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TOTAL EDGE IRREGULARITY STRENGTH OF SERIES PARALLEL GRAPHS

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Abstract: Given a graph $G(V, E)$ a labeling $\partial : V \cup E \to \{1, 2, ..., k\}$ is called an e*dge irregular total k-labeling* if for every pair of distinct edges uv and xy, $\partial(u) + \partial(uv) + \partial(v) \neq \partial(x) + \partial(xy) + \partial(y)$. The minimum k for which G has an edge irregular total k-labeling is called the total edge irregularity strength. In this paper we consider series composition of uniform theta graphs and obtain its total edge irregularity strength.

We have determined the exact value of the total edge irregularity strength of this graph. We have further given an algorithm to prove the result.

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1. Introduction

A basic feature for a system is that its components are connected together by

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physical communication links to transmit information according to some pattern. Moreover, it is undoubted that the power of a system is highly dependent upon the connection pattern of components in the system. A connection pattern of the components in a system is called an *interconnection network*, or *network*, of the system.

Interconnection networks are becoming increasingly pervasive in many different applications with the operational costs and characteristics of these networks depending considerably on the application. For some applications, interconnection networks have been studied in depth for decades. This is the case for telephone networks, computer networks (telecommunication) and backplane buses. However in the last fifteen years we have seen rapid evolution of the interconnection network technology that is currently being infused into a new generation of multiprocessor systems.

Some interconnection network topologies are designed and some borrow from nature. For example hypercubes, complete binary trees, butterflies and torus networks are some of the *designed* architectures. Grids, hexagonal networks, honeycomb networks and diamond networks, for instance, bear resemblance to atomic or molecular lattice structures. They are called *natural* architectures.

The advancement of large scale integrated circuit technology has enabled the construction of complex interconnection networks. Graph theory provide a fundamental tool for designing and analyzing such networks. Graph Theory and Interconnection Networks provides a thorough understanding of these interrelated topics. One of the main objectives of researchers is the application of Graph Theory to the study and design of interconnection networks. The problems usually considered include the analysis of characteristic parameters of the network (diameter, connectivity measures, etc.), the study of special substructures (rings, trees, etc), routing algorithms, modularity properties and specific networks (symmetric networks, permutation networks, loop networks, etc).

Graph labelings, have often been *motivated* by *practical considerations* such as *coding*, *X-ray crystallography*, *radar tracking*, *remote control*, *radio astronomy*, *communication networks*, *network flows* etc.. Their *theoretical applications* too are numerous, not only within the theory of graphs but also in other areas of mathematics such as *combinatorial number theory*, *linear algebra* and *group theory* admitting a given type of labeling [7]. They are also of interest on their own right due to their abstract mathematical properties arising from various structural considerations of the underlying graphs. An enormous body of literature has grown around the theme. For a dynamic survey of various graph labelings along with an extensive bibliography, one may refer to Gallian

[7].

Motivated by the notion of the *irregularity strength* and *irregular assignments* of a graph introduced by Chartrand et al. (refer [4]) in 1988 and various kinds of other total labelings, the *total edge irregularity strength* of a graph was introduced by Bača, Jendrol, Miller and Ryan [1] as follows: For a graph $G(V, E)$ a labeling $\partial : V \cup E \to \{1, 2, ..., k\}$ is called an *edge irregular total k-labeling* if for every pair of distinct edges uv and xy, $\partial(u) + \partial(uv) + \partial(v) \neq \partial(x) + \partial(xy) + \partial(y)$. Similarly, ∂ is called an *vertex irregular total k-labeling* if for every pair of distinct vertices u and v, $\partial(u) + \sum \partial(e)$ over all edges e incident to $u \neq \partial(v) + \sum \partial(e)$ over all edges e incident to v.

The minimum k for which G has an *edge irregular total k-labeling* is called the *total edge (vertex) irregularity strength* of G. The *total edge (vertex) irregular strength* of G is denoted by $tes(G)$ $(tvs(G))$.

We begin with few known results on tes(G).

Theorem 1. (see [1]) *Let G be a graph with m edges. Then tes(G)* $\geq \left\lceil \frac{m+2}{3} \right\rceil$.

Theorem 2. (see [1]) *Let G be a graph with maximum degree* Δ. *Then tes(G)* $\geq \left\lceil \frac{\Delta+1}{2} \right\rceil$.

Theorem 3. (see [3]) *A graph G(V, E) of order n, size m, and maximum degree* $0 < \Delta < \frac{m\sqrt{10^{-3}}}{8n}$ *satisfies tes(G)* $= \left\lceil \frac{m+2}{3} \right\rceil$.

Theorem 4. (see [3]) *Every graph G(V, E) of order n, minimum degree* $\delta > 0$, *and maximum degree* Δ *such that* $\frac{\Delta}{\delta} < \frac{\sqrt{n}\sqrt{10^{-3}}}{4\sqrt{2}}$ *satisfies tes(G)* $= \left\lceil \frac{m+2}{3} \right\rceil$.

Theorem 5. (see [3]) *For every integer* $\Delta \geq 1$, *there is some* $n(\Delta)$ *such that every graph G(V, E) without isolated vertices with order* $n \geq n(\Delta)$, *size m and maximum degree at most* Δ *satisfies tes(G)* $= \left\lceil \frac{m+2}{3} \right\rceil$.

Conjecture. (see [9]) *For every graph G with size m and maximum degree* Δ *that is different from* K_5, *the total edge irregularity strength equals* $max\{\left\lceil \frac{m+2}{3} \right\rceil, \left\lceil \frac{\Delta+1}{2} \right\rceil\}$.

For K_5, the maximum of the lower bounds is 4 while $tes(K_5) = 5$. Conjecture has been verified for trees by Ivančo and Jendrol [9] and for complete graphs and complete bipartite graphs by Jendrol et al. in [10].

In this paper we prove that the bound on tes given in Theorem1 is sharp for the Series parallel graph.

2. Series Parallel Graph

In graph theory, *series-parallel graphs* are graphs with two distinguished vertices called *terminals*, formed recursively by two simple composition operations. They can be used to model *series* and *parallel electric circuits*.

There are several ways to define series-parallel graphs. The following definition basically follows the one used by David Eppstein [6]. A series-parallel graph (*sp* graph) is usually defined recursively by using parallel and series compositions. This classical definition justifies another name of these graphs, 2-*terminal sp graphs*, since we assume that every such graph has two nodes distinguished as poles and denoted by S (for South) and N (for North).

Definition 1. *A sp graph G with poles S and N is defined as either:*

(i) an edge (S, N)
or can be constructed as in (ii) or (iii)

(ii) G is a parallel composition of at least two sp graphs $G_1, G_2, ..., G_l (l \geq 2)$, *denoted by* $G = \|G_1\| \parallel G_2 \parallel ... \|G_l\|$. *This operation identifies the South poles* S_i *of the component graphs into the South pole S of G, and similarly the North pole* N_i *become N of G.*

(iii) G is a series composition of atleast two sp graphs $G_1, G_2, ..., G_k (k \geq 2)$, *denoted by* $G = G_1 \circ G_2 \circ ... \circ G_k$. *This operation identifies* N_i *and* S_{i+1} *for* $i = 1, ..., k - 1$ *and assigns* S_1 *to S and* N_k *to N.*

In this paper we concentrate on series composition of uniform Θ-graphs.

Series parallel graphs can be characterized in many ways. The oldest and the most popular characterization due to Duffin [5] provides a Kuratowski-like condition which states that the graph G is series-parallel if and only if it contains no subgraph homeomorphic to K_4, the complete graph on four nodes (also known as Wheatstone bridge). Some recently invented characterization of *sp* graphs are given in [11].

Every series-parallel graph has treewidth at most 2 and branchwidth at most 2. The maximal series-parallel graphs, graphs to which no additional edges can be added without destroying their series-parallel structure, are exactly the 2-trees, [5, 2].

SPGs may be recognized in linear time [2] and their series-parallel decomposition may be constructed in linear time as well. Besides being a model of certain types of electric networks, these graphs are of interest in computational complexity theory, because a number of standard graph problems are solvable in linear time on SPGs [8], including finding the maximum matching, maxi-

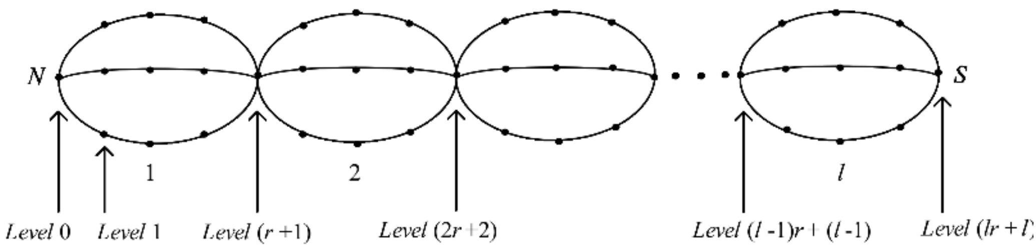

Figure 1: Levels of $sp(m, r, l)$

mum independent set, minimum dominating set and Hamiltonian completion in graphs. Some of these problems are NP-complete for general graphs. The solution capitalizes on the fact that if the answers for one of these problems are known for two SP-graphs, then one can quickly find the answer for their series and parallel compositions.

Definition 2. *A generalized theta graph* $\Theta(n, m)$ *or simply a theta graph with* n *vertices has two vertices N and S of degree* m *such that every other vertex is of degree 2 and lies in one of the* m *paths joining the vertices N and S. The two vertices N and S are called North Pole and South Pole respectively. A path between the South Pole and North Pole is called a longitude. A longitude is denoted by L. In the literature* $\Theta(n, 3)$ *is called a theta graph.*

A *theta graph* $\Theta(n, l)$ is said to be uniform if $|L_1| = |L_2| = ... = |L_l|$, where L_i is a longitude of $\Theta(n, l)$. As our study is on series composition of uniform theta graphs we shall use the following notation hereafter.

Notation 1. *The series-parallel graph* $G = G_1 \circ G_2 \circ G_3 ... G_l$, *where* $G_i = \Theta(rm + 2, m, r)$, *with* m *the number of longitudes and* r *vertices on each longitude,* $i = 1, 2, ..., l$ *is denoted by* $sp(m, r, l)$. *The levels of* G *are addressed as Level* 0, *Level* $1, ...,$ *Level* $(r + 1),$ *Level* $(r + 2), ...,$ *Level* $(lr + l - 1)$ *and*

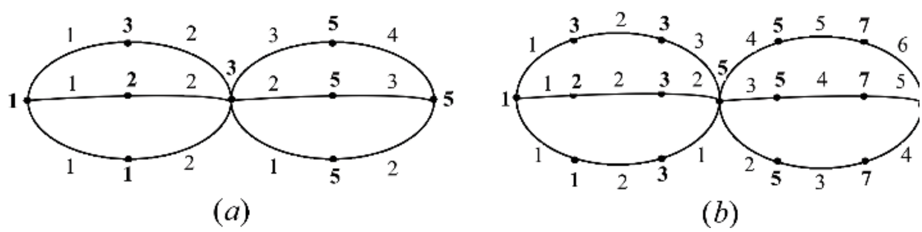

Figure 2: a) $tes(sp(3, 1, 2)) = 5$; b) $tes(sp(3, 2, 2)) = 7$

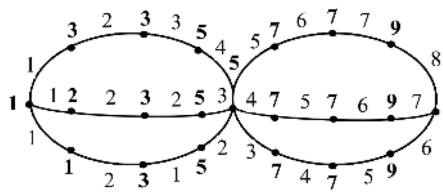

Figure 3: $tes(sp(3,3,2)) = 9$

Level $(lr + l)$ *as shown in Figure 1.*

$sp(m,r,l)$ has $lm(r+1)$ edges, and $(lr+l)$ *levels, where* $r = 1, 2, ..., p$ (for some finite p). By Theorem1[1], we have $tes(sp(m,r,l)) \geq \left\lceil \frac{lm(r+1)+2}{3} \right\rceil$. As the first result in this section we prove that the lower bound is sharp for $sp(m,r,l)$. We begin with $l = 2$.

3. Main Results

Lemma 1. $tes(sp(3,1,2)) = 5$.

Proof. Let $sp(3,1,2)$ be labeled as in Figure 2 (a). It is easy to check that $tes(sp(3,1,2)) = 5$.

We now consider $sp(3,r,2)$, $r \geq 2$.

Procedure $tes(sp(3,r,2))$

Input: Series-parallel graph, $sp(3,r,2)$, $r \geq 2$.

Algorithm: Let $k(r) = tes(sp(3,r,2))$.

(1) Label the vertices and edges of $sp(3,1,2)$ as in Lemma 1.

(2) Having labeled $sp(3,1,2)$, label $sp(3,r,2)$, $r \geq 2$ as follows:

The graph $sp(3,r,2)$, $r \geq 2$ is obtained by introducing r vertices on each longitude and hence the vertex at level r of $sp(3,r-1,2)$ are duplicated as vertices at level r of $sp(3,r,2)$ and the vertices at level $r+1$ of $sp(3,r-1,2)$ are merged into the vertex at level $r+1$ of $sp(3,r,2)$.

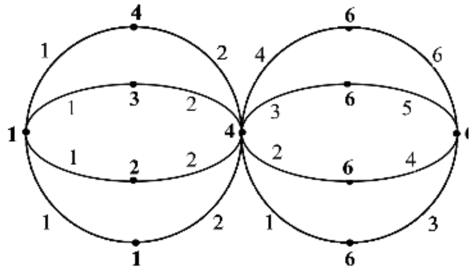

Figure 4: $tes(sp(4,1,2)) = 6$

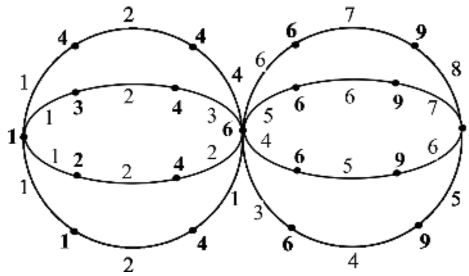

Figure 5: $tes(sp(4,2,2)) = 9$

(i) Label vertices and edges upto level $2r$ as in $sp(3, r-1, 2)$.

(ii) Label all the vertices of $sp(3, r, 2)$ at level $2r+1$ and the south pole as the tes value of $sp(3, r, 2)$.

(iii) The edges $e_j = (u_j, w_j)$, $1 \leq j \leq 3$, with vertex labels $l(u_j)$ and $l(w_j)$, between level $2r$ and $2r+1$ are labeled from bottom to top as $3k(r-1) - 1 + j - (l(u_j) + l(w_j))$.

(iv) The edges $e_i = (u_i, w_i)$, $1 \leq i \leq 3$, with vertex labels $l(u_i)$ and $l(w_i)$, connecting south pole to vertices at level $2r+1$ are labeled as $3k(r-1) + 2 + i - (l(u_i) + l(w_i))$.

End Procedure $tes(sp(3, r, 2))$.

Output: $tes(sp(3, r, 2)) = \left\lceil \frac{6(r+1)+2}{3} \right\rceil$.

Proof of Correctness: We prove the result by induction on r. When $r = 1$, the result is true by Lemma 1. Assume the result for $r-1$. Consider $sp(3, r, 2)$.

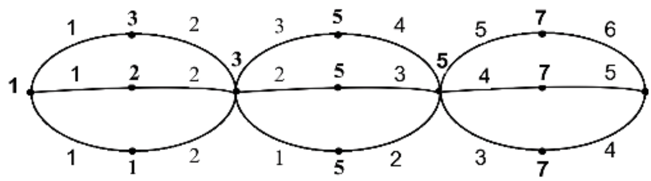

Figure 6: $tes(sp(3,1,3)) = 7$

Since the labeling of $sp(3, r-1, 2)$ is an edge irregular $k-$labeling, it is clear that the labeling of vertices and edges of $sp(3, r, 2)$ obtained by adding consecutive integers as in step 2 is also an edge irregular $k-$labeling. We know by actual verification that the edge sum labels obtained in Lemma 1 are distinct. Hence the edge sum labels of $sp(3, r, 2)$ are also distinct. Labeling of $sp(3, 2, 2)$ and $sp(3, 3, 2)$ are shown in Figure 2 (b) and Figure 3. □

Labeling of $sp(4, 1, 2)$ and $sp(4, 2, 2)$ are shown in Figure 4 and Figure 5. Thus we have the following theorem.

Theorem 2. *Let $sp(3, r, 2)$ be a series parallel graph. Then* $tes(sp(3, r, 2)) = \left\lceil \frac{6(r+1)+2}{3} \right\rceil$, $r \geq 1$.

We now proceed to obtain tes value for $sp(m, r, 3)$. To prove the exact result for $sp(m, r, l)$, $l \geq 2$, we prove that for $sp(3, r, 3)$ the result holds good.

Lemma 2. $tes(sp(3, 1, 3)) = 7$.

Proof. Let $sp(3, 1, 3)$ be labeled as in Figure 6. It is easy to check that $tes(sp(3, 1, 3)) = 7$.

The following algorithm yields the total edge irregularity strength of $sp(3, r, 3)$.

Procedure $tes(sp(3, r, 3))$

Input: Series-parallel graph, $sp(3, r, 3)$.

Algorithm: Let $k(r) = tes(sp(3, r, 3))$.

(1) Label the vertices and edges of $sp(3, 1, 3)$ as in Lemma 2.

(2) Having labeled $sp(3, 1, 3)$, label $sp(3, r, 3)$, $r \geq 2$ as follows:

(i) Label vertices and edges upto level $3r$ as in $sp(3, r-1, 3)$.

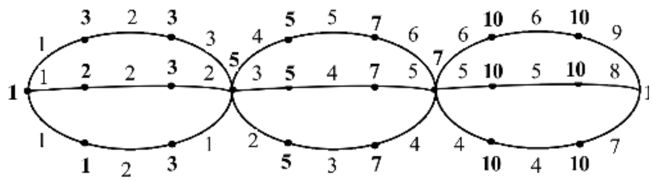

Figure 7: $tes(sp(3,2,3)) = 10$

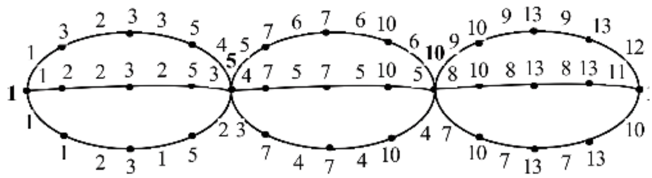

Figure 8: $tes(sp(3,3,3)) = 13$

(ii) The unlabeled vertices of $sp(3,r,3)$ and the south pole receive the tes value of $sp(3,r,3)$.

(iii) The edges $e_j = (u_j, w_j)$, $1 \leq j \leq 3$, with vertex labels $l(u_j)$ and $l(w_j)$, between level $3r$ and $3r+1$ are labeled from bottom to top as $3k(r-1) - 1 + j - (l(u_j) + l(w_j))$.

(iv) The edges $e_j = (v_j, q_j)$, $1 \leq j \leq 3$, with vertex labels $l(v_j)$ and $l(q_j)$, between level $3r+1$ and $3r+2$ are labeled from bottom to top as $l(u_3)$ $+ \; l(w_3) + l(u_3 w_3) + j - (l(v_j) + l(q_j))$.

(iv) Label the edges e_i connecting the vertices to the south pole from bottom to top as $3k(r-1) + 5 + i - 2k(r)$.

End Procedure $tes(sp(3,r,s))$.

Output: $tes(sp(3,r,3)) = \left\lceil \frac{9(r+1)+2}{3} \right\rceil$.

Proof of Correctness: We prove the result by induction on r.When $r = 1$, the result is true by Lemma 2. Assume the result for $r-1$. Consider $sp(3,r,3)$. Since the labeling of $sp(3,r-1,3)$ is an edge irregular $k-$labeling, it is clear that the labeling of vertices and edges of $sp(3,r,3)$ obtained by adding consecutive integers as in step 2 is also an edge irregular $k-$labeling. We know by actual verification that the edge sum labels obtained in Lemma 2 are distinct. Hence

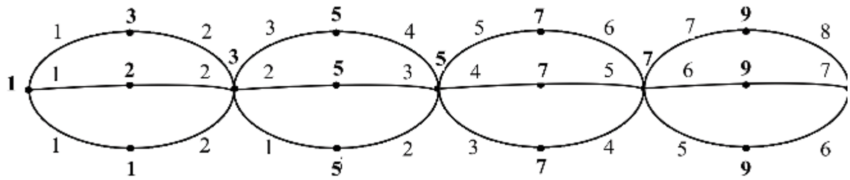

Figure 9: $tes(sp(3,1,4)) = 9$

the edge sum labels of $sp(3,r,3)$ are also distinct. Labeling of $sp(3,2,3)$ and $sp(3,3,3)$ are shown in Figure 7 and Figure 8. □

Theorem 3. *Let $sp(3,r,3)$ be a series parallel graph. Then* $tes(sp(3,r,3)) = \left\lceil \frac{9(r+1)+2}{3} \right\rceil$, $r \geq 1$.

As we labeled the vertices and edges of $sp(3,r,3)$, we observed the following which we give as a remark.

Remark 1. *By labeling $sp(3,r,2)$, $r \geq 2$ we noted that there where 2 levels of vertices and edges that were yet to be labeled. Similarly while labeling $sp(3,r,3)$, $r \geq 2$ we found that there were 3 levels of vertices and edges to be labeled. Hence we can conclude that for $sp(3,r,l)$, $r \geq 2$ there would be l levels to be labeled.*

By Theorem 2 and 3 we can generalise the result for l, for which the base case $sp(3,1,l)$ is obtained as follows. The labeling of $sp(3,1,l)$ is obtained from $sp(3,1,l-1)$. The remaining vertices are labeled as $tes(sp(3,1,l))$ and the edges are labeled from bottom to top as in step 2 (iii), (iv) and (v) of Procedure $tes(sp(3,r,3))$ respectively so that the sums received are consecutive at l levels. By the above procedure we get the following result.

Theorem 4. *Let $sp(m,r,l)$, $l \geq 2$ be a series parallel graph. Then* $tes(sp(m,r,l)) = \left\lceil \frac{lm(r+1)+2}{3} \right\rceil$, $r \geq 1$.

Labeling of $sp(3,1,4)$ is shown in Figure 9.

4. Conclusion

In this paper, we consider series-parallel graphs $sp(m,r,l)$ of uniform theta graphs and prove that they are total edge irregular and its optimal tes value is sharp, for $l \geq 2$. Further our study of total edge irregularity strength is extended to the special case of $l = 1$.

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