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# Subordination results for a class of analytic functions

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Abstract. In this paper, we derive several subordination results and integral means result for certain class of analytic functions with complex order defined by means of q-differential operator. Some interesting corollaries and consequences of our results are also considered

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## 1. Introduction and definitions

Let $\mathcal{A}$ be the class of functions which are analytic in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

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We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions which are analytic, univalent in $\mathbb{U}$ and normalized by $f(0) = 0 = f'(0) - 1$. The well known subclasses of $\mathcal{S}$ are the class of starlike functions $\mathcal{S}^*$ and convex functions $\mathcal{K}$. For given $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{S}$, the Hadamard product of $f$ and $g$ is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z). \tag{1.2}$$

We note that $f * g \in \mathcal{S}$ is analytic and univalent in the open disc $\mathbb{U}$.

For two analytic functions $f, g \in \mathcal{A}$ we say that $f$ is subordinate to $g$, denoted by $f \prec g$, if there exists a Schwarz function $\omega(z)$ which is analytic in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ for all $z \in \mathbb{U}$, such that $f(z) = g(\omega(z))$ for $z \in \mathbb{U}$. Note that, if the function $g$ is univalent in $\mathbb{U}$, due to Miller and Mocanu [28](see[18]), we have

$$f(z) \prec g(z) \Longleftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

Now we recall here the notion of *q-operator* i.e. *q-difference operator* that play vital role in the theory of hypergeometric series, quantum physics and in the operator theory. The application of *q-calculus* was initiated by Jackson [23], recently Kanas and Răducanu [24] (also see [1, 2, 16, 21, 22, 23, 33, 34]) have used the fractional *q-calculus operators* in investigations of certain classes of functions which are analytic in $\mathbb{U}$.

Let $0 < q < 1$. For any non-negative integer $n$, the *q-integer number n* is defined by

$$[n]_q = \frac{1 - q^n}{1 - q} = 1 + q + \cdots + q^{n-1}, \qquad [0]_q = 0. \tag{1.3}$$

In general, we will denote

$$[x]_q = \frac{1 - q^x}{1 - q}$$

for a non-integer number $x$. Also the *q-number shifted factorial* is defined by

$$[n]_q! = [n]_q [n-1]_q ... [2]_q [1]_q, \qquad [0]_q! = 1. \tag{1.4}$$

Clearly,

$$\lim_{q \to 1^-} [n]_q = n \qquad \text{and} \qquad \lim_{q \to 1^-} [n]_q! = n!.$$

For $0 < q < 1$, the Jackson's *q-derivative operator* (or *q-difference operator*) of a function $f \in \mathcal{A}$ given by (1.1) defined as follows [23]:

$$\mathfrak{D}_q f(z) = \begin{cases} \dfrac{f(z) - f(qz)}{(1-q)z} & \text{for } \ z \neq 0 \\ \\ f'(0) & \text{for } \ z = 0 \end{cases}, \tag{1.5}$$

$\mathfrak{D}_q^0 f(z) = f(z)$, and $\mathfrak{D}_q^m f(z) = \mathfrak{D}_q(\mathfrak{D}_q^{m-1} f(z))$, $m \in \mathbb{N} = \{1, 2, \ldots\}$. From (1.5), we have

$$\mathfrak{D}_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1} \qquad (z \in \mathbb{U}), \tag{1.6}$$

where $[n]_q$ is given by (1.3). For a function $\psi(z) = z^n$, we obtain

$$\mathfrak{D}_q\psi(z) = \mathfrak{D}_q z^n = \frac{1-q^n}{1-q}z^{n-1} = [n]_q z^{n-1}$$

and

$$\lim_{q\to 1^-} \mathfrak{D}_q\psi(z) = \lim_{q\to 1^-} \left([n]_q z^{n-1}\right) = nz^{n-1} = \psi'(z),$$

where $\psi'$ is the ordinary derivative.

Let $t \in \mathbb{R}$ and $n \in \mathbb{N}$. The *q-generalized Pochhammer symbol* is defined by

$$[t;n]_q = [t]_q[t+1]_q[t+2]_q...[t+n-1]_q \tag{1.7}$$

and for $t > 0$ the *q-gamma function* is defined by

$$\Gamma_q(t+1) = [t]_q\Gamma_q(t) \qquad \text{and} \qquad \Gamma_q(1) = 1. \tag{1.8}$$

Using the *q-difference operator*, Kanas and Raducanu [24] defined the Ruscheweyh *q-differential operator* as below: For $f \in \mathcal{A}$,

$$\mathcal{R}_q^\delta f(z) = f(z) * F_{q,\delta+1}(z) \qquad (\delta > -1,\, z \in \mathbb{U}) \tag{1.9}$$

where

$$F_{q,\delta+1}(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma_q(n+\delta)}{[n-1]_q!\,\Gamma_q(1+\delta)}z^n = z + \sum_{n=2}^{\infty}\frac{[\delta+1;n]_q}{[n-1]_q!}z^n. \tag{1.10}$$

Making use of (1.9) and (1.10), Aldweby and Darus[1] defined the $q-$analogue of Ruschewey operator $\mathcal{R}_q^\delta : \mathcal{A} \to \mathcal{A}$ as follows:

$$\begin{aligned}\mathcal{R}_q^\delta f(z) &= z + \sum_{n=2}^{\infty} \frac{\Gamma_q(n+\delta)}{[n-1]_q!\,\Gamma_q(1+\delta)}a_n z^n \qquad (z \in \mathbb{U}). \\ &= z + \sum_{n=2}^{\infty} \Theta_n(q,\delta)a_n z^n \qquad (z \in \mathbb{U}).\end{aligned} \tag{1.11}$$

where

$$\Theta_n := \Theta_n(q,\delta) = \frac{\Gamma_q(n+\delta)}{[n-1]_q!\,\Gamma_q(1+\delta)}. \tag{1.12}$$

As $q \to 1^-$ ,we note that

$$\begin{aligned}\mathcal{R}_q^0 f(z) &= f(z), \\ \mathcal{R}_q^1 f(z) &= z\mathfrak{D}_q f(z) = zf'(z),\end{aligned}$$

It is easy to check that

$$z\mathfrak{D}_q(F_{q,\delta+1}(z)) = \left(1 + \frac{[\delta]_q}{q^\delta}\right)F_{q,\delta+2}(z) - \frac{[\delta]_q}{q^\delta}F_{q,\delta+1}(z) \quad (z \in \mathbb{U}). \tag{1.13}$$

Making use of (1.9), (1.13) and the properties of Hadamard product, we obtain the following equality( see [24])

$$z\mathfrak{D}_q(\mathcal{R}_q^\delta f(z)) = \left(1+\frac{[\delta]_q}{q^\delta}\right)\mathcal{R}_q^{\delta+1} f(z)) - \frac{[\delta]_q}{q^\delta}\mathcal{R}_q^\delta f(z)) \quad (z\in\mathbb{U}). \tag{1.14}$$

From (1.11), we note that

$$\lim_{q\to 1^-} F_{q,\delta+1}(z) = \frac{z}{(1-z)^{\delta+1}}, \qquad \lim_{q\to 1^-} \mathcal{R}_q^\delta f(z) = f(z) * \frac{z}{(1-z)^{\delta+1}}.$$

Thus, when $q\to 1^-$ we can say that Ruscheweyh $q-$differential operator reduces to the differential operator defined by Ruscheweyh [32] (see[6, 9, 10, 11, 12]) and (1.14) gives the well known recurrent formula for Ruscheweyh differential operator. With the help of the differential operator $\mathcal{R}_q^\delta$, given by (1.11) we say that a function $f\in\mathcal{A}$ is said to be in the class $\mathcal{F}_{\lambda,q}(b,M)$ if it satisfies

$$\left|\frac{b-1+\frac{z(\mathcal{R}_q^\delta f(z))'}{\mathcal{R}_q^\delta f(z)}}{b} - M\right| < M \tag{1.15}$$

where $0<q<1, \delta>-1$, $M>\frac{1}{2}$, and $b\in\mathbb{C}^* = \mathbb{C}\setminus\{0\}$. It follows from [38] that $g\in\mathcal{F}(1,m) = \lim_{q\to 1^-}\mathcal{F}_{0,q}(1,M)$ and only if for $z\in\mathbb{U}$

$$\frac{zg'(z)}{g(z)} = \frac{1+w(z)}{1-mw(z)}, \quad (m = 1-\frac{1}{M}, M>\frac{1}{2}, w\in\Omega). \tag{1.16}$$

One can easily show that $f\in\mathcal{F}_{\lambda,q}(b,M)$ if and only if there is a function $g\in\mathcal{F}(1,M)$ such that

$$\mathcal{R}_q^\delta f(z) = z\left(\frac{g(z)}{z}\right)^b. \tag{1.17}$$

Thus from(1.16) and (1.17) it follows that $f\in\mathcal{F}_{\delta,q}(b,M)$ if and only if

$$\frac{z(\mathcal{R}_q^\delta f(z))'}{\mathcal{R}_q^\delta f(z)} = \frac{1+[b(1+m)-m]w(z)}{1-mw(z)}, \quad (z\in\mathbb{U}) \tag{1.18}$$

where $0\le q<1, \delta>-1$, $M>\frac{1}{2}$, and $b\in\mathbb{C}^*$ and $w\in\Omega$.

By giving specific values of $\delta, q, b$ and $M$, we obtain the following important subclasses studied by various authors in the earlier works:

(1) $\lim_{q\to 1^-}\mathcal{F}_{\delta,q}(b,M) = \mathcal{F}_\delta(b,M)$ (Kumar et al. [26])

(2) $\lim_{q\to 1^-}\mathcal{F}_{0,q}(b,M) = \mathcal{F}(b,M)$ and $\mathcal{F}(b,\infty) = \mathcal{S}(b)$ (Nasr and Aouf [30] and [31])

(3) $\lim_{q\to 1^-}\mathcal{F}_{1,q}(b,M) = \mathcal{G}_\lambda(b,M) = \mathcal{G}(b,\infty) = \mathcal{C}(b)$ (Nasr and Aouf [29])

(4) $\lim_{q\to 1^-}\mathcal{F}_{0,q}(cos\gamma e^{-i\gamma},M)=\mathcal{F}_{\gamma,M}$ and $\lim_{q\to 1^-}\mathcal{F}_{1,q}(cos\gamma e^{-i\gamma},M)=\mathcal{G}_{\gamma,M}(|\gamma|<\frac{\pi}{2})$ (Kulshrestha [25])

(5) $\lim_{q\to 1^-}\mathcal{F}_{0,q}(1,M)=\mathcal{F}(1,M)$ (Singh and Singh [39])

(6) $\mathcal{F}_{\delta,q}((1-\beta)cos\alpha e^{-i\alpha},M)=\mathcal{F}_{\delta,q}(\alpha,\beta,M);\ (0\le\beta<1,|\alpha|<\frac{\pi}{2})$ if

$$=\left\{f\in\mathcal{A}:\left|\frac{e^{i\alpha}\frac{z((\mathcal{R}_q^{\delta}f(z))'}{\mathcal{R}_q^{\delta}f(z)}-\beta cos\alpha-isin\alpha}{(1-\beta)cos\alpha}-M\right|<M,\quad z\in\mathbb{U}\right\},$$

$\lim_{q\to 1^-}\mathcal{F}_{0,q}(\alpha,\beta,M)=\mathcal{F}(\alpha,\beta,M)$ and $\lim_{q\to 1^-}\mathcal{F}_{1,q}(\alpha,\beta,M)=\mathcal{G}(\alpha,\beta,M)$ (see Aouf[4, 5, 8]).

From the definitions of the classes $\mathcal{F}_{\delta,q}(b,M)$ and $\mathcal{F}(b,M)$ we observe that

$$f\in\mathcal{F}_{\delta,q}(b,M)\Leftrightarrow\mathcal{R}_q^{\delta}f(z)\in\mathcal{F}(b,M).$$

Before we state and prove our main result we need the following definitions and lemmas.

**Definition 1.1.** [41] ***( Subordinating Factor Sequence)***. *A sequence $\{b_n\}_{n=1}^{\infty}$ of complex numbers is called a subordinating factor sequence if, whenever $f$ is analytic , univalent and convex in $\mathbb{U}$, we have the subordination given by*

$$\sum_{n=2}^{\infty}b_na_nz^n\prec f(z)\qquad(z\in\mathbb{U},\ a_1=1).\tag{1.19}$$

**Lemma 1.1.** [41] *The sequence $\{b_n\}_{n=1}^{\infty}$ is a subordinating factor sequence if and only if*

$$\Re\left(1+2\sum_{n=1}^{\infty}b_nz^n\right)>0\qquad(z\in\mathbb{U}).\tag{1.20}$$

Now we prove the following lemma which gives a sufficient condition for function $f\in\mathcal{F}_{\delta,q}(b,M)$

**Lemma 1.2.** *If*

$$\sum_{n=2}^{\infty}[(n-1)+|b(1+m)+m(n-1)|]\Theta_n(q,\delta)\,|a_n|\le|b(1+m)|\tag{1.21}$$

*then , $f\in\mathcal{F}_{\lambda,q}(b,M)$where* $0<q<1,\delta>-1,m=1-\frac{1}{M},(M>\frac{1}{2})$ *and* $b\in\mathbb{C}^*$.

*Proof.* Suppose that the inequality (1.21) holds.Then for $z\in\mathbb{U}$, we have

$$\left|z(\mathcal{R}_q^{\delta}f(z))'-\mathcal{R}_q^{\delta}f(z)\right|-\left|b(1+m)\mathcal{R}_q^{\delta}f(z)+m[z(\mathcal{R}_q^{\delta}f(z))'-\mathcal{R}_q^{\delta}f(z)]\right|.$$

We have

$$\left|\sum_{n=2}^{\infty}(n-1)\Theta_n(q,\delta)a_nz^n\right|$$

$$
\begin{aligned}
& -\left|b(1+m)(z+\sum_{n=2}^{\infty}\Theta_n(q,\delta)a_nz^n)+m\sum_{n=2}^{\infty}(n-1)\Theta_n(q,\delta)a_nz^n\right| \\
\leq\ & \sum_{n=2}^{\infty}(n-1)\Theta_n(q,\delta)|a_n|r^n \\
& -\left\{|b(1+m)|r-\sum_{n=2}^{\infty}|b(1+m)|+m(n-1)|\Theta_n(q,\delta)|a_n|r^n\right\} \\
=\ & \sum_{n=2}^{\infty}[(n-1)+|b(1+m)+m(n-1)|]\Theta_n(q,\delta)|a_n|r^n-|b(1+m)|r.
\end{aligned}
$$

Letting $r\to 1^-$ we have

$$
\begin{aligned}
& \left|z((\mathcal{R}_q^{\delta}f(z))'-\mathcal{R}_q^{\delta}f(z)\right|-\left|b(1+m)\mathcal{R}_q^{\delta}f(z)+m[z(\mathcal{R}_q^{\delta}f(z))'-\mathcal{R}_q^{\delta}f(z)]\right| \\
\leq\ & \sum_{n=2}^{\infty}[(n-1)+|b(1+m)+m(n-1)|]\Theta_n(q,\delta)|a_n|-|b(1+m)| \\
\leq\ & 0,\ by\ (1.21).
\end{aligned}
$$

Hence, it follows that

$$
\left|\frac{\frac{z(\mathcal{R}_q^{\delta}f(z))'}{\mathcal{R}_q^{\delta}f(z)}-1}{b(1+m)+m\frac{z(\mathcal{R}_q^{\delta}f(z))'}{\mathcal{R}_q^{\delta}f(z)}-1}\right|<1, z\in\mathbb{U}.
$$

Letting

$$
w(z)=\frac{\frac{z(\mathcal{R}_q^{\delta}f(z))'}{\mathcal{R}_q^{\delta}f(z)}-1}{b(1+m)+m\frac{z(\mathcal{R}_q^{\delta}f(z))'}{\mathcal{R}_q^{\delta}f(z)}-1},
$$

then $w(0)=0$, $w(z)$ is analytic in $\mathbb{U}$ and $|w(z)|<1$. Hence we have

$$
\frac{z(\mathcal{R}_q^{\delta}f(z))'}{\mathcal{R}_q^{\delta}f(z)}=\frac{1+[b(1+m)-m]w(z)}{1-mw(z)},\quad (z\in\mathbb{U})
$$

which shows that $f\in\mathcal{F}_{\delta,q}(b,M)$.

**Corollary 1.1.** *Let the function $f$ be defined by (1.1) be in the class $\mathcal{F}_{\delta,q}(b,M)$. Then*

$$
|a_n|\leq\frac{|b(1+m)|}{[(n-1)+|b(1+m)+m(n-1)|]\Theta_n(q,\delta)},\quad (n\geq 2).
$$

*The result is sharp for the function*

$$
f_n(z)=z+\frac{|b(1+m)|}{[(n-1)+|b(1+m)+m(n-1)|]\Theta_n(q,\delta)}z^n,\quad (n\geq 2).
$$

*That is*

$$f_2(z) = z + \frac{|b(1+m)|}{[1+|b(1+m)+m|]\Theta_2(q,\delta)} z^2, \quad (n \geq 2).$$

Let $\mathcal{F}^*_{\delta,q}(b,M)$ denote the class of functions $f \in \mathcal{A}$ whose coefficients satisfy the condition (1.21). We note that $\mathcal{F}^*_{\delta,q}(b,M) \subseteq \mathcal{F}_{\delta,q}(b,M)$.

## 2. Main Theorem

Employing the techniques used earlier by Attiya [17],Frasin [20], Singh [38] Srivastava and Attiya [40] and others([7, 13, 14, 15, 20, 21], we obtain subordination relation involving the function classes $\mathcal{F}^*_{\delta,q}(b,M)$, $\mathcal{F}^*_{\delta}(b,M)$, $\mathcal{F}^*(b,M)$, $\mathcal{G}^*(b,M)$ and $\mathcal{F}^*_{\delta,q}(\alpha,\beta,M)$.

**Theorem 2.1.** *Let the function $f$ be defined by (1.1) be in the class $\mathcal{F}^*_{\delta,q}(b,M)$, where $0<q<1, \delta > -1$, $M > \frac{1}{2}$, and $b \in \mathbb{C}^*$ with $\Re(b) > -\frac{m}{2(1+m)}(m>0)$ and $\Re(b) < -\frac{m}{2(1+m)}(m<0)$. Then*

$$\frac{[1+|b(1+m)+m|]\Theta_2(q,\delta)}{2[(1+|b(1+m)+m|)\Theta_2(q,\delta)+|b(1+m)|]}(f*g)(z) \prec g(z) \qquad (z \in \mathbb{U};\ g \in \mathcal{K}), \tag{2.1}$$

*and*

$$\Re(f(z)) > -\frac{[1+|b(1+m)+m|]\Theta_2(q,\delta)+|b(1+m)|}{[1+|b(1+m)+m|]\Theta_2(q,\delta)}, \qquad (z \in \mathbb{U}). \tag{2.2}$$

The constant $\frac{[1+|b(1+m)+m|]\Theta_2(q,\delta)}{2[(1+|b(1+m)+m|)\Theta_2(q,\delta)+|b(1+m)|]}$ is the best estimate.

*Proof.* Let $\mathcal{F}^*_{\delta,q}(b,M)$ and let $g(z) = z + \sum\limits_{n=2}^{\infty} c_n z^n \in \mathcal{K}$. Then

$$\begin{aligned} &\frac{[1+|b(1+m)+m|]\Theta_2(q,\delta)}{2[(1+|b(1+m)+m|)\Theta_2(q,\delta)+|b(1+m)|]}(f*g)(z) \\ =\ &\frac{[1+|b(1+m)+m|]\Theta_2(q,\delta)}{2[(1+|b(1+m)+m|)\Theta_2(q,\delta)+|b(1+m)|]}\left(z + \sum_{n=2}^{\infty} a_n c_n z^n\right). \end{aligned} \tag{2.3}$$

Thus, by Definition 1.1 , the assertion of our theorem will hold if the sequence

$$\left\{\frac{[1+|b(1+m)+m|]\Theta_2(q,\delta)}{2[(1+|b(1+m)+m|)\Theta_2(q,\delta)+|b(1+m)|]} a_n\right\}_{n=1}^{\infty} \tag{2.4}$$

is a subordinating factor sequence, with $a_1 = 1$. In view of Lemma 1.1, this will be the case if and only if

$$\Re\left(1 + \sum_{n=1}^{\infty} \frac{[1+|b(1+m)+m|]\Theta_2(q,\delta)}{(1+|b(1+m)+m|)\Theta_2(q,\delta)+|b(1+m)|} a_n z^n\right) > 0 \ (z \in \mathbb{U}). \tag{2.5}$$

Now because $\{[(n-1)+|b(1+m)+m(n-1)|]\Theta_n(q,\delta)\}$ $(n \geq 2, \delta > -1, 0 < q < 1)$ is increasing function of $n (n \geq 2)$ we have

$$\begin{aligned}
&\Re\left(1+\sum_{n=1}^{\infty}\frac{[1+|b(1+m)+m|]\Theta_2(q,\delta)}{(1+|b(1+m)+m|)\Theta_2(q,\delta)+|b(1+m)|}a_n z^n\right)\\
=\ &\Re\left(1+\frac{[1+|b(1+m)+m|]\Theta_2(q,\delta)}{(1+|b(1+m)+m|)\Theta_2(q,\delta)+|b(1+m)|}z\right.\\
&\qquad\left.+\frac{\sum\limits_{n=2}^{\infty}[1+|b(1+m)+m|]\Theta_2(q,\delta)a_n z^n}{(1+|b(1+m)+m|)\Theta_2(q,\delta)+|b(1+m)|}\right)\\
\geq\ &\left(1-\frac{[1+|b(1+m)+m|]\Theta_2(q,\delta)}{(1+|b(1+m)+m|)\Theta_2(q,\delta)+|b(1+m)|}r\right.\\
&\left.-\frac{\sum\limits_{n=1}^{\infty}[(n-1)+|b(1+m)+m(n-1)|]\Theta_n(q,\delta)a_n r^n}{(1+|b(1+m)+m|)\Theta_2(q,\delta)+|b(1+m)|}\right)\\
\geq\ &\left(1-\frac{[1+|b(1+m)+m|]\Theta_2(q,\delta)}{(1+|b(1+m)+m|)\Theta_2(q,\delta)+|b(1+m)|}r\right.\\
&\qquad\left.-\frac{|b(1+m)|]}{(1+|b(1+m)+m|)\Theta_2(q,\delta)+|b(1+m)|}r\right)=1-r>0 \qquad (|z|=r).
\end{aligned}$$

Thus (2.5) holds true in $\mathbb{U}$. This proves the inequality (2.1). The inequality (2.2) follows by taking the convex function $g(z)=\frac{z}{1-z}=z+\sum\limits_{n=2}^{\infty}z^n$ in (2.1). To prove the sharpness of the constant $\frac{[1+|b(1+m)+m|]\Theta_2(q,\delta)}{2[(1+|b(1+m)+m|)\Theta_2(q,\delta)+|b(1+m)|]}$ we consider the function $f_0$ given by

$$f_0(z)=z-\frac{|b(1+m)|}{[1+|b(1+m)+m|]\Theta_2(q,\delta)+|b(1+m)|}z^2, (z\in\mathbb{U})$$

which is a member of the class $\mathcal{F}^*_{\delta,q}(b,M)$. Thus from (2.1), we have

$$\frac{[1+|b(1+m)+m|]\Theta_2(q,\delta)}{2[(1+|b(1+m)+m|)\Theta_2(q,\delta)+|b(1+m)|]}f_0(z)\prec\frac{z}{1-z}. \tag{2.6}$$

It can easily verified that

$$\min_{|z|\leq r}\left\{\Re\left(\frac{[1+|b(1+m)+m|]\Theta_2(q,\delta)}{2[(1+|b(1+m)+m|)\Theta_2(q,\delta)+|b(1+m)|]}f_0(z)\right)\right\}=-\frac{1}{2} \qquad (z\in\mathbb{U}). \tag{2.7}$$

This shows that the constant $\frac{[1+|b(1+m)+m|]\Theta_2(q,\delta)}{2[(1+|b(1+m)+m|)\Theta_2(q,\delta)+|b(1+m)|]}$ cannot be replaced by a larger one, which completes the proof .

Letting $q\to 1^-$ in Theorem 2.1, we obtain the following corollary.

**Corollary 2.1.** *Let the function $f$ be defined by (1.1) be in the class $\mathcal{F}_{\delta}^{*}(b,M)$, where $M>\frac{1}{2}$, and $b\in\mathbb{C}^{*}$ with $\Re(b)>-\frac{m}{2(1+m)}(m>0)$; $\Re(b)<-\frac{m}{2(1+m)}(m<0)$ and satisfy the condition*

$$\sum_{n=2}^{\infty}[(n-1)+|b(1+m)+m(n-1)|]\Theta_{n}(1,\delta)\,|a_{n}|\leq|b(1+m)|$$

*where $\Theta_2(1,\delta)=\frac{\Gamma_1(2+\delta)}{[2-1]_1!\,\Gamma_1(1+\delta)}=1+\delta$.Then*

$$\frac{[1+|b(1+m)+m|](1+\delta)}{2[(1+|b(1+m)+m1|)(1+\delta)+|b(1+m)|]}(f*g)(z)\prec g(z)\qquad(z\in\mathbb{U};\ g\in\mathcal{K}),\tag{2.8}$$

*and*

$$\Re(f(z))>-\frac{[1+|b(1+m)+m|](1+\delta)+|b(1+m)|}{[1+|b(1+m)+m|](1+\delta)},\qquad(z\in\mathbb{U}).\tag{2.9}$$

*The constant $\frac{[1+|b(1+m)+m|](1+\delta)}{2[(1+|b(1+m)+m1|)(1+\delta)+|b(1+m)|]}$ is the best estimate.*

Letting $q\to1^{-}$ and taking $\delta=0$ in Theorem 2.1, we obtain the following corollary.

**Corollary 2.2.** *Let the function $f$ be defined by (1.1) be in the class $\mathcal{F}^{*}(b,M)$, where $M>\frac{1}{2}$, and $b\in\mathbb{C}^{*}$ with $\Re(b)>-\frac{m}{2(1+m)}(m>0)$; $\Re(b)<-\frac{m}{2(1+m)}(m<0)$ and satisfy the condition*

$$\sum_{n=2}^{\infty}[(n-1)+|b(1+m)+m(n-1)|]\,|a_{n}|\leq|b(1+m)|.$$

*Then*

$$\frac{[1+|b(1+m)+m|]}{(1+|b(1+m)+m1|)+|b(1+m)|}(f*g)(z)\prec g(z)\qquad(z\in\mathbb{U};\ g\in\mathcal{K}),\tag{2.10}$$

*and*

$$\Re(f(z))>-\frac{[1+|b(1+m)+m|]+|b(1+m)|}{1+|b(1+m)+m|},\qquad(z\in\mathbb{U}).\tag{2.11}$$

*The constant $\frac{[1+|b(1+m)+m|]}{(1+|b(1+m)+m1|)+|b(1+m)|}$ is the best estimate.*

Letting $q\to1^{-}$ and taking $\delta=1$ in Theorem 2.1, we obtain the following corollary.

**Corollary 2.3.** *Let the function $f$ be defined by (1.1) be in the class $\mathcal{G}^{*}(b,M)$, where $M>\frac{1}{2}$, and $b\in\mathbb{C}^{*}$ with $\Re(b)>-\frac{m}{2(1+m)}(m>0)$; $\Re(b)<-\frac{m}{2(1+m)}(m<0)$ and satisfy the condition*

$$\sum_{n=2}^{\infty}n[(n-1)+|b(1+m)+m(n-1)|]\,|a_{n}|\leq|b(1+m)|.$$

*Then*

$$\frac{[1+|b(1+m)+m|]}{2[2(1+|b(1+m)+m1|)+|b(1+m)|]}(f*g)(z)\prec g(z)\qquad(z\in\mathbb{U};\ g\in\mathcal{K}),\tag{2.12}$$

*and*

$$\Re(f(z)) > -\frac{2[1+|b(1+m)+m|]+|b(1+m)|}{1+|b(1+m)+m|}, \qquad (z \in \mathbb{U}). \tag{2.13}$$

*The constant* $\frac{[1+|b(1+m)+m|]}{2[2(1+|b(1+m)+m1|)+|b(1+m)|]}$ *is the best estimate.*

If we put $b = (1-\beta)\cos\alpha e^{-i\alpha}(0 \leq \beta < 1, |\alpha| < \frac{\pi}{2})$, in Theorem 2.1, we obtain the next two result.

**Corollary 2.4.** *Let the function $f$ be defined by (1.1) be in the class $\mathcal{F}_{\delta}^{*}(\alpha,\beta,M)$ and satisfy the condition*

$$\sum_{n=2}^{\infty}[(n-1)+|(1+m)(1-\beta)\cos\alpha e^{-i\alpha}+m(n-1)|]\Theta_n(q,\delta)\,|a_n| \leq |1+m|(1-\beta)\cos\alpha,$$

*where* $0<q<1; M>\frac{1}{2}$,*with* $(1-\beta)\cos^2\alpha > -\frac{m}{2(1+m)}(m>0)$ *and* $(1-\beta)\cos^2\alpha < -\frac{m}{2(1+m)}(m<0)$.*Then*

$$\frac{[1+|(1+m)(1-\beta)\cos\alpha e^{-i\alpha}+m|]\Theta_2(q,\delta)}{2[(1+|(1+m)(1-\beta)\cos\alpha e^{-i\alpha}+m|)\Theta_2(q,\delta)+|1+m|(1-\beta)\cos\alpha]}(f*g)(z) \prec g(z) \tag{2.14}$$

$(z \in \mathbb{U};\ g \in \mathcal{K})$ *and*

$$\Re(f(z)) > -\frac{[1+|(1+m)(1-\beta)\cos\alpha e^{-i\alpha}+m|]\Theta_2(q,\delta)+|(1+m)(1-\beta)\cos\alpha e^{-i\alpha}|}{[1+|(1+m)(1-\beta)\cos\alpha e^{-i\alpha}+m|]\Theta_2(q,\delta)}, (z \in \mathbb{U}). \tag{2.15}$$

*The constant* $\frac{[1+|(1+m)(1-\beta)\cos\alpha e^{-i\alpha}+m|]\Theta_2(q,\delta)}{2[(1+|(1+m)(1-\beta)\cos\alpha e^{-i\alpha}+m|)\Theta_q(q,\delta)+|1+m|(1-\beta)\cos\alpha]}$ *is the best estimate.*

## 3. Integral Means Inequalities

**Lemma 3.1.** [27] *If the functions $f$ and $g$ are analytic in $\Delta$ with $g \prec f$, then for $\eta > 0$, and $0 < r < 1$,*

$$\int_0^{2\pi}\left|g(re^{i\theta})\right|^{\eta}d\theta \leq \int_0^{2\pi}\left|f(re^{i\theta})\right|^{\eta}d\theta. \tag{3.1}$$

In [35], Silverman found that the function $f_2(z) = z - \frac{z^2}{2}$ is often extremal over the family $\mathcal{T}$denote the subset of $\mathcal{A}$ comprising of functions

$$f(z) = z - \sum_{n=2}^{\infty}|a_n|z^n \qquad (z \in \mathbb{U}) \tag{3.2}$$

and applied this function to resolve his integral means inequality, conjectured in [36] and settled in [37], that

$$\int_0^{2\pi}\left|f(re^{i\theta})\right|^{\eta}d\theta \leq \int_0^{2\pi}\left|f_2(re^{i\theta})\right|^{\eta}d\theta,$$

for all $f \in \mathcal{T}$, $\eta > 0$ and $0 < r < 1$. In [37], Silverman also proved his conjecture for the subclasses of starlike and convex functions of order $\alpha(0 \leq \alpha < 1)$.

Applying Lemma 3.1 and Lemma 1.2 , we prove the following result.

**Theorem 3.1.** *Suppose* $f \in \mathcal{F}^*_{\delta,q}(b, M)$, $\eta > 0$, *and* $f_2(z)$ *is defined by*

$$f_2(z) = z - \frac{|b(1+m)|}{[1+|b(1+m)+m|]\Theta_2(q,\delta)} z^2.$$

*Then for* $z = re^{i\theta}$, $0 < r < 1$, *we have*

$$\int_0^{2\pi} |f(z)|^\eta \, d\theta \leq \int_0^{2\pi} |f_2(z)|^\eta \, d\theta. \tag{3.3}$$

*Proof.* For $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$, (3.3) is equivalent to proving that

$$\int_0^{2\pi} \left| 1 - \sum_{n=2}^{\infty} |a_n| z^{n-1} \right|^\eta d\theta \leq \int_0^{2\pi} \left| 1 - \frac{|b(1+m)|}{[1+|b(1+m)+m|]\Theta_2(q,\delta)} z \right|^\eta d\theta.$$

By Lemma 3.1, it suffices to show that

$$1 - \sum_{n=2}^{\infty} |a_n| z^{n-1} \prec 1 - \frac{|b(1+m)|}{[1+|b(1+m)+m|]\Theta_2(q,\delta)} z.$$

Setting

$$1 - \sum_{n=2}^{\infty} |a_n| z^{n-1} = 1 - \frac{|b(1+m)| w(z)}{[(n-1)+|b(1+m)+m(n-1)|]\Theta_n(q,\delta)}, \tag{3.4}$$

and using (1.21), we obtain $w(z)$ is analytic in $\Delta$, $w(0) = 0$, and

$$\begin{aligned} |w(z)| &= \left| \sum_{n=2}^{\infty} \frac{[1+|b(1+m)+m|]\Theta_2(q,\delta)}{|b(1+m)|} |a_n| z^{n-1} \right| \\ &\leq |z| \sum_{n=2}^{\infty} \frac{[(n-1)+|b(1+m)+m(n-1)|]\Theta_n(q,\delta)}{|b(1+m)|} |a_n| \\ &\leq |z|. \end{aligned}$$

This completes the proof of Theorem 3.1.

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