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The b-chromatic number of some special families of graphs

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Abstract. Given G, $b-$coloring is a proper k coloring of G in which each and every color class has at least one $b-$vertex that has a neighbour in other $k-1$ color classes. The largest integer k is the $b-$chromatic number $b(G)$ for which G having a $b-$coloring using k colors. In this paper, we constructed some family of graphs and found its $b-$chromatic number.

1. Introduction

All graphs we consider are simple, finite and undirected graphs. Let $G=(V,E)$ be a graph. Then the set of vertices denoted by $V(G)$ with order n and set of edges denoted by $E(G)$ with size m. A proper vertex $k-$coloring of G is a nonempty partition $P=\{V_1,V_2,...,V_k\}$ produce a color class, each V_i is an independent set of G. The minimum integer k is the chromatic number $\chi(G)$ for which G has a $k-$colorable. A $b-$coloring is a proper $k-$coloring in which each and every color class V_i contains at least one vertex that has a neighbour in other $k-1$ color classes. A vertex which is satisfying the above property is called a $b-$vertex. A set of all vertices in S_0 are $b-$vertices is called a $b-$system such that every $b-$vertex belongs to different color classes. The largest integer k is the $b-$chromatic number $b(G)$ for which G having a $b-$coloring using k colors. First Irving and Manlove [3] introduced the concept of $b-$chromatic number and also they derived the upper bound, $b(G)\leq\Delta(G)+1$. In particular, they remark that, G having a $b-$chromatic coloring using k colors and in G should have at least k vertices having a degree $k-1$. Effantin and Kheddouci discussed the $b-$chromatic number of some power graphs [2]. On $b-$coloring of regular graphs studied by Blidia, Maffray and Zoham [1]. The $b-$chromatic number of some path related graphs discussed by Vaidya and Rakhimol [5] also they investigated the $b-$chromatic number of the degree splitting graphs of the path, shell and gear graph in [4]. In general, the corona of any two graphs G and H denoted by $G\odot H$. Vernold Vivin and Venkatachalam [7] have found the $b-$chromatic number of corona product of any graph G with path, cycle and complete graph also Vivin et al[6] investigated the $b-$chromatic number of star graph families.

2. Main Results

In the main section, we describe few particular families of graphs and obtained its $b-$chromatic number.

2.1. Definition
Let $H = K_{1,n-3}$ be a star graph on $n-2$ vertices and let $V(K_{1,n-3}) = \{u_1, u_2, ..., u_{n-3}, c\}$, where c is the central vertex of H. The graph F_1 is constructed from C_n by adding a copy of graph H to every vertex v_i of C_n. Clearly the order of F_1 is $n + n(n-3)$.
The following family of graphs $\{F_1^0, F_1^1, F_1^2, ..., F_1^k\}$ are constructed from F_1 such that $F_1^i, i = 0,1,2,...,k$ is obtained by adding i number of edges to every copy of H.
$F_1 = F_1^0 = C_n \odot K_{1,n-3}$
$F_1^1 = C_n \odot \left(K_{1,n-3} + \{e_1\}\right)$
$F_1^2 = C_n \odot \left(K_{1,n-3} + \{e_1, e_2\}\right)$
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$F_1^k = C_n \odot \left(K_{1,n-3} + \{e_1, e_2, ..., e_k\}\right), 1 \le k \le \frac{(n-4)(n-3)}{2}$
Let $\mathscr{F}(C_n) = \{F_1^0, F_1^1, F_1^2, ..., F_1^k\}$ be denote the family of graphs and the order of every graph in $\mathscr{F}(C_n)$ is $n + n(n-3)$.

2.2. Theorem
For any graph of $\mathscr{F}(C_n)$, the $b-$ chromatic number is n.
Proof
Let $F_1 \in \mathscr{F}(C_n)$ and let $V(F_1) = \left\{v_i, u_i^j, 1 \le i \le n, 1 \le j \le n-3\right\}$. The order of F_1 is $n + n(n-3)$. Suppose we assume the $b-$ chromatic number of F_1 is greater than or equal to n that is $b(F_1) \ge n$. Therefore, we have the existence of a $b-$ system S_0 such that $|S_0| \ge n+1$. This means that, in F_1 having $b-$ system S_0 and that $b-$ system contains $n+1$ vertices of degree at least n. But here F_1 having only n vertices of degree $n-1$ and the remaining vertices are of degree at most $n-3$, which contradicts our assumption and hence $b(F_1) \le n$.
Now we define the following mapping $C : V(F_1) \to \{1,2,3,...,n\}$ to vertices as follows.

$$C(v_i) = i \qquad 1 \le i \le n,$$

$$C(u_i^j) = \begin{cases} i+j+1 & i = 1,2,\ 1 \le j \le n-3 \\ i+j+1 & 3 \le i \le n-1,\ 1 \le j \le n-(i+1) \\ j & 3 \le i \le n-1,\ 1 \le j \le i-2 \\ j & i = n,\ 2 \le j \le i-2 \end{cases}$$

Thus we get a proper $b-$coloring of C. Therefore $b(F_1) \ge n$ and hence $b(F_1) = n$.

2.3. Definition
Let $H = K_{1,2}$ be a star graph with 3 vertices and let $V(K_{1,2}) = \{u_1, u_2, c\}$, where c is the central vertex of H. The graph F_2 is constructed from $\overline{C_n}$ by adding a copy of graph H to every vertex v_i of $\overline{C_n}$. Clearly the order of F_2 is $n + 2n = 3n$.

The following family of graphs $\{F_2^0, F_2^1\}$ are constructed from F_2 such that $F_2^i, i = 0,1,2...,k$ is obtained by adding i number of edges to every copy of H .

$F_2 = F_2^0 = \overline{C_n} \odot K_{1,2}$

$F_2^1 = \overline{C_n} \odot (K_{1,2} + \{e_1\})$

Let $\mathscr{F}(\overline{C_n}) = \{F_2^0, F_2^1\}$ be denote the family of graphs and the order of every graph in $\mathscr{F}(\overline{C_n})$ is $3n$.

2.4. Theorem

For any graph of $\mathscr{F}(\overline{C_n})$, $n \geq 5$ the $b-$chromatic number is n.

Proof

Let $F_2 \in \mathscr{F}(\overline{C_n})$, $n \geq 5$ and let $V(F_2) = \left\{v_i, u_i^j, 1 \leq i \leq n, j = 1,2\right\}$. The order of F_2 is $3n$. Suppose we assume the $b-$chromatic number of F_2 is greater than or equal to n that is $b(F_2) \geq n$. Therefore, we have the existence of a $b-$system S_0 such that $|S_0| \geq n+1$. This means that, in F_2 having $b-$system S_0 and that $b-$system contains $n+1$ vertices of degree at least n. But here F_2 having only n vertices of degree $n-1$ and the remaining vertices are of degree at most 2, which contradicts our assumption and hence $b(F_2) \leq n$.

Now we define the following mapping $C : V(F_2) \rightarrow \{1,2,3,...,n\}$ to vertices as follows.

$$C(v_i) = i \quad 1 \leq i \leq n$$

$$C(u_i^1) = \begin{cases} n & i = 1 \\ i-1 & i \geq 2 \end{cases}$$

$$C(u_i^2) = \begin{cases} i+1 & 1 \leq i \leq n-1 \\ 1 & i = n \end{cases}$$

Thus we get a proper $b-$coloring of C. Therefore $b(F_2) \geq n$ and hence $b(F_2) = n.$

2.5. Definition

Let $H_1 = K_{1,m-1}$, $H_2 = K_{1,n-1}$ and let $V(K_{1,m-1}) = \{u_1, u_2, ..., u_{m-1}, c\}$, $V(K_{1,n-1}) = \{v_1, v_2, ..., v_{n-1}, c'\}$ where c, c' are central vertex of H_1 and H_2 . Let $K_{m,n}$, $m < n$ be a complete bipartite graph with bipartitions V_1 and V_2 .The graph F_3 is constructed from $K_{m,n}$, $m < n$ by adding m copy of graph H_1 to every vertex $v_i \in V_1(K_{m,n})$ and n copy of graph H_2 to every vertex of $v_i \in V_2(K_{m,n})$. Clearly the order of F_3 is $(m+n) + m(m-1) + n(n-1)$.

The following family of graphs $\{F_3^0, F_3^1, F_3^2, ..., F_3^k\}$ is constructed from F_3 such that $F_3^i, i = 0,1,2,...,k$ is obtained by adding i number of edges to every copy of H_1 and H_2 .

$F_3 = F_3^0 = K_{m,n} \odot (H_1, H_2)$

$F_3^1 = K_{m,n} \odot (H_1 + \{e_1\}, H_2 + \{e_1\})$

$F_3^2 = K_{m,n} \odot (H_1 + \{e_1, e_2\}, H_2 + \{e_1, e_2\})$

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$F_3^k = K_{m,n} \odot (H_1 + \{e_1, e_2, ..., e_k\}, H_2 + \{e_1, e_2, ..., e_l\}), 1 \le k \le \frac{(m-1)(m-2)}{2}, 1 \le l \le \frac{(n-1)(n-2)}{2}$

Let $\mathscr{F}(K_{m,n}) = \{F_3^0, F_3^1, F_3^2, ..., F_3^k\}$ be denote the family of graphs and the order of every graph in $\mathscr{F}(K_{m,n})$ is $(m+n) + m(m-1) + n(n-1)$.

2.6. Theorem
For any graph of $\mathscr{F}(K_{m,n})$, the $b-$chromatic number is $m+n$.
Proof
Let $F_3 \in \mathscr{F}(K_{m,n})$ and let $V(F_3) = \{V_1 \cup V_2 \cup V_3\}$ where $V_1 = \{v_1, v_2, ..., v_m\}$, $V_2 = \{v_{m+1}, v_{m+2}, ..., v_{m+n}\}$ and $V_3 = \left\{\begin{matrix} u_i^j, & 1 \le i \le m, 1 \le j \le m-1 \\ v_i^j, & m+1 \le i \le m+n, 1 \le j \le n-1 \end{matrix}\right\}$. The order of F_3 is $(m+n) + m(m-1) + n(n-1)$.
Suppose we assume the $b-$chromatic number of F_3 is greater than or equal to $m+n$ that is $b(F_2) \ge m+n$. Therefore, we have the existence of a $b-$system S_0 such that $|S_0| \ge m+n+1$. This means that, in F_3 having $b-$system S_0 and that $b-$system contains $m+n+1$ vertices of degree at least $m+n$. But here F_3 having only $m+n$ vertices of degree $m+n-1$ and the remaining vertices are of degree at most $m-1$ in H_1 and $n-1$ in H_2, which contradicts our assumption and hence $b(F_3) \le m+n$.
Now we define the following mapping $C: V(F_3) \to \{1, 2, 3, ..., (m+n)\}$ to vertices as follows,

$$C(v_i) = i \quad 1 \le i \le m+n$$

$$C(u_i^j) = \begin{cases} m & i = j \quad 1 \le i \le m \\ j & i \ne j \quad 1 \le j \le m-1 \end{cases}$$

$$C(v_{m+i}^j) = \begin{cases} m+n & i = j \quad 1 \le i \le n \\ m+j & i \ne j \quad 1 \le j \le n-1 \end{cases}$$

Thus we get a proper $b-$coloring of C. Therefore $b(F_3) \ge m+n$ and hence $b(F_3) = m+n$.

2.7. Definition
Let $H = K_{1,n-4}$ be a star graph on $n-3$ vertices and $V(K_{1,n-4}) = \{u_1, u_2, ..., u_{n-4}, c\}$ where c is the central vertex of H. Let $W_n, n \ge 4$ be the wheel graph with $V(W_n) = \{v_1, v_2, v_3 ..., v_n\}, v_1$ is central vertex. The graph F_4 is constructed from W_n by adding a copy of graph H to every vertex $v_i (2 \le i \le n)$ of W_n. Clearly the order of F_4 is $n + (n-1)(n-4)$.
The following family of graphs $\{F_4^0, F_4^1, F_4^2, ..., F_4^k\}$ is constructed from F_4 such that $F_4^i, i = 0, 1, 2, ..., k$ is obtained by adding i number of edges to every copy of H.

$F_4 = F_4^0 = W_n \odot K_{1,n-4}$

$F_4^1 = W_n \odot \left(K_{1,n-4} + \{e_1\}\right)$

$F_4^2 = W_n \odot \left(K_{1,n-4} + \{e_1, e_2\}\right)$

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$F_4^k = C_n \odot \left(K_{1,n-4} + \{e_1, e_2, ..., e_k\}\right), 1 \le k \le \frac{(n-4)(n-5)}{2}$

Let $\mathscr{F}(W_n) = \{F_4^0, F_4^1, F_4^2, ..., F_4^k\}$ be denote the family of graphs and the order of every graph in $\mathscr{F}(W_n)$ is $n+(n-1)(n-4)$.

2.8. Theorem

For any graph of $F_4 \in \mathscr{F}(W_n)$, the $b-$chromatic number is n.

Proof

Let $F_4 \in \mathscr{F}(W_n)$ and let $V(F_4) = \{v_1 \cup v_i \cup u_i^j, 2 \le i \le n, 1 \le j \le n-4\}$.The order of F_4 is $n+(n-1)(n-4)$. Suppose we assume the $b-$chromatic number of F_4 is greater than or equal to n that is $b(F_4) \ge n$. Therefore, we have the existence of a $b-$system S_0 such that $|S_0| \ge n+1$. This means that, in F_4 having $b-$system S_0 and that $b-$system contains $n+1$ vertices of degree at least n. But here F_4 having only n vertices of degree $n-1$ and the remaining vertices are of degree at most $n-4$, which contradicts our assumption and hence $b(F_4) \le n$.

Now we define the following mapping $C: V(F_4) \to \{1,2,3,...,n\}$ to vertices as follows,

$$C(v_i) = 1 \qquad i = 1$$
$$C(v_i) = i \qquad 2 \le i \le n$$
$$C(u_i^j) = \begin{cases} i+j+1 & i = 2,3,\ 1 \le j \le n-3 \\ i+j+1 & 4 \le i \le n-1,\ 1 \le j \le n-(i+1) \\ j & 4 \le i \le n-1,\ 2 \le j \le i-2 \\ j & i = n,\quad 3 \le j \le i-2 \end{cases}$$

Thus we get a proper $b-$coloring of C. Therefore $b(F_4) \ge n$ and hence $b(F_4) = n$

3. Conclusion

In this paper, we defined some particular family of graphs such as $\mathscr{F}(C_n)$, $\mathscr{F}(\overline{C_n})$, $\mathscr{F}(K_{m,n})$ $\mathscr{F}(W_n)$ and obtained its $b-$chromatic number. The $b-$chromatic numbers of central, middle and total graphs of above family of graphs are still open.

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