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Trigonometric approximation of functions belonging to Lipschitz class by matrix $(C^1 \cdot N_p)$ operator of conjugate series of Fourier series

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Abstract

In the present paper, a new theorem on the degree of approximation of a function $\widetilde{f}$, conjugate to a 2π periodic function f belonging to the Lip α $(0 < \alpha \leq 1)$ class without the monotonicity condition on the generating sequence $\{p_n\}$ has been established, which in turn generalizes the results of Lal (Appl. Math. Comput. 209: 346-350, 2009) on a Fourier series.

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1 Introduction

The degree of approximation of functions belonging to Lip α, Lip(α, r), Lip$(\xi(t), r)$ and $W(L_r, \xi(t))$, $(r \geq 1)$-classes through trigonometric Fourier approximation using different summability matrices with monotone rows has been proved by various investigators like Khan [1], Mittal *et al.* [2, 3], Mittal, Rhoades and Mishra [4], Qureshi [5], Chandra [6], Leindler [7], Rhoades *et al.* [8]. Recently Lal [9] has proved a theorem on the degree of approximation of a function f belonging to the Lip α $(0 < \alpha \leq 1)$ class by $C^1 \cdot N_p$ summability method of its Fourier series. Lal [9] has assumed monotonicity on the generating sequence $\{p_n\}$. The approximation of a function $\widetilde{f}(x)$, conjugate to a 2π periodic function to $f \in$ Lip α $(0 < \alpha \leq 1)$ using product $(C^1 \cdot N_p)$-summability has not been studied so far. In this paper, we obtain a new theorem on the degree of approximation of a function $\widetilde{f}$, conjugate to a 2π periodic function $f \in$ Lip α $(0 < \alpha \leq 1)$ class without monotonicity condition on the generating sequence $\{p_n\}$.

Let $\sum_{n=0}^{\infty} a_n$ be a given infinite series with the sequence of nth partial sums $\{s_n\}$. Let $\{p_n\}$ be a non-negative sequence of constants, real ($\mathbf{R}$) or complex, and let us write

$$P_n = \sum_{k=0}^{n} p_k \neq 0 \quad \forall n \geq 0, p_{-1} = 0 = P_{-1} \text{ and } P_n \to \infty \text{ as } n \to \infty.$$

The sequence to sequence transformation $t_n^N = \sum_{\nu=0}^{n} p_{n-\nu} s_\nu / P_n$ defines the sequence $\{t_n^N\}$ of Nörlund means of the sequence $\{s_n\}$, generated by the sequence of coefficients

$\{p_n\}$. The series $\sum_{n=0}^{\infty} a_n$ is said to be N_p summable to the sum s if $\lim_{n\to\infty} t_n^N$ exists and is equal to a finite number s. In the special case, in which

$$p_n = \binom{n+\alpha-1}{\alpha-1} = \frac{(n+\alpha)}{(n+1)(\alpha)} \quad (\alpha > 0),$$

the Nörlund summability N_p reduces to the familiar C^α summability.

The product of C^1 summability with a N_p summability defines $C^1 \cdot N_p$ summability. Thus the $C^1 \cdot N_p$ mean is given by $t_n^{CN} = \frac{1}{n+1}\sum_{k=0}^{n} P_k^{-1} \sum_{\nu=0}^{k} p_{k-\nu} s_\nu$.

If $t_n^{CN} \to s$ as $n \to \infty$, then the infinite series $\sum_{n=0}^{\infty} a_n$ or the sequence $\{s_n\}$ is said to be summable $C^1 \cdot N_p$ to the sum s if $\lim_{n\to\infty} t_n^{CN}$ exists and is equal to s.

$$\begin{aligned} s_n \to s \quad &\Rightarrow \quad N_p(s_n) = t_n^N = P_n^{-1} \sum_{\nu=0}^{n} p_{n-\nu} s_\nu \to s, \quad \text{as } n \to \infty, N_p \text{ method is regular,} \\ &\Rightarrow \quad C^1\big(N_p(s_n)\big) = t_n^{CN} \to s, \quad \text{as } n \to \infty, C^1 \text{ method is regular,} \\ &\Rightarrow \quad C^1 \cdot N_p \quad \text{method is regular.} \end{aligned}$$

Let $f(x)$ be a 2π-periodic function and Lebesgue integrable. The Fourier series of $f(x)$ is given by

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \equiv \sum_{n=0}^{\infty} A_n(x), \quad \forall n \geq 0, \tag{1.1}$$

with $(n+1)$th partial sum $s_n(f;x)$ called the trigonometric polynomial of degree (order) n of the first $(n+1)$ terms of the Fourier series of f.

The conjugate series of Fourier series (1.1) is given by

$$\sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx) \equiv \sum_{n=1}^{\infty} B_n(x). \tag{1.2}$$

A function $f(x) \in \operatorname{Lip}\alpha$ if

$$f(x+t) - f(x) = O\big(|t^\alpha|\big) \quad \text{for } 0 < \alpha \leq 1, t > 0.$$

L_∞-norm of a function $f : R \to R$ is defined by $\|f\|_\infty = \sup\{|f(x)| : x \in R\}$.

The degree of approximation of a function $f : R \to R$ by the trigonometric polynomial t_n of order n under the sup norm $\| \ \|_\infty$ is defined by [10]

$$\|t_n - f\|_\infty = \sup\big\{|t_n(x) - f(x)| : x \in R\big\}$$

and $E_n(f)$ of a function $f \in L_r$ is given by $E_n(f) = \min_n \|t_n - f\|_r$.

The conjugate function $\widetilde{f}(x)$ is defined for almost every x by

$$\begin{aligned} \widetilde{f}(x) &= -\frac{1}{2\pi}\int_0^\pi \psi(t) \cot t/2 \, dt \\ &= \lim_{h\to 0}\left(-\frac{1}{2\pi}\int_h^\pi \psi(t) \cot t/2 \, dt\right) \quad \text{(see [11, Definition 1.10]).} \end{aligned}$$

We note that t_n^N and t_n^{CN} are also trigonometric polynomials of degree (or order) n.

Abel's transformation: The formula

$$\sum_{k=m}^{n} u_k v_k = \sum_{k=m}^{n-1} U_k(v_k - v_{k+1}) - U_{m-1}v_m + U_n v_n, \tag{1.3}$$

where $0 \leq m \leq n$, $U_k = u_0 + u_1 + u_2 + \cdots + u_k$, if $k \geq 0$, $U_{-1} = 0$, which can be verified, is known as Abel's transformation and will be used extensively in what follows.

If $v_m, v_{m+1}, \ldots, v_n$ are non-negative and non-increasing, the left-hand side of (1.3) does not exceed $2v_m \max_{m-1\leq k\leq n} |U_k|$ in absolute value. In fact,

$$\begin{aligned}\left|\sum_{k=m}^{n} u_k v_k\right| &\leq \max |U_k| \left\{\sum_{k=m}^{n-1}(v_k - v_{k+1}) + v_m + v_n\right\} \\ &= 2v_m \max |U_k|. \end{aligned} \tag{1.4}$$

We write throughout the paper

$$\begin{aligned}&\psi_x(t) = \psi(t) = f(x+t) - f(x-t), \\ &(\widetilde{CN})_n(t) = \frac{1}{2\pi(n+1)} \sum_{k=0}^{n} P_k^{-1} \sum_{\nu=0}^{k} p_\nu \frac{\cos(k-\nu+1/2)t}{\sin t/2}, \end{aligned} \tag{1.5}$$

$\tau = [1/t]$, where τ denotes the greatest integer not exceeding $1/t$, $P_\tau = P[1/t]$, $\Delta p_k = p_k - p_{k+1}$.

2 Known results

In a recent paper Lal [9] obtained a theorem on the degree of approximation for a function belonging to the Lipschitz class $\text{Lip}\,\alpha$ using Cesàro-Nörlund $(C^1 \cdot N_p)$-summability means of its Fourier series with non-increasing weights $\{p_n\}$. He proved the following theorem.

Theorem 2.1 *Let N_p be a regular Nörlund method defined by a sequence $\{p_n\}$ such that*

$$P_\tau \sum_{\nu=\tau}^{n} P_\nu^{-1} = O(n+1). \tag{2.1}$$

Let $f \in L^1[0, 2\pi]$ be a 2π-periodic function belonging to $\text{Lip}\,\alpha$ $(0 < \alpha \leq 1)$, *then the degree of approximation of f by $C^1 \cdot N_p$ means of its Fourier series* (1.1) *is given by*

$$\sup_{0\leq x\leq 2\pi} \left|t_n^{CN}(x) - f(x)\right| = \left\|t_n^{CN} - f\right\|_\infty = \begin{cases} O((n+1)^{-\alpha}), & 0 < \alpha < 1, \\ O(\log(n+1)\pi e/(n+1)), & \alpha = 1. \end{cases} \tag{2.2}$$

Remark 1 In the proof of Theorem 2.1 of Lal [5, p.349], the estimate for the case $\alpha = 1$ is obtained as

$$O\left(\frac{1}{n+1}\right) + O\left(\frac{\log(n+1)\pi}{n+1}\right) = O\left(\frac{\log e}{n+1}\right) + O\left(\frac{\log(n+1)\pi}{n+1}\right) = O\left(\frac{\log(n+1)\pi e}{n+1}\right).$$

Since $1/(n+1) \leq \log((n+1)\pi)/(n+1)$, the e is not needed in (2.2) for the case $\alpha = 1$ (*cf.* [8, p.6870]).

Remark 2 Lal [9] has used the monotonicity condition on the generating sequence $\{p_n\}$ in the proof of Theorem 2.1 but has not mentioned it in the statement.

3 Main theorem

The theory of approximation is a very extensive field and the study of theory of trigonometric approximation is of great mathematical interest and of great practical importance. It is well known that the theory of approximations, *i.e.*, TFA, which originated from a well-known theorem of Weierstrass, has become an exciting interdisciplinary field of study for the last 130 years. These approximations have assumed important new dimensions due to their wide applications in signal analysis [12] in general and in digital signal processing [13] in particular, in view of the classical Shannon sampling theorem. Mittal *et al.* [2–4, 14] have obtained many interesting results on TFA using summability methods without monotonicity on the rows of the matrix T: a digital filter. Broadly speaking, signals are treated as functions of one variable and images are represented by functions of two variables. But till now, nothing seems to have been done so far to obtain the degree of approximation of conjugate of a function using $C^1 \cdot N_p$ product summability method of its conjugate series of a Fourier series. The observations of Remarks 1 and 2 motivated us to determine a proper set of conditions to prove Theorem 2.1 on the conjugate series of its Fourier series. The series, conjugate to a Fourier series, is not necessarily a Fourier series. Hence a separate study of conjugate series is desirable, which attracted the attention of researchers.

Therefore, the purpose of present paper is to establish a quite new theorem on the degree of approximation of a function $\tilde{f}(x)$, conjugate to a 2π-periodic function f belonging to the $\operatorname{Lip}\alpha$ $(0 < \alpha \leq 1)$ class by $C^1 \cdot N_p$ means of conjugate series of its Fourier series without monotonicity on the generating sequence $\{p_n\}$ (that is, weakening the conditions on the filter, we improve the quality of a digital filter [2, p.4485]). More precisely, we prove the following theorem.

Theorem 3.1 *Let N_p be the regular Nörlund summability matrix generated by the non-negative $\{p_n\}$ such that*

$$(n+1)p_n = O(P_n), \quad \forall n \geq 0. \tag{3.1}$$

Let $f \in L^1[0, 2\pi]$ be a 2π-periodic signal (function). Then the degree of approximation of $\tilde{f}(x)$, conjugate to $f \in \operatorname{Lip}\alpha$ $(0 < \alpha \leq 1)$ by $C^1 \cdot N_p$ means of conjugate series of its Fourier series, is given by

$$\begin{aligned} \left\| \tilde{t}_n^{CN}(f;x) - \tilde{f}(x) \right\|_\infty &= \sup_{0 \leq x \leq 2\pi} \left| \tilde{t}_n^{CN}(f;x) - \tilde{f}(x) \right| \\ &= \begin{cases} O((n+1)^{-\alpha}), & 0 < \alpha < 1, \\ O(\frac{\log(n+1)}{n+1}), & \alpha = 1. \end{cases} \end{aligned} \tag{3.2}$$

Remark 3 For a non-increasing sequence $\{p_n\}$, we get

$$P_n = \sum_{k=0}^{n} p_k \geq p_n \sum_{k=0}^{n} 1 = (n+1)p_n, \quad \textit{i.e.}\ (n+1)p_n = O(P_n).$$

Thus the condition (3.1) holds for a non-increasing sequence $\{p_n\}$. Hence our Theorem 3.1 generalizes Theorem 2.1 on conjugate series of its Fourier series.

Note 1 The product transform $C^1 \cdot N_p$ plays an important role in signal theory as a double digital filter [14].

4 Lemmas

We need the following lemmas for the proof of our theorem.

Lemma 1 *If P_n is positive and $P_n^{-1} \geq P_{n+1}^{-1}$ $\forall n \geq 0$, then for $0 \leq a < b \leq \infty$, $0 < t \leq \pi$ and for any n, we have*

$$\left|\sum_{k=a}^{b} P_k^{-1} e^{i(n-k)t}\right| = \begin{cases} O(t^{-1}) & \text{for any } a, \\ O(t^{-1}P_a^{-1}) & \text{for } a \geq [t^{-1}]. \end{cases}$$

Proof Let $\tau = [t^{-1}]$. Then

$$\left|\sum_{k=a}^{b} P_k^{-1} e^{i(n-k)t}\right| = \left|e^{int}\sum_{k=a}^{b} P_k^{-1} e^{-ikt}\right| \leq \left|\sum_{k=a}^{\tau-1} P_k^{-1} e^{-ikt}\right| + \left|\sum_{k=\tau}^{b} P_k^{-1} e^{-ikt}\right|;$$

but

$$\left|\sum_{k=a}^{\tau-1} P_k^{-1} e^{-ikt}\right| \leq \left|e^{-iat}\right|\left|\sum_{k=a}^{\tau-1} P_k^{-1}\right| \leq \left|\sum_{k=0}^{\tau-1} P_k^{-1}\right| \leq \frac{\tau}{p_0} = O\left(t^{-1}\right),$$

and, by (1.4), we have

$$\begin{aligned}\left|\sum_{k=\tau}^{b} P_k^{-1} e^{-ikt}\right| &\leq 2P_\tau^{-1} \max_{\tau+1\leq k\leq b}\left|\frac{1-e^{-i(k+1)t}}{1-e^{-it}}\right| \leq 4P_\tau^{-1}\left|\frac{e^{it/2}}{e^{it/2}-e^{-it/2}}\right| \\ &\leq 2P_\tau^{-1}\left(\frac{1}{\sin(t/2)}\right) = O\left(t^{-1}P_\tau^{-1}\right).\end{aligned}$$

Since $P_n > 0$ and $P_n^{-1} \geq P_{n+1}^{-1}$ $\forall n \geq 0$, we have

$$t^{-1}P_{t^{-1}}^{-1} \leq \left(\left[t^{-1}\right]+1\right)P_{[t^{-1}]}^{-1} \leq P_{[t^{-1}]}^{-1} \leq P^{-1}\left(t^{-1}\right),$$

and, in case $a \geq [t^{-1}]$, we would have

$$\left|\sum_{k=a}^{b} P_k^{-1} e^{-ikt}\right| \leq 2P_a^{-1} \max_{a\leq k\leq b}\left|\frac{1-e^{-i(k+1)t}}{1-e^{-it}}\right| \leq Ct^{-1}P_a^{-1} = O\left(t^{-1}P_a^{-1}\right).$$

This completes the proof of Lemma 1. □

Lemma 2 $|(\widetilde{CN})_n(t)| = O[1/t]$ *for* $0 < t \leq \pi/(n+1)$.

Proof For $0 < t \leq \pi/(n+1)$, $\sin(t/2) \geq (t/\pi)$ and $|\cos nt| \leq 1$.

$$\begin{aligned}
|(\widetilde{CN})_n(t)| &= \left| \frac{1}{2\pi(n+1)} \sum_{k=0}^{n} P_k^{-1} \sum_{\nu=0}^{k} p_\nu \frac{\cos(k-\nu+1/2)t}{\sin t/2} \right| \\
&\leq \frac{1}{2\pi(n+1)} \sum_{k=0}^{n} P_k^{-1} \sum_{\nu=0}^{k} p_\nu \frac{|\cos(k-\nu+1/2)t|}{|\sin t/2|} \\
&\leq \frac{1}{2t(n+1)} \sum_{k=0}^{n} P_k^{-1} \sum_{\nu=0}^{k} p_\nu \\
&= \frac{1}{2t(n+1)} \sum_{k=0}^{n} P_k^{-1} P_k \\
&= O[\tau].
\end{aligned}$$

This completes the proof of Lemma 2. □

Lemma 3 *Let* $\{p_n\}$ *be a non-negative sequence satisfying* (3.1), *then*

$$|(\widetilde{CN})_n(t)| = O\left(\frac{\tau^2}{(n+1)} + \tau \right), \quad \textit{uniformly in } \frac{\pi}{(n+1)} < t \leq \pi.$$

Proof For $\pi/(n+1) < t \leq \pi$, we have

$$\begin{aligned}
(\widetilde{CN})_n(t) &= \frac{1}{2\pi(n+1)} \sum_{k=0}^{n} P_k^{-1} \sum_{\nu=0}^{k} p_\nu \frac{\cos(k-\nu+1/2)t}{\sin t/2} \\
&= \frac{1}{2\pi(n+1)} \left(\sum_{k=0}^{\tau} + \sum_{k=\tau+1}^{n} \right) P_k^{-1} \sum_{\nu=0}^{k} p_\nu \frac{\cos(k-\nu+1/2)t}{\sin t/2} \\
&= \tilde{J}_1(n,t) + \tilde{J}_2(n,t), \quad \text{say},
\end{aligned} \tag{4.1}$$

where

$$\begin{aligned}
|\tilde{J}_1(n,t)| &= \left| \frac{1}{2\pi(n+1)} \sum_{k=0}^{\tau} P_k^{-1} \sum_{\nu=0}^{k} p_\nu \frac{\cos(k-\nu+1/2)t}{\sin t/2} \right| \\
&\leq \frac{1}{2\pi(n+1)} \sum_{k=0}^{\tau} P_k^{-1} \sum_{\nu=0}^{k} p_\nu \frac{|\cos(k-\nu+1/2)t|}{|\sin t/2|} \\
&\leq \frac{1}{2t(n+1)} \sum_{k=0}^{\tau} P_k^{-1} \sum_{\nu=0}^{k} p_\nu \\
&= O\left(\frac{\tau^2}{(n+1)} \right),
\end{aligned} \tag{4.2}$$

in view of $(\sin t/2)^{-1} \leq \pi/t$, for $0 < t \leq \pi$.

Again, using $(\sin t/2)^{-1} \leq \pi/t$, for $0 < t \leq \pi$ and changing the order of summation, we find

$$\begin{aligned}
|\tilde{J}_2(n,t)| &= \left| \frac{1}{2\pi(n+1)} \sum_{k=\tau+1}^{n} P_k^{-1} \sum_{\nu=0}^{k} p_\nu \frac{\cos(k-\nu+1/2)t}{\sin t/2} \right| \\
&\leq \frac{1}{2t(n+1)} \left| \sum_{k=\tau+1}^{n} P_k^{-1} \sum_{\nu=0}^{k} p_\nu \cos(k-\nu+1/2)t \right| \\
&= O\left(\frac{\tau}{n+1}\right) \left| \left(\sum_{\nu=0}^{\tau+1} p_\nu \sum_{k=\tau+1}^{n} P_k^{-1} \cos(k-\nu+1/2)t \right.\right. \\
&\quad \left.\left. + \sum_{\nu=\tau+1}^{n} p_\nu \sum_{k=\nu}^{n} P_k^{-1} \cos(k-\nu+1/2)t \right) \right|. \qquad (4.3)
\end{aligned}$$

Using Lemma 1, we have

$$\begin{aligned}
&\left| \sum_{\nu=0}^{\tau+1} p_\nu \sum_{k=\tau+1}^{n} P_k^{-1} \cos(k-\nu+1/2)t \right| \\
&\quad \leq \left(\sum_{\nu=0}^{\tau+1} p_\nu \left| \sum_{k=\tau+1}^{n} P_k^{-1} e^{i(k-\nu)t} \cdot e^{it/2} \right| \right) \\
&\quad = \sum_{\nu=0}^{\tau+1} p_\nu \left| \sum_{k=\tau+1}^{n} P_k^{-1} e^{i(k-\nu)t} \right| = \sum_{\nu=0}^{\tau+1} p_\nu O\left(\tau P_{\tau+1}^{-1}\right) = O(\tau). \qquad (4.4)
\end{aligned}$$

Using Abel's transformation, we obtain

$$\begin{aligned}
\sum_{k=\nu}^{n} P_k^{-1} \cos(k-\nu+1/2)t &= \sum_{k=\nu}^{n-1} \left(\Delta P_k^{-1}\right) \sum_{\gamma=0}^{k} \cos(k-\gamma+1/2)t + P_n^{-1} \sum_{\gamma=0}^{n} \cos(k-\gamma+1/2)t \\
&\quad - P_\nu^{-1} \sum_{\gamma=0}^{\nu-1} \cos(k-\gamma+1/2)t.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\left| \sum_{k=\nu}^{n} P_k^{-1} \cos(k-\nu+1/2)t \right| \\
&\quad \leq \sum_{k=\nu}^{n-1} \left|\Delta P_k^{-1}\right| \left| \sum_{\gamma=0}^{k} \cos(k-\gamma+1/2)t \right| + P_n^{-1} \left| \sum_{\gamma=0}^{n} \cos(k-\gamma+1/2)t \right| \\
&\qquad + P_\nu^{-1} \left| \sum_{\gamma=0}^{\nu-1} \cos(k-\gamma+1/2)t \right| \\
&\quad = O(\tau) \left(\sum_{k=\nu}^{n-1} \left|\Delta P_k^{-1}\right| + P_n^{-1} + P_\nu^{-1} \right) = O(\tau)\left(P_n^{-1} + P_\nu^{-1}\right), \qquad (4.5)
\end{aligned}$$

by virtue of the fact that $\sum_{k=\lambda}^{\mu} \exp(-ikt) = O(\tau)$, $0 \leq \lambda \leq k \leq \mu$, and $P_n \geq P_{n-1}$ $\forall n \geq 0$.

On combining (4.3) to (4.5), we get

$$\begin{aligned}
|\tilde{J}_2(n,t)| &= O\left(\frac{\tau^2}{n+1}\right)\left(1+\sum_{\nu=\tau+1}^{n} p_\nu\left(P_n^{-1}+P_\nu^{-1}\right)\right) \\
&= O\left(\frac{\tau^2}{n+1}\right)\left(1+P_n^{-1}\sum_{\nu=0}^{n} p_\nu+\sum_{\nu=\tau+1}^{n}\frac{p_\nu}{P_\nu}\right) \\
&= O\left(\frac{\tau^2}{n+1}\right)\left(1+1+\sum_{\nu=\tau+1}^{n}\frac{1}{\nu+1}\right) = O\left(\frac{\tau^2}{n+1}\right)\left(2+O\left(\frac{n-\tau}{\tau+1}\right)\right) \\
&= O\left(\frac{\tau^2}{n+1}\right)+O\left(\left(\frac{\tau^2}{n+1}\right)\cdot\left(\frac{n}{\tau+1}\right)\right) = O\left(\frac{\tau^2}{n+1}+\tau\right),
\end{aligned} \tag{4.6}$$

in view of (3.1) and $\tau \le 1/t < \tau + 1$.

Finally, collecting (4.1), (4.2) and (4.6) yields Lemma 3.

This completes the proof of Lemma 3. □

5 Proof of the theorem

Let $\tilde{s}_n(f;x)$ denote the partial sum of series (1.2), then we have

$$\tilde{s}_n(f;x) - \tilde{f}(x) = \frac{1}{2\pi}\int_0^\pi \psi_x(t)\frac{\cos(n+1/2)t}{\sin t/2}\,dt.$$

Denoting $C^1 \cdot N_p$ means of $\{\widetilde{s_n}(f;x)\}$ by $\tilde{t}_n^{CN}$, we write

$$\begin{aligned}
\tilde{t}_n^{CN}(f;x) - \tilde{f}(x) &= \int_0^\pi \psi_x(t)\frac{1}{2\pi(n+1)}\sum_{k=0}^{n}P_k^{-1}\sum_{\nu=0}^{k}p_\nu\frac{\cos(k-\nu+1/2)t}{\sin t/2}\,dt \\
&= \int_0^\pi \psi_x(t)(\widetilde{CN})_n(t)\,dt \\
&= \left[\int_0^{\pi/(n+1)} + \int_{\pi/(n+1)}^{\pi}\right]\psi_x(t)(\widetilde{CN})_n(t)\,dt = I_1 + I_2 \quad \text{(say)}
\end{aligned} \tag{5.1}$$

If $f(x) \in \operatorname{Lip}\alpha$, then

$$\begin{aligned}
\left|\psi_x(t+h) - \psi_x(t)\right| &= \left|f(x+t+h) - f(x+t) + f(x-t) - f(x-t-h)\right| \\
&\le \left|f(x+t+h) - f(x+t)\right| + \left|f(x-t) - f(x-t-h)\right| \\
&\le C|h|^\alpha.
\end{aligned}$$

Therefore $\psi_x(t) \in \operatorname{Lip}\alpha$.

Now, using Lemma 2, we have

$$|I_1| = O\left(\int_0^{\pi/(n+1)} t^\alpha \cdot \frac{1}{t}\,dt\right) = O\left((n+1)^{-\alpha}\right). \tag{5.2}$$

Using Lemma 3, we obtain

$$|I_2| = O\left\{\int_{\pi/(n+1)}^{\pi} t^\alpha\left(\frac{\tau^2}{n+1}+\tau\right)dt\right\} = O(I_{21}) + O(I_{22}), \quad \text{say,} \tag{5.3}$$

where

$$I_{21} = \frac{1}{n+1}\int_{\pi/(n+1)}^{\pi} t^{\alpha-2}\,dt = \begin{cases} O((n+1)^{-\alpha}), & 0 < \alpha < 1, \\ O(\frac{\log(n+1)}{n+1}), & \alpha = 1, \end{cases} \tag{5.4}$$

and

$$I_{22} = \int_{\pi/(n+1)}^{\pi} t^{\alpha-1}\,dt = O\bigl((n+1)^{-\alpha}\bigr). \tag{5.5}$$

On combining (5.1) with (5.5) and using the inequality $1/(n+1) \leq \log(n+1)/(n+1)$, for higher values of n, we have

$$\bigl|\tilde{t}_n^{CN}(f;x) - \tilde{f}(x)\bigr| = \begin{cases} O((n+1)^{-\alpha}), & 0 < \alpha < 1, \\ O(\frac{\log(n+1)}{n+1}), & \alpha = 1. \end{cases}$$

Hence,

$$\bigl\|\tilde{t}_n^{CN}(f;x) - \tilde{f}(x)\bigr\|_\infty = \sup_{0\leq x\leq 2\pi}\bigl|\tilde{t}_n^{CN}(f;x) - \tilde{f}(x)\bigr| = \begin{cases} O((n+1)^{-\alpha}), & 0 < \alpha < 1, \\ O(\frac{\log(n+1)}{n+1}), & \alpha = 1. \end{cases}$$

This completes the proof of Theorem 3.1.

6 Conclusion

Several results concerning the degree of approximation of periodic signals (functions) belonging to the Lipschitz class by Matrix Operator have been reviewed and the condition of monotonicity on the generating sequence $\{p_n\}$ has been relaxed. Further, a proper set of conditions has been discussed to rectify the errors and applications pointed out in Remarks 1 and 2. Some interesting applications of the operator used in this paper were pointed out in Note 1.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

LNM computed lemmas and established the main theorem in this direction. LNM and VNM conceived of the study and participated in its design and coordination. LNM, VNM contributed equally and significantly in writing this paper. All the authors drafted the manuscript, read and approved the final manuscript.

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