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T-Normed Fuzzy TM-Subalgebra of TM-Algebras

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ABSTRACT

The concept of T-normed fuzzy TM-subalgebras is introduced by applying the notion of t-norm to fuzzy TM-algebra and its properties are investigated. The ideas based on minimum t-norm are generalized to all widely accepted t-norms in a fuzzy TM-subalgebra.The characteristics of an idempotent T-normed fuzzy TM-subalgebra are studied. The properties of image and the inverse image of a T-normed fuzzy TM-subalgebra under homomorphism is discussed. The T-direct product and T-product of T-normed fuzzy TM-subalgebras are also considered.

1. INTRODUCTION

Triangular norms (abbreviation t-norms) were first appeared in the background of statistical metric spaces, introduced by K. Menger [1] and studied later by Schweizer and Sklar [2,3]. Klement *et al.* [4–6] conducted a systematic study on the related properties of t-norms. The concept of fuzzy sets were introduced by Zadeh [7]. Rosenfeld [8] applied this concept to group theory and introduced fuzzy subgroups leading to the fuzzification of different algebraic structures. Alsina *et al.* [9,10] and Prade [11] suggested to use a t-norm for fuzzy intersection and its t-conorm for fuzzy union, following some attempts of Hohle [12] in introducing t-norms into the area of fuzzy logics. This was extended by combining the notions of fuzzy sets and t-norm to different algebraic structures such as group [13–17], BCK-algebra [18], BCC-algebra [19], B-algebra [20], KU-algebra [21,22], BG-algebra [23], and so on, and defined different types of product of fuzzy substructures on them.

TM-algebra is a class of logical algebra based on propositional calculus, introduced by Megalai and Tamilarasi [24]. They have investigated several characterizations of it and relation between TM-algebras and other algebras. They [25] applied the concept of fuzzy set to TM-algebra and studied the properties of the newly obtained algebraic structure called fuzzy TM-algebra. Some operations on fuzzy TM-subalgebra were discussed and fuzzy ideals were also defined. Several fuzzy substructures in TM-algebras were considered by many researchers (see [26–28]).

Speaking in terms of t-norm, fuzzy TM-subalgebra was actually defined using the concept of minimum t-norm. Hence we generalize this concept by taking an arbitrary t-norm. The whole paper is arranged as follows: Relevant definitions and theorems needed in sequel are included in Section 2. In Section 3, we introduced the notion of T-normed fuzzy TM-subalgebra with suitable examples and the characteristics are studied. An idempotent T-normed fuzzy TM-subalgebra is defined depending on which whether the image set of the membership function becomes a subset of the subsemigroup of idempotents of the semigroup ([0, 1], T) or not and its properties are studied. The properties of image and the inverse image of a T-normed fuzzy TM-subalgebra under homomorphism are investigated. In Section 4, some properties of the T-product and T-direct product of T-normed fuzzy TM-subalgebras and the relationship between them are also considered. The conclusion and a comparison with the existing results are given in the last section.

2. PRELIMINARIES

We recall some definitions and results that will be required in the sections that follow:

Definition 1. [24] A TM-algebra is a triple $(X, *, \theta)$, where $X\,(\neq \phi)$ is a set with a fixed element θ and $*$ is a binary operation such that the conditions

i. $x * \theta = x$

ii. $(x * y) * (x * z) = z * y$

hold for all $x, y, z \in X$.

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A nonempty subset S of a TM-algebra X is called a TM-subalgebra of X if $x * y \in S$ for all $x, y \in S$.

Definition 2. [29] Let $(X_1, *_1, \theta_1)$ and $(X_2, *_2, \theta_2)$ be two TM-algebras. The direct product $X = X_1 \times X_2$ is also a TM-algebra with the binary operation $*$ defined as $(x_1, x_2) * (y_1, y_2) = (x_1 *_1 y_1, x_2 *_2 y_2)$ for all $(x_1, x_2), (y_1, y_2) \in X_1 \times X_2$ and $\theta = (\theta_1, \theta_2)$.

Definition 3. [7] A fuzzy set A in a set X is a pair (X, μ_A), where the function $\mu_A: X \to [0,1]$ is called the membership function of A. For $\alpha \in [0,1]$, the set $U(\mu_A; \alpha) := \{x \in X | \mu_A(x) \geqslant \alpha\}$ is called an upper level set of A.

Definition 4. [7] Let $A = (X, \mu_A)$ and $B = (Y, \eta_B)$ are fuzzy sets in X and Y, respectively, and f is a mapping defined from X into Y. Then $f(A)$ is a fuzzy set in $f(X)$, where $\mu_{f(A)}$ is defined by

$$f(\mu_A)(y) = \begin{cases} sup\{\mu_A(x) | x \in f^{-1}(y) \neq \phi\} \\ 0 \text{ if } f^{-1}(y) = \phi \end{cases}$$

for all $y \in f(X)$ and is called the image of A under f. A is said to have *sup* property if, for every subset $P \subseteq X$, there exists $p_0 \in P$ such that $\mu_A(p_0) = sup\{\mu_A(p) | p \in P$. The inverse image $f^{-1}(B)$ in X is also a fuzzy set of X, where $\eta_{f^{-1}(B)}$ is defined by $f^{-1}(\eta_B)(x) = \eta_B(f(x))$ for all $x \in X$ is also a fuzzy set of X.

When X is taken as a TM-algebra, then we have the following definition:

Definition 5. [25] A fuzzy set $A = (X, \mu_A)$ of a TM-algebra X is called a fuzzy TM-subalgebra of X if $\mu_A(x * y) \geqslant min\{\mu_A(x), \mu_A(y)\}$, for all $x, y \in X$.

Theorem 1. *[25] Let $f : X \to Y$ be a homomorphism from a TM-algebra X onto a TM-algebra Y. If $A = (X, \mu_A)$ is a fuzzy TM-subalgebra of X, then the image $f(A) = (Y, f(\mu_A))$ of A under f is a fuzzy TM-subalgebra of Y.*

Now we recall some preliminary ideas on t-norm.

Definition 6. [5] A t-norm is a function $T : [0,1] \times [0,1] \to [0,1]$ that satisfies

i. $T(x, 1) = x$
ii. $T(x, y) = T(y, x)$
iii. $T(x, T(y, z)) = T(T(x, y), z)$
iv. $T(x, y) \leqslant T(x, z)$ whenever $y \leqslant z$, for all $x, y, z \in [0,1]$.

A t-norm T on $[0,1]$ is called a continuous t-norm if T is a continuous function from $[0,1] \times [0,1]$ to $[0,1]$ with respect to the usual topology.

Some examples of t-norm are the following:

i. Lukasiewicz t-norm $T_{Luk}(x, y) = max\{x + y - 1, 0\}$ for all $x, y \in [0,1]$.
ii. Minimum t-norm $T_{min}(x, y) = min(x, y)$ for all $x, y \in [0,1]$.
iii. Product t-norm $T_P(x, y) = x \cdot y$ for all $x, y \in [0,1]$.
iv. Drastic t-norm $T_D(x, y) = \begin{cases} y & \text{if } x = 1 \\ x & \text{if } y = 1 \\ 0 & \text{otherwise} \end{cases}$ for all $x, y \in [0,1]$.

Some useful properties of a t-norm T used in the sequel are the following:

i. $T(x, 0) = 0$ for all x in $[0,1]$.
ii. $T_D(x, y) \leqslant T(x, y) \leqslant T_{min}(x, y)$ for any t-norm T and all x, y in $[0,1]$.
iii. $T(T(x, y), T(z, t)) = T(T(x, z), T(y, t)) = T(T(x, t), T(y, z))$ for all x, y, z and t in $[0,1]$.

Definition 7. Let T be a t-norm. Denote by E_T the set of all idempotents with respect to T, that is, $E_T = \{x \in [0,1] | T(x, x) = x\}$. A fuzzy set A in X is called an idempotent T-normed fuzzy set if $Im(\mu_A) \subseteq E_T$.

Definition 8. [16] A t-norm T_1 dominates a t-norm T_2, or equivalently, T_2 is dominated by T_1, and write $T_1 >> T_2$ if $T_1(T_2(x, y), T_2(a, b)) \geqslant T_2(T_1(x, a), T_1(y, b))$ for all x, y, a, $b \in [0,1]$.

We can extend these concepts by generalizing the domain of t-norm to $\prod_{i=1}^{n} [0,1]$ to define the function t_n-norm.

Definition 9. [16] The function $T_n : \prod_{i=1}^{n} 0, 1 \to [0,1]$ is defined by $T_n(x_1, x_2, \cdots, x_n) = T(x_i, T_{n-1}(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n))$ for all $1 \leqslant i \leqslant n$, where $n \geqslant 2$, $T_2 = T$ and $T_1 = i_d$ (identity).

For a t-norm T and every $x_i, y_i \in [0,1]$, where $1 \leqslant i \leqslant n$ and $n \geqslant 2$, we have $T_n(T(x_1, y_1), T(x_2, y_2), \cdots, T(x_n, y_n)) = T(T_n(x_1, x_2, \cdots, x_n), T_n(y_1, y_2, \cdots, y_n))$.

3. T-NORMED FUZZY TM-SUBALGEBRA OF A TM-ALGEBRA

We first apply the notion of t-norm to obtain a new fuzzy substructure called T-normed fuzzy TM-subalgebra in a TM-algebra.

Definition 10. Let $(X, *, \theta)$ be a TM-algebra and $A = (X, \mu_A)$ be a fuzzy set in X. Then the set A is a T-normed fuzzy TM-subalgebra over the binary operation $*$ if it satisfies $\mu_A(x * y) \geqslant T\{\mu_A(x), \mu_A(y)\}$ for all $x, y \in X$.

Example 1. Define a fuzzy set A in the TM-algebra X given in Table 1, by $\mu_A(\theta) = 0.5$, $\mu_A(a) = 0.3$, $\mu_A(b) = 0.7$, and $\mu_A(c) = 0.6$. Then A is a T_{Luk}-normed fuzzy TM-subalgebra of X.

Definition 11. A T-normed fuzzy TM-subalgebra A is called an idempotent T-normed fuzzy TM-subalgebra of X if $Im(\mu_A) \subseteq E_T$.

Table 1 Cayley Table

$*$	θ	a	b	c
θ	θ	a	c	b
a	a	θ	b	c
b	b	c	θ	a
c	c	b	a	θ

Example 2. Consider a TM-algebra $X = \{\theta, a, b, c\}$ defined in Table 1. Define a fuzzy set A in X by $\mu_A(x) = 0$, if $x \in \{\theta, a\}$ and $\mu_A(x) = 0.6$, if $x \in \{b, c\}$. Consider a t-norm T_k defined in [30] by

$$T_k(x, y) = \begin{cases} min\{x, y\} & \text{if } max\{x, y\} = 1 \\ 0 & \text{if } max\{x, y\} < 1, x + y \leqslant 1 + k \\ k & \text{otherwise} \end{cases}$$

for all $x, y \in [0, 1]$. Take $k = 0.6$. It is easy to check that $\mu_A(x * y) \geqslant T_k\{\mu_A(x), \mu_A(y)\}$ for all $x, y \in X$. Also $Im(\mu_A) \subseteq E_{T_k}$. Hence A is an idempotent T_k-normed fuzzy TM-subalgebra of X when $k = 0.6$.

Proposition 2. *If A is an idempotent T-normed fuzzy TM-subalgebra of TM-algebra X, then we have the following results for all $x \in X$:*

i. $\mu_A(\theta) \geqslant \mu_A(x)$

ii. $\mu_A(\theta * x) \geqslant \mu_A(x)$

iii. *If there exists a sequence x_n in X such that $\lim_{n\to\infty} \mu_A(x_n) = 1$ then $\mu_A(\theta) = 1$*

Proof. Let $x \in X$.

i. Then by using the two conditions in Definition 1, we get $\mu_A(\theta) = \mu_A(\theta * \theta) = \mu_A((x * \theta) * (x * \theta)) = \mu_A(x * x) \geqslant T\{\mu_A(x), \mu_A(x)\} = \mu_A(x)$.

ii. $\mu_A(\theta * x) \geqslant T\{\mu_A(\theta), \mu_A(x)\} = T\{\mu_A(x * x), \mu_A(x)\} \geqslant T\{T\{\mu_A(x), \mu_A(x)\}, \mu_A(x)\} = \mu_A(x)$ since it is idempotent.

iii. By (i), $\mu_A(\theta) \geqslant \mu_A(x)$ for all $x \in X$, therefore $\mu_A(\theta) \geqslant \mu_A(x_n)$ for every positive integer n. Consider, $1 \geqslant \mu_A(\theta) \geqslant \lim_{n\to\infty} \mu_A(x_n) = 1$. Hence, $\mu_A(\theta) = 1$.

Theorem 3. *Let A_1 and A_2 be two T-normed fuzzy TM-subalgebras of X. Then $A_1 \cap A_2$ is a T-normed fuzzy TM-subalgebra of X.*

Proof. Let $x, y \in A_1 \cap A_2$. Then $x, y \in A_1$ and A_2. Now,

$$\begin{aligned} &\mu_{A_1 \cap A_2}(x * y) \\ &= min\{\mu_{A_1}(x * y), \mu_{A_2}(x * y)\} \\ &\geqslant min\{T\{\mu_{A_1}(x), \mu_{A_1}(y)\}, T\{\mu_{A_2}(x), \mu_{A_2}(y)\}\} \\ &\geqslant T\{min\{\mu_{A_1}(x), \mu_{A_2}(x)\}, min\{\mu_{A_1}(y), \mu_{A_2}(y)\}\} \\ &= T\{\mu_{A_1 \cap A_2}(x), \mu_{A_1 \cap A_2}(y)\}. \end{aligned}$$

Hence, $A_1 \cap A_2$ is a T-normed fuzzy TM-subalgebra of X.

This can be generalized to obtain the following theorem:

Theorem 4. *Let $\{A_i | i \in I\}$ be a family of T-normed fuzzy TM-subalgebras of a TM-algebra X. Then $\cap_{i\in I} A_i$ is also a T-normed fuzzy TM-subalgebra of X, where $\cap_{i\in I} A_i = \{< x, \inf_{i\in I} \mu_{A_i}(x) >: x \in X\}$.*

Proof. For any $x, y \in X$, we have $\mu_{A_i}(x) \geqslant \inf_{i\in I} \mu_{A_i}(x)$ and $\mu_{A_i}(y) \geqslant \inf_{i\in I} \mu_{A_i}(y)$. Hence for every $i \in I$, $T(\mu_{A_i}(x), \mu_{A_i}(y)) \geqslant T(\inf_{i\in I} \mu_{A_i}(x), \inf_{i\in I} \mu_{A_i}(y))$, and so $\inf_{i\in I} T(\mu_{A_i}(x), \mu_{A_i}(y)) \geqslant T(\inf_{i\in I} \mu_{A_i}(x), \inf_{i\in I} \mu_{A_i}(y))$. It follows that

$$\begin{aligned} \mu_{\cap_{i\in I} A_i}(x * y) &= \inf_{i\in I} \mu_{A_i}(x * y) \\ &\geqslant \inf_{i\in I} T(\mu_{A_i}(x), \mu_{A_i}(y)) \\ &\geqslant T(\inf_{i\in I} \mu_{A_i}(x), \inf_{i\in I} \mu_{A_i}(y)) \\ &= T(\mu_{\cap_{i\in I} A_i}(x), \mu_{\cap_{i\in I} A_i}(y)). \end{aligned}$$

This completes the proof.

Theorem 5. *Let T be a t-norm and let A be a fuzzy set in a TM-algebra X with $Im(\mu_A) = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$, where $\alpha_i < \alpha_j$ whenever $i > j$. Suppose that there exists an ascending chain of subalgebras $S_0 \subset S_1 \subset \cdots \subset S_n = X$ of X such that $\mu_A(\widetilde{S_k}) = \alpha_k$, where $\widetilde{S_k} = S_k \backslash S_{k-1}$ for $k = 1, \cdots, n$ and $\widetilde{S_0} = S_0$. Then A is a T-normed fuzzy TM-subalgebra of X.*

Proof. Let $x, y \in X$. If x and y belong to the same $\widetilde{S_k}$, then $\mu_A(x) = \mu_A(y) = \alpha_k$ and $x * y \in S_k$. Hence $\mu_A(x * y) \geqslant \alpha_k = min\{\mu_A(x), \mu_A(y)\} \geqslant T(\mu_A(x), \mu_A(y))$. Assume that $x \in \widetilde{S_i}$ and $y \in \widetilde{S_j}$ for every $i \neq j$. Without loss of generality we may assume that $i > j$. Then $\mu_A(x) = \alpha_i < \alpha_j = \mu_A(y)$ and $x * y \in G_i$. It follows that $\mu_A(x * y) \geqslant \alpha_i = min\{\mu_A(x), \mu_A(y)\} \geqslant T(\mu_A(x), \mu_A(y))$. Consequently, A is a T-normed fuzzy TM-subalgebra of X.

Theorem 6. *Let A be an idempotent T-normed fuzzy TM-subalgebra of X, then the set $I_{\mu_A} = \{x \in X | \mu_A(x) = \mu_A(\theta)\}$ is a TM-subalgebra of X.*

Proof. Let $x, y \in I_{\mu_A}$. Then $\mu_A(x) = \mu_A(\theta) = \mu_A(y)$ and so, $\mu_A(x * y) \geqslant T\{\mu_A(x), \mu_A(y)\} = T\{\mu_A(\theta), \mu_A(\theta)\} = \mu_A(\theta)$. By using Proposition 2, we know that $\mu_A(x * y) \leqslant \mu_A(\theta)$. Hence $\mu_A(x * y) = \mu_A(\theta)$ or equivalently $x * y \in I_{\mu_A}$. Therefore, the set I_{μ_A} is TM-subalgebra of X.

Theorem 7. *If A is a TM-subalgebra of X, then the characteristic function χ_A is a T-normed fuzzy TM-subalgebra of X.*

Proof. Let $x, y \in X$. We consider here three cases:

Case (i). If $x, y \in A$, then $x * y \in A$ since A is a TM-subalgebra of X. Then $\chi_A(x * y) = 1 \geqslant T\{\chi_A(x), \chi_A(y)\}$.

Case (ii). If $x, y \notin A$, then $\chi_A(x) = 0 = \chi_A(y)$. Thus $\chi_A(x * y) \geqslant 0 = min\{0, 0\} = T\{0, 0\} = T\{\chi_A(x), \chi_A(y)\}$.

Case (iii). If $x \in A$ and $y \notin A$ (or $x \notin A$ and $y \in A$), then $\chi_A(x) = 1, \chi_A(y) = 0$. Thus $\chi_A(x * y) \geqslant 0 = T\{0, 1\} = T\{1, 0\} = T\{\chi_A(x), \chi_A(y)\}$.

Therefore, the characteristic function χ_A is a T-normed fuzzy TM-subalgebra of X.

Theorem 8. *Let A be a non-empty subset of X. If χ_A satisfies $\chi_A(x * y) \geqslant T\{\chi_A(x), \chi_A(y)\}$, then A is a TM-subalgebra of X.*

Proof. Let $x, y \in A$. Then $\chi_A(x * y) \geqslant T\{\chi_A(x), \chi_A(y)\} = T\{1, 1\} = 1$ so that $\chi_A(x * y) = 1$, i.e., $x * y \in A$. Hence, A is a TM-subalgebra of X.

Proposition 9. *Let Y be a TM-subalgebra of X and A be a fuzzy set in X defined by*

$$\mu_A(x) = \begin{cases} \lambda, & if\, x \in Y \\ \tau, & otherwise \end{cases}$$

for all $\lambda, \tau \in [0,1]$ with $\lambda \geqslant \tau$. Then A is a T_{Luk}-normed fuzzy subalgebra of X. In particular if $\lambda = 1$ and $\tau = 0$ then A is an idempotent T_{Luk}-normed fuzzy subalgebra of X. Moreover, $I_{\mu_A} = Y$.

Proof. Let $x, y \in X$. We consider here three cases:

Case (i). If $x, y \in Y$, then

$$\begin{aligned} T_{Luk}\left(\mu_A(x), \mu_A(y)\right) &= T_{Luk}(\lambda, \lambda) \\ &= max(2\lambda - 1, 0) \\ &= \begin{cases} 2\lambda - 1 & \text{if } \lambda \geqslant \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \\ &\leqslant \lambda \\ &= \mu_A(x * y) \end{aligned}$$

Case (ii). If $x \in Y$ and $y \notin Y$ (or, $x \notin Y$ and $y \in Y$), then

$$\begin{aligned} T_{Luk}\left(\mu_A(x), \mu_A(y)\right) &= T_{Luk}(\lambda, \tau) \\ &= max(\lambda + \tau - 1, 0) \\ &= \begin{cases} \lambda + \tau - 1 & \text{if } \lambda + \tau \geqslant 1 \\ 0 & \text{otherwise} \end{cases} \\ &\leqslant \tau \\ &= \mu_A(x * y) \end{aligned}$$

Case (iii). If $x, y \notin Y$, then

$$\begin{aligned} T_{Luk}\left(\mu_A(x), \mu_A(y)\right) &= T_{Luk}(\tau, \tau) \\ &= max(2\tau - 1, 0) \\ &= \begin{cases} 2\tau - 1 & \text{if } \tau \geqslant \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \\ &\leqslant \tau \\ &= \mu_A(x * y). \end{aligned}$$

Hence, A is an T_{Luk}-normed fuzzy TM-subalgebra of X.

Assume that $\lambda = 1$ and $\tau = 0$. Then $T_{Luk}(\lambda, \lambda) = max(\lambda + \lambda - 1, 0) = 1 = \lambda$ and $T_{Luk}(\tau, \tau) = max(\tau + \tau - 1, 0) = 0 = \tau$. Thus $\lambda, \tau \in E_{T_{Luk}}$, that is, $Im(\mu_A) \subseteq E_{T_{Luk}}$. So, A is an idempotent T_{Luk}-normed fuzzy TM-subalgebra of X.

Also,

$$I_{\mu_A} = \{x \in X | \mu_A(x) = \mu_A(\theta)\} = \{x \in X, \mu_A(x) = \lambda\} = Y.$$

Therefore, $I_{\mu_A} = Y$.

Theorem 10. *Let A be a T-normed fuzzy TM-subalgebra of X and $\alpha \in [0,1]$. Then if $\alpha = 1$, the upper level set $U(\mu_A; \alpha)$ is either empty or a TM-subalgebra of X.*

Proof. Let $\alpha = 1$. Suppose $U(\mu_A; \alpha)$ is not empty and let $x, y \in U(\mu_A; \alpha)$. Then $\mu_A(x) \geqslant \alpha = 1$ and $\mu_A(y) \geqslant \alpha = 1$. It follows that $\mu_A(x * y) \geqslant T\left(\mu_A(x), \mu_A(y)\right) \geqslant T(1,1) = 1$ so that $x * y \in U(\mu_A; \alpha)$. Hence, $U(\mu_A; \alpha)$ is a TM-subalgebra of X when $\alpha = 1$.

Theorem 11. *If A is an idempotent T-normed fuzzy TM-subalgebra of X, then the upper level set $U(\mu_A; \alpha)$ of A is a TM-subalgebra of X.*

Proof. Assume that $x, y \in U(\mu_A; \alpha)$. Then $\mu_A(x) \geqslant \alpha$ and $\mu_A(y) \geqslant \alpha$. It follows that $\mu_A(x * y) \geqslant T\{\mu_A(x), \mu_A(y)\} \geqslant T(\alpha, \alpha) = \alpha$ so that $x * y \in U(\mu_A; \alpha)$. Hence, $U(\mu_A; \alpha)$ is a TM-subalgebra of X.

Theorem 12. *Let A be a fuzzy set in X such that the set $U(\mu_A; \alpha)$ is a TM-subalgebra of X for every $\alpha \in [0,1]$. Then A is a T-normed fuzzy TM-subalgebra of X.*

Proof. Let for every $\alpha \in [0,1]$, $U(\mu_A; \alpha)$ is a TM-subalgebra of X. In contrary, let $x_0, y_0 \in X$ be such that $\mu_A(x_0 * y_0) < T\{\mu_A(x_0), \mu_A(y_0)\}$. Let us consider, $\alpha_0 = \frac{1}{2}\left[\mu_A(x_0 * y_0) + T\{\mu_A(x_0), \mu_A(y_0)\}\right]$. Then $\mu_A(x_0 * y_0) < \alpha_0 \leqslant T\{\mu_A(x_0), \mu_A(y_0)\} \leqslant T_{min}\{\mu_A(x_0), \mu_A(y_0)\}$ and so $x_0 * y_0 \notin U(\mu_A; \alpha_0)$ but $x_0, y_0 \in U(\mu_A; \alpha_0)$. This is a contradiction and hence μ_A satisfies the inequality $\mu_A(x * y) \geqslant T\{\mu_A(x), \mu_A(y)\}$ for all $x, y \in X$.

Theorem 13. *Let $f : X \to Y$ be a homomorphism of TM-algebras $(X, *, \theta)$ onto $(Y, *_1, \theta_1)$. If $B = (Y, \mu_B)$ is a T-normed fuzzy TM-subalgebra of Y, then the pre-image $f^{-1}(B) = \left(X, f^{-1}(\mu_B)\right)$ of B under f is a T-normed fuzzy TM-subalgebra of X.*

Proof. Assume that B is a T-normed fuzzy TM-subalgebra of Y and let $x, y \in X$. Then

$$\begin{aligned} f^{-1}(\mu_B)(x * y) &= \mu_B\left(f(x * y)\right) \\ &= \mu_B\left(f(x) *_1 f(y)\right) \\ &\geqslant T\{\mu_B(f(x)), \mu_B(f(y)))\} \\ &= T\left\{f^{-1}(\mu_B)(x), f^{-1}(\mu_B)(y)\right\}. \end{aligned}$$

Therefore, $f^{-1}(B)$ is a T-normed fuzzy TM-subalgebra of X.

Theorem 14. *Let T be a continuous t-norm and let f be an epimorphism of TM-algebras $(X, *, \theta)$ onto $(Y, *_1, \theta_1)$. If A is a T-normed fuzzy TM-subalgebra of X, then $f(A)$ is a T-normed fuzzy TM-subalgebra of Y.*

Proof. Let $y_1, y_2 \in Y$. Take $A_1 = f^{-1}(y_1), A_2 = f^{-1}(y_2)$ and $A_3 = f^{-1}(y_1 *_1 y_2)$.

Consider the set

$A_1 * A_2 = \{x \in X | x = a_1 * a_2$ for some $a_1 \in A_1$ and $a_2 \in A_2\}$.

If $x \in A_1 * A_2$, then $x = x_1 * x_2$ for some $x_1 \in A_1$ and $x_2 \in A_2$ and so $f(x) = f(x_1 * x_2) = f(x_1) *_1 f(x_2) = y_1 *_1 y_2$, that is, $x \in f^{-1}(y_1 *_1 y_2) = A_3$. Thus $A_1 * A_2 \subseteq A3$. It follows that

$$\begin{aligned} \mu_{f(A)}(y_1 *_1 y_2) &= \sup_{x \in f^{-1}(y_1 *_1 y_2)} \mu_A(x) \\ &= \sup_{x \in A_3} \mu_A(x) \\ &\geqslant \sup_{x \in A_1 * A_2} \mu_A(x) \\ &\geqslant \sup_{x_1 \in A_1, x_2 \in A_2} \mu_A(x_1 * x_2) \\ &\geqslant \sup_{x_1 \in A_1, x_2 \in A_2} T(\mu_A(x_1), \mu_A(x_2)). \end{aligned}$$

Since T is continuous, for every $\varepsilon > 0$ there exists a number $\delta > 0$ such that if

$\sup_{x_1 \in A_1} \mu_A(x_1) - x_1^* \leqslant \delta$ and $\sup_{x_2 \in A_2} \mu_A(x_2) - x_2^* \leqslant \delta$, then

$$T\left(\sup_{x_1 \in A_1} \mu_A(x_1), \sup_{x_2 \in A_2} \mu_A(x_2)\right) - T\left(x_1^*, x_2^*\right) \leqslant \varepsilon.$$

Choose $a_1 \in A_1$ and $a_2 \in A_2$ such that

$\sup_{x_1 \in A_1} \mu_A(x_1) - \mu_A(a_1) \leqslant \delta$ and $\sup_{x_2 \in A_2} \mu_A(x_2) - \mu_A(a_2) \leqslant \delta$.

Then
$T\left(\sup_{x_1\in A_1}\mu_A(x_1), \sup_{x_2\in A_2}\mu_A(x_2)\right) - T(\mu_A(a_1), \mu_A(a_2)) \leqslant \varepsilon.$

Consequently

$$\begin{aligned}\mu_{f(A)}(y_1 *_1 y_2) &\geqslant \sup_{x_1\in A_1, x_2\in A_2} T(\mu_A(x_1), \mu_A(x_2))\\ &\geqslant T\left(\sup_{x_1\in A_1}\mu_A(x_1), \sup_{x_2\in A_2}\mu_A(x_2)\right)\\ &= T\left(\mu_{f(A)}(y_1), \mu_{f(A)}(y_2)\right)\end{aligned}$$

which shows that $f(A)$ is a T-normed fuzzy TM-subalgebra of Y.

Theorem 15. *Let $f : X \to Y$ be an epimorphism from a TM-algebra X onto a TM-algebra Y. If A is an idempotent T-normed fuzzy TM-subalgebra of X, then the image $f(A)$ of A under f is a T-normed fuzzy TM-subalgebra of Y.*

Proof. Let A be an idempotent T-normed fuzzy TM-subalgebra of X. By Theorem 11, $U(\mu_A;\alpha)$ is TM-subalgebra of X for every $\alpha \in [0,1]$. Therefore by Theorem 1, $f(U(\mu_A;\alpha))$ is a TM-subalgebra of Y. But $f(U(\mu_A;\alpha)) = U(f(\mu_A);\alpha)$. Hence $U(f(\mu_A);\alpha)$ is a TM-subalgebra of X for every $\alpha \in [0,1]$. By Theorem 12, $f(A)$ is a T-normed fuzzy TM-subalgebra of Y.

4. PRODUCT OF T-NORMED FUZZY TM-SUBALGEBRAS

We will define a concept called T-product in TM-algebra using a t-norm T, analogue to the pointwise product of functions.

Definition 12. Let $A_1 = (X, \mu_{A_1})$ and $A_2 = (X, \mu_{A_2})$ be two fuzzy sets of a TM-algebra X and T be a t-norm. Then the T-product of A_1 and A_2 denoted by $[A_1 \cdot A_2]_T = \left(X, \mu_{[A_1\cdot A_2]_T}\right)$ and is defined by $\mu_{[A_1\cdot A_2]_T}(x) = T\left(\mu_{A_1}(x), \mu_{A_2}(x)\right)$ for all $x \in X$. Also $\mu_{[A_1\cdot A_2]_T} = \mu_{[A_2\cdot A_1]_T}$.

Theorem 16. *Let A_1 and A_2 be two T-normed fuzzy TM-subalgebras of X. If T^* is a t-norm such that $T^* >> T$, then the T^*-product of A_1 and A_2, $[A_1 \cdot A_2]_{T^*}$ is a T-normed fuzzy TM-subalgebra of X.*

Proof. For any $x, y \in X$, we have

$$\begin{aligned}\mu_{[A_1\cdot A_2]_{T^*}}(x*y) &= T^*\left(\mu_{A_1}(x*y), \mu_{A_2}(x*y)\right)\\ &\geqslant T^*\left(T\left(\mu_{A_1}(x), \mu_{A_1}(y)\right), T\left(\mu_{A_2}(x), \mu_{A_2}(y)\right)\right)\\ &\geqslant T\left(T^*\left(\mu_{A_1}(x), \mu_{A_2}(x)\right), T^*\left(\mu_{A_1}(y), \mu_{A_2}(y)\right)\right)\\ &= T\left(\mu_{[A_1\cdot A_2]_{T^*}}(x), \mu_{[A_1\cdot A_2]_{T^*}}(y)\right).\end{aligned}$$

Hence, $[A_1 \cdot A_2]_{T^*}$ is a T-normed fuzzy TM-subalgebra of X.

Corollary 17. *Let $f : X \to Y$ be an epimorphism of TM-algebras. Let T and T^* be t-norms such that $T^* >> T$. If A_1 and A_2 be two T-normed fuzzy TM-subalgebras of Y, then the pre-images $f^{-1}(A_1), f^{-1}(A_2)$ and $f^{-1}\left([A_1 \cdot A_2]_{T^*}\right)$ are T-normed fuzzy TM-subalgebras of X.*

Proof. Since every epimorphic pre-image of a T-normed fuzzy TM-subalgebra is again a T-normed fuzzy TM-subalgebra, their T^*-product is also T-normed fuzzy TM-subalgebra by the previous theorem.

The relationship of $f^{-1}\left(\mu_{[A_1\cdot A_2]_{T^*}}\right)$ with the T^*-product of $f^{-1}(\mu_{A_1})$ and $f^{-1}(\mu_{A_2})$ can be viewed by the following theorem:

Theorem 18. *Let $f : X \to Y$ be an epimorphism of TM-algebras. Let T and T^* be t-norms such that $T^* >> T$. Let A_1 and A_2 be two T-normed fuzzy TM-subalgebra of Y. If $[A_1 \cdot A_2]_{T^*}$ is the T^*-product of A_1 and A_2 and $\left[f^{-1}(A_1) . f^{-1}(A_2)\right]_{T^*}$ is the T^*-product of $f^{-1}(A_1)$ and $f^{-1}(A_2)$, then*

$$f^{-1}\left(\mu_{[A_1\cdot A_2]_{T^*}}\right) = \left[f^{-1}(\mu_{A_1}) \cdot f^{-1}(\mu_{A_2})\right]_{T^*}.$$

Proof. For any $x \in X$ we get,

$$\begin{aligned}f^{-1}\left(\mu_{[A_1\cdot A_2]_{T^*}}\right)(x) &= \mu_{[A_1\cdot A_2]_{T^*}}(f(x))\\ &= T^*\left(\mu_{A_1}(f(x)), \mu_{A_2}(f(x))\right)\\ &= T^*\left(f^{-1}(\mu_{A_1})(x), f^{-1}(\mu_{A_2})(x)\right)\\ &= \left[f^{-1}(\mu_{A_1}) \cdot f^{-1}(\mu_{A_2})\right]_{T^*}(x).\end{aligned}$$

Hence the proof.

Remark 1. Now let us consider about the image of T-product of T-normed fuzzy TM-subalgebras.

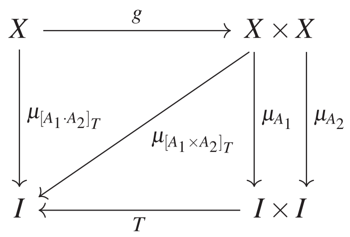
Let $f : X \to Y$ be an epimorphism of TM-algebras. Let T and T^* be t-norms such that $T^* >> T$, where T is a continuous t-norm. If A_1 and A_2 be two T-normed fuzzy TM-subalgebras of X, then the images $f(A_1), f(A_2), f\left([A_1 \cdot A_2]_{T^*}\right)$, and $\left[f(A_1) \cdot f(A_2)\right]_{T^*}$ are T-normed fuzzy TM-subalgebras of Y by Theorems 14 and 16.

Theorem 19. *Let T and T^* be t-norms such that $T^* \gg T$, where T is a continuous t-norm. Let A_1 and A_2 be two T-normed fuzzy TM-subalgebras of a TM-algebra X and $f : X \to Y$ be an epimorphism of TM-algebras. Then $\mu_{f\left([A_1\cdot A_2]_{T^*}\right)} \subset \mu_{[f(A_1)\cdot f(A_2)]_{T^*}}$.*

Proof. For each y in Y,

$$\begin{aligned}\mu_{f\left([A_1\cdot A_2]_{T^*}\right)}(y) &= f\left(\mu_{[A_1\cdot A_2]_{T^*}}\right)(y)\\ &= \sup_{x\in f^{-1}(y)} \mu_{[A_1\cdot A_2]_{T^*}}(x)\\ &= \sup_{x\in f^{-1}(y)} T^*\left(\mu_{A_1}(x), \mu_{A_2}(x)\right)\\ &\leqslant \left(\sup_{x\in f^{-1}(y)}\mu_{A_1}(x), \sup_{x\in f^{-1}(y)}\mu_{A_2}(x)\right)\\ &= T^*\left(\mu_{f(A_1)}(y), \mu_{f(A_2)}(y)\right)\\ &= \mu_{[f(A_1)\cdot f(A_2)]_{T^*}}(y).\end{aligned}$$

Next we consider the T^*-direct product of two T-normed fuzzy TM-subalgebras.



Definition 13. Let $A_1 = \left(X_1, \mu_{A_1}\right)$ and $A_2 = \left(X_2, \mu_{A_2}\right)$ be two T-normed fuzzy TM-subalgebras of TM-algebras X_1 and X_2, respectively, and T^* be a t-norm. Then the T^*-direct product of A_1 and A_2 denoted by $[A_1 \times A_2]_{T^*} = \left(X_1 \times X_2, \mu_{[A_1 \times A_2]_{T^*}}\right)$ and is defined by $\mu_{[A_1 \times A_2]_{T^*}}(x_1, x_2) = T^*\left(\mu_{A_1}(x_1), \mu_{A_2}(x_2)\right)$ for all $(x_1, x_2) \in X_1 \times X_2$.

Remark 2. Let X be a TM-algebra and $I = [0, 1]$. Define a map $g : X \times X \to X$ by $g(x) = (x, x)$ for all $x \in X$. We can see the relationship between T-product and the T-direct product of two T-normed fuzzy TM-subalgebras in the following diagram:

Clearly, $[A_1 \cdot A_2]_T$ is the preimage of $[A_1 \times A_2]_T$ under the map g.

Theorem 20. *Let $X = X_1 \times X_2$ be the direct product of TM-algebras X_1 and X_2. If $A_1 = \left(X, \mu_{A_1}\right)$ and $A_2 = \left(X, \mu_{A_2}\right)$ be two T- normed fuzzy TM-subalgebras of X_1 and X_2, respectively, then $A = (X, \mu_A)$ is a T-normed fuzzy TM-subalgebra of X defined by $\mu_A(x_1, x_2) = \mu_{[A_1 \times A_2]_T}(x_1, x_2) = T\left(\mu_{A_1}(x_1), \mu_{A_2}(x_2)\right)$ for all $(x_1, x_2) \in X_1 \times X_2$.*

Proof. Let $x = (x_1, x_2)$ and $y = \left(y_1, y_2\right)$ be any elements of X. We have

$$\begin{aligned}
\mu_A\left(x * y\right) &= \mu_A\left((x_1, x_2) * \left(y_1, y_2\right)\right) \\
&= \mu_A\left(x_1 * y_1, x_2 * y_2\right) \\
&= \mu_{[A_1 \times A_2]_T}\left(x_1 * y_1, x_2 * y_2\right) \\
&= T\left(\mu_{A_1}\left(x_1 * y_1\right), \mu_{A_2}\left(x_2 * y_2\right)\right) \\
&\geqslant T\left(T\left(\mu_{A_1}(x_1), \mu_{A_1}\left(y_1\right)\right), T\left(\mu_{A_2}(x_2), \mu_{A_2}\left(y_2\right)\right)\right) \\
&= T\left(T\left(\mu_{A_1}(x_1), \mu_{A_2}(x_2)\right), T\left(\mu_{A_1}\left(y_1\right), \mu_{A_2}\left(y_2\right)\right)\right) \\
&= T\left(\left(\mu_{[A_1 \times A_2]_T}\right)(x_1, x_2), \left(\mu_{[A_1 \times A_2]_T}\right)\left(y_1, y_2\right)\right) \\
&= T\left(\mu_A(x), \mu_A\left(y\right)\right).
\end{aligned}$$

Hence, $A = (X, \mu_A)$ is a T-normed fuzzy TM-subalgebra of X.

We can generalize previous theorem to the product of n T-normed fuzzy TM-algebras using the function T_n defined in Definition 9.

Theorem 21. *Let T be a t-norm and let $\{X_i\}_{i=1}^{n}$ be the finite collection of TM-algebras and $X = \prod_{i=1}^{n} X_i$ the direct product TM-algebras of $\{X_i\}$. Let A_i be a T-normed fuzzy TM-subalgebra of X_i, where $1 \leqslant i \leqslant n$. Then $A = \prod_{i=1}^{n} A_i$ defined by $\mu_A(x_1, x_2, \cdots, x_n) = \mu_{\left[\prod_{i=1}^{n} A_i\right]_T}(x_1, x_2, \cdots, x_n) = T_n\left(\mu_{A_1}(x_1), \mu_{A_2}(x_2), \cdots, \mu_{A_n}(x_n)\right)$ is a T-normed fuzzy TM-subalgebra of the TM-algebra X.*

Proof. Let $x = (x_1, x_2, \cdots, x_n)$ and $y = \left(y_1, y_2, \cdots, y_n\right)$ be any elements of $X = \prod_{i=1}^{n} X_i$. Then,

$$\begin{aligned}
\mu_A\left(x * y\right) &= \mu_A\left(x_1 * y_1, x_2 * y_2, \cdots, x_n * y_n\right) \\
&= T_n\left(\mu_{A_1}\left(x_1 * y_1\right), \mu_{A_2}\left(x_2 * y_2\right), \cdots, \mu_{A_n}\left(x_n * y_n\right)\right) \\
&\geqslant T_n\left(T\left(\mu_{A_1}(x_1), \mu_{A_1}\left(y_1\right)\right), T\left(\mu_{A_2}(x_2), \mu_{A_2}\left(y_2\right)\right),\right. \\
&\qquad \left.\cdots, T\left(\mu_{A_n}(x_n), \mu_{A_n}\left(y_n\right)\right)\right) \\
&= T\left(T_n\left(\mu_{A_1}(x_1), \mu_{A_2}(x_2), \cdots, \mu_{A_n}(x_n)\right),\right. \\
&\qquad \left.T_n\left(\mu_{A_1}\left(y_1\right), \mu_{A_2}\left(y_2\right), \cdots, \mu_{A_n}\left(y_n\right)\right)\right) \\
&= T\left(\mu_A(x_1, x_2, \cdots, x_n), \mu_A\left(y_1, y_2, \cdots, y_n\right)\right) \\
&= T\left(\mu_A(x), \mu_A\left(y\right)\right).
\end{aligned}$$

Hence A is a T-normed fuzzy TM-subalgebra of X.

5. CONCLUSION

The previous works related to fuzzy TM-subalgebra relied on the conventional min/max t-norm/t-conorm dual combinations. But the literature on t-norms suggests that there exists other widely accepted t-norms. In this article, we put forth a new notion of T-normed fuzzy TM-subalgebra of TM-algebra by generalizing the concept of fuzzy TM-subalgebra (defined using minimum t-norm) introduced in [25]. We observed that our generalized concept satisfy most of the various theorems stated in the previous related works. The theorem (Theorem 14 of [25]) which is stated as "A is a fuzzy TM-subalgebra of a TM-algebra X if and only if its level set $U(\mu_A; \alpha)$ is either empty or a TM-subalgebra for all $\alpha \in [0, 1]$," is found to be different in our generalized case. Theorems 11 and 12 in this article shows that this may not hold in the case of a T-normed fuzzy TM-subalgebra in general, but the level set can be a T-normed fuzzy TM-subalgebra when the corresponding fuzzy set A is an idempotent T-normed fuzzy TM-subalgebra. The converse part of the theorem always holds.

Moreover, we studied the properties of image and the inverse image of a T-normed fuzzy TM-subalgebra under a homomorphism. The relationship between the T-direct product and T-product of T-normed fuzzy TM-subalgebras is also obtained. In this paper, we focused on the t-norms and so this can be extended by exploring the analogous observations for the t-conorms in a fuzzy TM-algebra based on the duality between these operators.

CONFLICT OF INTEREST

All authors have no conflict of interest to report.

AUTHORS' CONTRIBUTIONS

All authors contributed equally in developing the concepts, analysing of the results and to the writing of the manuscript.

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