a_n|$ for $n \in N \setminus \{1,2\}$; $N := \{1,2,3,\ldots\}$ is presumably still an open problem.

The study of operators plays an important role in geometric function theory and its related fields. Many differential and integral operators can be written in terms of convolution of certain analytic functions. It is observed that this formalism brings an ease in further mathematical exploration and also helps to understand the geometric properties of such operators better.

The convolution or Hadamard product of two functions $f, h \in A$ is denoted by $f * h$ and is defined as

$$(f * h)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n \tag{4}$$

where $f(z)$ is given by Eq. (1) and $h(z) = z + \sum_{n=2}^{\infty} b_n z^n$

Now we briefly recall the definitions of the special functions and operators used in this paper.

For complex parameters $\alpha_1, \ldots, \alpha_p$ $(\frac{\alpha_j}{A_j} \neq 0, -1, \ldots; j = 1,2,\ldots p)$ and

$$\beta_1, \ldots, \beta_q \quad (\frac{\beta_j}{B_j} \neq 0, -1, \ldots; j = 1,2,\ldots q)$$

the Fox-H functions, by which we mean Wright generalized hypergeometric functions ${}_p\Psi_q$ with $A_j, B_j > 0$, aregiven by (rather general and typical examples of H- functions, not reducible to G- functions):

$${}_p\Psi_q\left(\begin{matrix}(\alpha_1, A_1), \ldots, (\alpha_p, A_p) \\ (\beta_1, B_1), \ldots, (\beta_q, B_q)\end{matrix} : z\right) = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha_1 + nA_1)\ldots\Gamma(\alpha_p + nA_p)}{\Gamma(\beta_1 + nB_1)\ldots\Gamma(\beta_q + nB_q)} \frac{z^n}{n!}$$

$$= H^{1,p}_{p,q+1}\left[-z \,\middle|\, \begin{matrix}(1-\alpha_1, A_1), \ldots, (1-\alpha_p, A_p) \\ (0,1), (1-\beta_1, B_1), \ldots, (1-\beta_q, B_q)\end{matrix}\right] \tag{5}$$

with $1+\sum_{n=1}^{q} B_n - \sum_{n=1}^{p} A_n \geq 0$, $(p,q \in N = 1,2,3,...)$ and for suitably bounded values of $|z|$.

Note that when $A_1 = \ldots = A_p = B_1 = \ldots = B_q = 1$, they turn into the generalized hypergeometric functions

$$ {}_p\Psi_q \left(\begin{matrix} (\alpha_1,1),\ldots,(\alpha_p,1) \\ (\beta_1,1),\ldots,(\beta_q,1) \end{matrix} : z \right) = \left[\frac{\prod_{j=1}^{p}\Gamma(\alpha_j)}{\prod_{i=1}^{q}\Gamma(\beta_j)} \right]_p F_q(\alpha_1,\ldots,\alpha_p : \beta_1,\ldots,\beta_q : z), $$

where $p \leq q+1; p,q \in N_0 = N \cup \{0\}; z \in U$.

Now we recall the linear operator due to Srivastava [8] (see Dziok & Raina [9]) and Wright [10] in terms of the Hadamard product (or convolution) involving the generalized hypergeometric function. Let $l,m \in N$ and suppose that the parameters $\alpha_1, A_1 \ldots, \alpha_l, A_l$ and $\beta_1, B_1 \ldots, \beta_m, B_m$ are also positive real numbers. Then, corresponding to a function

$$ {}_l\Phi_m[(\alpha_j, A_j)_{1,l};(\beta_j, B_j)_{1,m};z] $$

defined by

$$ {}_l\Phi_m[(\alpha_j, A_j)_{1,l};(\beta_j, B_j)_{1,m};z] = \Omega z\, {}_l\Psi_m[(\alpha_j, A_j)_{1,l}(\beta_j, B_j)_{1,m};z] \tag{6} $$

where $\Omega = \left(\prod_{j=1}^{l}\Gamma(\alpha_j)\right)^{-1}\left(\prod_{j=1}^{m}\Gamma(\beta_j)\right)$ and we consider a linear operator

$$ \mathrm{W}[(\alpha_j, A_j)_{1,l};(\beta_j, B_j)_{1,m}] : \mathrm{A} \to \mathrm{A} $$

defined by the following Hadamard product (or convolution)

$$ \mathrm{W}[(\alpha_j, A_j)_{1,l};(\beta_j, B_j)_{1,m}](f)(z) := z\, {}_l\Phi_m[(\alpha_j, A_j)_{1,l};(\beta_j, B_j)_{1,m};z] * f(z). $$

We observe that, for $f(z)$ of the form Eq. (1), we have

$$ \mathrm{W}[(\alpha_j, A_j)_{1,l};(\beta_j, B_j)_{1,m}]f(z) = z + \sum_{n=2}^{\infty}\phi_n\, a_n z^n \tag{7} $$

where

$$\phi_n = \frac{\Omega\Gamma(\alpha_1 + A_1(n-1))\ldots\Gamma(\alpha_l + A_l(n-1))}{(n-1)!\Gamma(\beta_1 + B_1(n-1))\ldots\Gamma(\beta_m + B_m(n-1))} \tag{8}$$

If, for convenience, we write

$$\mathsf{W}_m^l f(z) = \mathsf{W}[(\alpha_1, A_1),\ldots,(\alpha_l, A_l);(\beta_1, B_1),\ldots,(\beta_m, B_m)]f(z). \tag{9}$$

We state the following remark due to Srivastava [8] (see Dziok & Raina [9]) and Wright [10].

Remark 1.1 Other interesting and useful special cases of the Fox-Wright generalized hypergeometric function include (for example) the generalized Bessel function

$${}_0\Psi_1(-;(\nu+1,\mu);-z) \equiv \mathsf{J}_\mu^\nu = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma(n\mu+\nu+1)}$$

For $\mu = 1,$ corresponds essentially to the *classical Bessel function* $J^\nu(z),$ and the generalized Mittag-Leffler function

$${}_1\Psi_1((1,1);(\mu,\lambda);z) \equiv E_\mu^\lambda = \sum_{n=0}^{\infty} \frac{(z)^n}{\Gamma(n\lambda+\mu)}$$

Remark 1.2 By setting $A_j = 1(j = 1,...,l)$ and $B_j = 1(j = 1,...,m)$ in Eq. (6), we are led immediately to the *generalized hypergeometric function* ${}_lF_m(z)$, which is defined by

$$\Omega_l F_m(z) \equiv \Omega_l F_m(\alpha_1,\ldots\alpha_l;\beta_1,\ldots,\beta_m;z) := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_l)_n}{(\beta_1)_n \ldots (\beta_m)_n} \frac{z^n}{n!} \tag{10}$$

$(l \le m+1; l,m \in N_0 := N \cup \{0\}; z \in U)$ where N denotes the set of all positive integers, $(\alpha)_n$ is the Pochhammer symbol.

In view of the relationship (10), the linear operator Eq. (7) includes the Dziok-Srivastava operator (see Dziok & Srivastava [11]), so that it includes (as its special cases) various other linear operators introduced and studied by Bernardi [12], Carlson & Shaffer [13], Libera [14], Livingston [15], Ruscheweyh [16] and Srivastava & Owa [17].

Motivated by the earlier work of Deniz [18] (see [19-21]) in the present paper we introduce a new subclass of the function class Σ of complex order $\gamma \in C \setminus \{0\}$, involving Wright hypergeometric functions W_m^l, and find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the new subclasses of function class Σ. Several related classes are also considered and connections to earlier known results are made.

Definition 1.3 A function $f \in \Sigma$ given by Eq. (1) is said to be in the class $G_{\Sigma}^{l,m}(\gamma,\lambda,\varphi)$ if the following conditions are satisfied:

$$1+\frac{1}{\gamma}\left(\frac{z(\mathrm{W}_m^l f(z))'}{(1-\lambda)\mathrm{W}_m^l f(z)+\lambda z(\mathrm{W}_m^l f(z))'}-1\right) \prec \varphi(z) \tag{11}$$

and

$$1+\frac{1}{\gamma}\left(\frac{w(\mathrm{W}_m^l g(w))'}{(1-\lambda)\mathrm{W}_m^l g(w)+\lambda z(\mathrm{W}_m^l g(w))'}-1\right) \prec \varphi(w) \tag{12}$$

where $(\gamma \in \mathrm{C}\setminus\{0\}; 0 \le \lambda < 1; z, w \in \mathrm{U})$, the function g is given by Eq. (2).

Example 1. For $\lambda = 0$ and $\gamma \in \mathrm{C}\setminus\{0\}$, a function $f \in \Sigma$, given by Eq. (1) is said to be in the class $S_{\Sigma}^{l,m}(\gamma,\varphi)$ if the following conditions are satisfied:

$$1+\frac{1}{\gamma}\left(\frac{z(\mathrm{W}_m^l f(z))'}{\mathrm{W}_m^l f(z)}-1\right) \prec \varphi(z) \tag{13}$$

and

$$1+\frac{1}{\gamma}\left(\frac{w(\mathrm{W}_m^l g(w))'}{\mathrm{W}_m^l g(w)}-1\right) \prec \varphi(w) \tag{14}$$

where $z, w \in U$ and the function g is given by Eq. (2).

On specializing the parameters *l, m* one can state the various new subclasses of Σ (or $\mathrm{G}_{\Sigma}^{l,m}(\gamma,\lambda,\varphi)$), as illustrations, we present some examples for the case with $A_j = 1 (j = 1,2,...,l)$; $B_j = 1 (j = 1,2,...,m)$.

Example 2. If $l \le m+1; l,m \in N_0 := N \cup \{0\}$, and $\gamma \in \mathrm{C}\setminus\{0\}$, then a function $f \in \Sigma$, given by Eq. (1) is said to be in the class $S_{\Sigma}^{l,m}(\gamma,\lambda,\varphi)$ if the following conditions are satisfied:

$$1+\frac{1}{\gamma}\left(\frac{z(\mathrm{H}_m^l f(z))'}{(1-\lambda)(\mathrm{H}_m^l f(z))+\lambda(\mathrm{H}_m^l f(z))'}-1\right) \prec \varphi(z)$$

and

$$1+\frac{1}{\gamma}\left(\frac{w(\mathrm{H}_m^l g(w))'}{(1-\lambda)(\mathrm{H}_m^l g(w))+\lambda(\mathrm{H}_m^l g(w))'}-1\right)\prec\varphi(w),$$

where $\mathrm{H}_m^l f(z):=\left(z+\sum_{n=2}^{\infty}\frac{(\alpha_1)_n\ldots(\alpha_l)_n}{(\beta_1)_n\ldots(\beta_m)_n}\frac{(z)^n}{n!}\right)*f(z)$

$$\equiv \mathrm{W}[(\alpha_t,1)_{1,l};(\beta_t,1)_{1,m}]f(z)$$

a well-known Dziok-Srivastava operator [11], the function ***g*** is given by Eq. (2), $0\le\lambda<1$ and $z,w\in\mathrm{U}$.

Example 3. If $l=2$ and $m=1$ with $\alpha_1=a(a>0)$, $\alpha_2=1$, $\beta_1=c(c>0)$, and $\gamma\in\mathrm{C}\setminus\{0\}$, a function $f\in\Sigma$, given by Eq. (1), is said to be in the class $S_\Sigma^{a,c}(\gamma,\lambda,\varphi)$, if the following conditions are satisfied:

$$1+\frac{1}{\gamma}\left(\frac{z(L(a,c)f(z))'}{(1-\lambda)(L(a,c)f(z))+\lambda(L(a,c)f(z))'}-1\right)\prec\varphi(z)$$

and

$$1+\frac{1}{\gamma}\left(\frac{w(L(a,c)g(w))'}{(1-\lambda)(L(a,c)g(w))+\lambda(L(a,c)g(w))'}-1\right)\prec\varphi(w),$$

where $L(a,c)f(z):=\left(z+\sum_{n=2}^{\infty}\frac{(a)_n}{(c)_n}z^n\right)*f(z)\equiv\mathrm{H}_1^2(a,1;c)f(z),$ a well-known Carlson-Shaffer operator [13] and the function ***g*** is given by Eq. (2) $0\le\lambda<1$; *and* $z,w\in\mathrm{U}$.

Example 4. If $l=2$ and $m=1$ with $\alpha_1=\delta+1(\delta\ge-1)$, $\alpha_2=1$, $\beta_1=1$, and $\gamma\in\mathrm{C}\setminus\{0\}$, a function $f\in\Sigma$, given by Eq. (1), is said to be in the class $S_\Sigma^{\delta}(\gamma,\lambda,\varphi)$ if the following conditions are satisfied:

$$1+\frac{1}{\gamma}\left(\frac{z(D^\delta f(z))'}{(1-\lambda)(D^\delta f(z))+\lambda(D^\delta f(z))'}-1\right)\prec\varphi(z)$$

and

$$1+\frac{1}{\gamma}\left(\frac{w(D^{\delta}g(w))'}{(1-\lambda)(D^{\delta}g(w))+\lambda(D^{\delta}g(w))'}-1\right)\prec\varphi(w),$$

where D^{δ} is called the Ruscheweyh derivative of order $\delta\,(\delta\geq -1)$ and $D^{\delta}f(z):=\frac{z}{(1-z)^{\delta+1}}*f(z)\equiv \mathrm{H}_1^2(\delta+1,1;1)f(z)$ and the function g is given by Eq. (2) $0\leq\lambda<1$; *and* $z,w\in\mathrm{U}$.

Example 5. If $l=2$ and $m=1$ with $\alpha_1=1$, $\alpha_2=1$, $\beta_1=1$, and $\gamma\in\mathrm{C}\setminus\{0\}$, a function $f\in\Sigma$, given by Eq. (1) is said to be in the class $S_{\Sigma}^{*}(\gamma,\lambda,\varphi)$ if the following conditions are satisfied:

$$1+\frac{1}{\gamma}\left(\frac{zf'(z)}{(1-\lambda)f(z)+\lambda f'(z)}-1\right)\prec\varphi(z)$$

and

$$1+\frac{1}{\gamma}\left(\frac{wg'(w)}{(1-\lambda)g(w)+\lambda g'(w)}-1\right)\prec\varphi(w),$$

where $0\leq\lambda<1$, $z,w\in U$ and the function ***g*** is given by Eq. (2).

Example 6. If $l=2$ and $m=1$ with $\alpha_1=1$, $\alpha_2=1$, $\beta_1=1$, and $\lambda=0;\gamma\in\mathrm{C}\setminus\{0\}$, a function $f\in\Sigma$, given by Eq. (1), is said to be in the class $S_{\Sigma}^{*}(\gamma,\varphi)$ if the following conditions are satisfied:

$$1+\frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)}-1\right)\prec\varphi(z)$$

and

$$1+\frac{1}{\gamma}\left(\frac{wg'(w)}{g(w)}-1\right)\prec\varphi(w),$$

where $0\leq\lambda<1$, $z,w\in U$ and the function g is given by Eq. (2).

Remark 1.4 For $\lambda=0$ and $\gamma\in\mathrm{C}\setminus\{0\}$, a function $f\in\Sigma$, given by Eq. (1), as in Example 1 one can state various analogous subclasses defined in Examples 2 to 5.

In the following section we find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the above-defined subclasses $\mathcal{G}_\Sigma^{l,m}(\gamma,\lambda,\varphi)$ of the function class Σ by employing the techniques used earlier by Deniz [7].

2 Coefficient Bounds for the Function Class $\mathcal{G}_\Sigma^{l,m}(\gamma,\lambda,\varphi)$

In order to derive our main results, we shall need the following lemma.

Lemma 2.1 (see [22]) If $h\in P,$ then $|c_k|\le 2$ for each *k*, where ***P*** is the family of all functions *h*, analytic in $\mathbb{U}$, for which

$$\Re\{h(z)\}>0 \qquad (z\in \mathrm{U}),$$

where

$$h(z)=1+c_1z+c_2z^2+\cdots \qquad (z\in\mathbb{U}) \tag{15}$$

We begin by finding the estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the class $\mathcal{G}_\Sigma^{l,m}(\gamma,\lambda,\varphi)$. Define the functions $p(z)$ and $q(z)$ by

$$p(z):=\frac{1+u(z)}{1-u(z)}=1+p_1z+p_2z^2+\cdots$$

and

$$q(z):=\frac{1+v(z)}{1-v(z)}=1+q_1z+q_2z^2+\cdots$$

or, equivalently,

$$u(z):=\frac{p(z)-1}{p(z)+1}=\frac{1}{2}\left[p_1z+\left(p_2-\frac{p_1^2}{2}\right)z^2+\cdots\right]$$

and

$$v(z):=\frac{q(z)-1}{q(z)+1}=\frac{1}{2}\left[q_1z+\left(q_2-\frac{q_1^2}{2}\right)z^2+\cdots\right]$$

Then $p(z)$ and $q(z)$ are analytic in $\mathbb{U}$ with $p(0)=1=q(0)$. Since $u,v:U\to U$, the functions $p(z)$ and $q(z)$ have a positive real part in $\mathbb{U}$, and $|p_i|\le 2$ and $|q_i|\le 2.$

Theorem 2.2 Let the function f (z) given by Eq. (1) be in the class $G_{\Sigma}^{l,m}(\gamma,\lambda,\varphi)$ Then

$$|a_2| \le \frac{|\gamma| B_1\sqrt{B_1}}{\sqrt{\left|[\gamma(\lambda^2-1)B_1^2+(1-\lambda)^2(B_1-B_2)]\phi_2^2+2\gamma(1-\lambda)B_1^2\phi_3\right|}} \tag{16}$$

and

$$|a_3| \le \frac{|\gamma|^2 B_1^2}{(1-\lambda)^2\phi_2^2}+\frac{|\gamma| B_1}{2(1-\lambda)\phi_3} \tag{17}$$

Proof. It follows from Eq. (11) and Eq. (12) that

$$1+\frac{1}{\gamma}\left(\frac{z(\mathsf{W}_m^l f(z))'}{(1-\lambda)\mathsf{W}_m^l f(z)+\lambda z(\mathsf{W}_m^l f(z))'}-1\right)=\varphi(u(z)) \tag{18}$$

and

$$1+\frac{1}{\gamma}\left(\frac{w(\mathsf{W}_m^l g(w))'}{(1-\lambda)\mathsf{W}_m^l g(w)+\lambda z(\mathsf{W}_m^l g(w))'}-1\right)=\varphi(v(w)) \tag{19}$$

where $p(z)$ and $q(w)$ in **P** and have the following forms:

$$\varphi(u(z))=\varphi\left(\frac{1}{2}\left[p_1 z+\left(p_2-\frac{p_1^2}{2}\right)z^2+\cdots\right]\right) \tag{20}$$

and

$$\varphi(v(w))=\varphi\left(\frac{1}{2}\left[q_1 w+\left(q_2-\frac{q_1^2}{2}\right)w^2+\cdots\right]\right)., \tag{21}$$

respectively. Now, equating the coefficients in Eq. (18) and Eq. (19), we get

$$\frac{(1-\lambda)}{\gamma}\phi_2 a_2=\frac{1}{2}B_1 p_1, \tag{22}$$

$$\frac{(\lambda^2-1)}{\gamma}\phi_2^2 a_2^2+\frac{2(1-\lambda)}{\gamma}\phi_3 a_3=\frac{1}{2}B_1(p_2-\frac{p_1^2}{2})+\frac{1}{4}B_2 p_1^2, \tag{23}$$

$$-\frac{(1-\lambda)}{\gamma}\phi_2 a_2=\frac{1}{2}B_1 q_1 \tag{24}$$

and

$$\frac{(\lambda^2-1)}{\gamma}\phi_2^2 a_2^2+\frac{2(1-\lambda)}{\gamma}\phi_3(2a_2^2-a_3)=\frac{1}{2}B_1(q_2-\frac{q_1^2}{2})+\frac{1}{4}B_2q_1^2. \tag{25}$$

From Eq. (22) and Eq. (24), we find that

$$a_2=\frac{\gamma B_1 p_1}{2(1-\lambda)\phi_2}=\frac{-\gamma B_1 q_1}{2(1-\lambda)\phi_2}, \tag{26}$$

which implies

$$p_1=-q_1. \tag{27}$$

and

$$8(1-\lambda)^2\phi_2^2 a_2^2=\gamma^2 B_1^2(p_1^2+q_1^2). \tag{28}$$

Adding Eq. (23) and Eq. (25), by using Eq. (26) and Eq. (27) ,we obtain

$$4\left([\gamma(\lambda^2-1)B_1^2+(1-\lambda)^2(B_1-B_2)]\phi_2^2+2\gamma(1-\lambda)B_1^2\phi_3\right)a_2^2=\gamma^2B_1^3(p_2+q_2) \tag{29}$$

Thus,

$$a_2^2=\frac{\gamma^2B_1^3(p_2+q_2)}{4\left([\gamma(\lambda^2-1)B_1^2+(1-\lambda)^2(B_1-B_2)]\phi_2^2+2\gamma(1-\lambda)B_1^2\phi_3\right)}. \tag{30}$$

Applying Lemma 2.1 for the coefficients p_2 and q_2, we immediately have

$$|a_2|^2\le\frac{|\gamma|^2\ B_1^3}{\left|[\gamma(\lambda^2-1)B_1^2+(1-\lambda)^2(B_1-B_2)]\phi_2^2+2\gamma(1-\lambda)B_1^2\phi_3\right|}. \tag{31}$$

Since $B_1>0,$ the last inequality gives the desired estimate on $|a_2|$ given in Eq. (16).

Next, in order to find the bound on $|a_3|$, by subtracting Eq. (25) from Eq. (23), we get

$$\frac{4(1-\lambda)}{\gamma}\phi_3a_3-\frac{4(1-\lambda)}{\gamma}\phi_3a_2^2=\frac{B_1}{2}(p_2-q_2)+\frac{B_2-B_1}{4}(p_1^2-q_1^2). \tag{32}$$

It follows from Eq. (26), Eq. (27) and Eq. (32) that

$$a_3=\frac{\gamma^2B_1^2(p_1^2+q_1^2)}{8(1-\lambda)^2\phi_2^2}+\frac{\gamma B_1(p_2-q_2)}{8(1-\lambda)\phi_3}.$$

Applying Lemma 2.1 once again for the coefficients p_2 and q_2, we readily get

$$|a_3| \leq \frac{|\gamma|^2 B_1^2}{(1-\lambda)^2 \phi_2^2} + \frac{|\gamma| B_1}{2(1-\lambda)\phi_3}, B_1 > 0.$$

This completes the proof of Theorem 2.2. Putting $\lambda = 0$ in Theorem 2.2, we have the following corollary.

Corollary 2.3 Let the function *f (z)* given by Eq. (1) be in the class $S_\Sigma^{l,m}(\gamma,\varphi)$. Then

$$|a_2| \leq \frac{|\gamma| B_1 \sqrt{B_1}}{\sqrt{\left|[(B_1 - B_2) - \gamma B_1^2]\phi_2^2 + 2\gamma B_1^2 \phi_3\right|}} \tag{33}$$

and

$$|a_3| \leq \frac{|\gamma|^2 B_1^2}{\phi_2^2} + \frac{|\gamma| B_1}{2\phi_3} \tag{34}$$

By setting $A_j = 1 (j = 1,...,l)$ and $B_j = 1 (j = 1,...,m)$, taking l = 2 and m = 1 and with $\alpha_1 = 1$, $\alpha_2 = 1$, $\beta_1 = 1$, in φ_n, we get $\varphi_n = 1$. Hence we state the following corollaries for the new classes defined in Example 5.

Corollary 2.4 Let the function *f (z)* given by Eq. (1) be in the class $S_\Sigma^*(\gamma,\lambda,\varphi)$. Then

$$|a_2| \leq \frac{|\gamma| B_1 \sqrt{B_1}}{(1-\lambda)\sqrt{\left|(B_1 - B_2) + \gamma B_1^2\right|}} \tag{35}$$

and

$$|a_3| \leq \frac{|\gamma|^2 B_1^2}{(1-\lambda)^2} + \frac{|\gamma| B_1}{2(1-\lambda)} \tag{36}$$

Taking $\lambda = 0$, we get

Corollary 2.5 Let the function $f(z)$ given by Eq. (1) be in the class $S_\Sigma^*(\gamma,\varphi)$. Then

$$|a_2| \leq \frac{|\gamma| B_1 \sqrt{B_1}}{\sqrt{\left|(B_1 - B_2) + \gamma B_1^2\right|}} \tag{37}$$

and

$$|a_3| \le |\gamma|^2 B_1^2 + \frac{|\gamma| B_1}{2} \tag{38}$$

3 Concluding Remarks

For the class of strongly starlike functions, the function ϕ is given by

$$\varphi(z) = \left(\frac{1+z}{1-z}\right)^{\alpha} = 1 + 2\alpha z + 2\alpha^2 z^2 + \cdots \quad (0 < \alpha \le 1) \tag{39}$$

which gives $B_1 = 2\alpha$ and $B_2 = 2\alpha^2$.

On the other hand if we take

$$\varphi(z) = \frac{1+(1-2\beta)z}{1-z} = 1 + 2(1-\beta)z + 2(1-\beta)^2 z^2 + \cdots \quad (0 \le \beta < 1), \tag{40}$$

then $B_1 = B_2 = 2(1-\beta)$.

From Corollary 2.4, when $\gamma = 1$, and $\varphi(z)$ given by Eq. (39) or by Eq. (40) we state the following results given in [21].

Corollary 3.1 Let $f(z)$ given by Eq. (1) be in the class $S_{\Sigma}^{*}(\alpha, \lambda, \varphi)$, and $B_1 = 2\alpha; B_2 = 2\alpha^2$ $(0 < \alpha \le 1; 0 \le \lambda < 1)$. Then

$$|a_2| \le \frac{2\alpha}{(1-\lambda)\sqrt{1+\alpha}} \tag{41}$$

and

$$|a_3| \le \frac{4\alpha^2}{(1-\lambda)^2} + \frac{\alpha}{1-\lambda} \tag{42}$$

From Corollary 2.4, when $\gamma = 1$, and we have

Corollary 3.2 Let $f(z)$ given by Eq. (1) be in the class $S_{\Sigma}^{*}(\beta, \lambda, \varphi)$, $B_1 = B_2 = 2(1-\beta)$ $(0 \le \beta < 1; 0 \le \lambda < 1)$. Then

$$|a_2| \le \frac{\sqrt{2(1-\beta)}}{1-\lambda} \tag{43}$$

and

$$|a_3| \le \frac{4(1-\beta)^2}{(1-\lambda)^2} + \frac{(1-\beta)}{1-\lambda}. \tag{44}$$

Remark 3.3 From Corollary 2.5, when $\gamma = 1; \lambda = 0$ and $B_1 = 2\alpha; B_2 = 2\alpha^2$ and for $f \in S_\Sigma^*(\alpha)$, [23] we get

$$|a_2| \le \frac{2\alpha}{\sqrt{\alpha+1}} \qquad \text{and} \qquad |a_3| \le 4\alpha^2 + \alpha.$$

and for $B_1 = B_2 = 2(1-\beta)$

$$|a_2| \le \sqrt{2(1-\beta)} \qquad \text{and} \qquad |a_3| \le 4(1-\beta)^2 + (1-\beta).$$

Similarly, we can prove the results obtained earlier; we mention them in the following remarks.

Remark 3.4 Putting $\gamma = 1$ in Corollary 2.5, we obtain the corresponding results obtained in [19].

Remark 3.5 If l = 2 and m = 1 with $\alpha_1 = \mu + 1 (\mu > -1)$, $\alpha_2 = 1$, $\beta_1 = \mu + 2$, where J_μ is a Bernardi operator [12] defined by

$$J_\mu f(z) := \frac{\mu+1}{z^\mu} \int_0^z t^{\mu-1} f(t) dt \equiv H_1^2(\mu+1, 1; \mu+2) f(z).$$

Note that the operator J_1 was studied earlier by Libera [14] and Livingston [15], various other interesting corollaries and consequences of our main results (which are asserted by Theorem 2.2 above) can be derived similarly. Further, by setting $A_j = 1 (j = 1, ..., l)$ and $B_j = 1 (j = 1, ..., m)$, and suitably choosing l, m, λ we state the various results for the new classes defined in Examples 2 to 5, the details involved may be left as an exercise for the interested reader.

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