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TWO GENERALIZED COMMON FIXED POINT THEOREMS INVOLVING COMPATIBILITY AND PROPERTY E.A.

Abstract. We prove two generalizations: the first to Das and Naik's theorem for a pair of compatible maps without continuity; and the next as an extension of our first result to three self-maps on a metric space X without compatibility, under a stronger contraction type inequality and restricting the completeness of X to its subspace. The latter is a significant generalization of a recent result of Pant et al.

1. Introduction

Let (X, d) be a metric space and T, a self-map on X. If $x \in X$, we write Tx for the image of x under T. The T-iterates $x, Tx, T^2x, \ldots$ define the T-orbit $O_T(x)$ at $x \in X$. A point a of X will be a contractive fixed point of T if $Ta = a$ to which every T-orbit converges. The well-known Banach contraction principle, also referred to as Banach–Caccioppoli's theorem [2] states that a contraction T on X with $d(Tx, Ty) \leq qd(x, y)$ for all $x, y \in X$ where $0 \leq q < 1$, has a unique fixed point, which indeed is a contractive fixed point of it, provided X is complete. Though classical, this result provides a technique for solving a great variety of applied problems in mathematical sciences and engineering. It may be noted that every contraction is continuous but converse is not true as the identity map suggests. There can be discontinuous self-maps which have fixed points. For instance, the Dirichlet function $f : \mathbb{R} \to \mathbb{R}$ given by $fx = 1$ if x is rational, 0 elsewhere, is nowhere continuous and hence is not a contraction but $x = 1$ is its fixed point. In 1968, Kannan [9] analyzed a substantially new type of contractive condition to ensure the existence of fixed point for maps that have discontinuity in its domain. There have been many theorems involving various linear, rational and general contractive type inequalities (see the survey articles by Ciric [3], Collaco & Silva [4], Danes [5], Kinces and Totok [10], Rhoades [16, 17] etc.).

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Motivated by the interdependence of the existence of commuting pair of mappings and their common fixed point, Gerald Jungck [7] proved the following result:

Theorem 1.1. *Let S and T be commuting self-maps on a complete metric space X such that*

$$T(X) \subset S(X), \tag{1.1}$$

satisfying the inequality

$$d(Tx, Ty) \leq qd(Sx, Sy), \quad for\ all \quad x, y \in X, \tag{1.2}$$

where $0 < q < 1$. If S is continuous, then S and T have a unique common fixed point.

This was generalized by several researchers either by weakening (1.2) and/or dropping the continuity of S. The following is one such result due to Das and Naik [6]:

Theorem 1.2. *Let S and T be commuting self-maps on a complete metric space X satisfying* (1.1) *and the inequality*

$$\begin{aligned} d(Tx, Ty) \leq q \max\{&d(Sx, Sy), d(Sx, Tx), d(Sy, Ty), \\ &d(Sx, Ty), d(Sy, Tx)\}, \quad for\ all \quad x, y \in X \end{aligned} \tag{1.3}$$

with $0 < q < 1$. If S is continuous then S and T will have a unique common fixed point.

Later, Nagaraja Rao and K. P. R. Rao [11] obtained the conclusion of Theorem 1.1 by replacing the continuity of S with the condition:

$$d(Tx, Sy) \leq d(y, Sx), \quad \text{for all} \quad x, y \in X \quad \text{with} \quad y \neq Sx. \tag{1.4}$$

Further they claimed that the condition (1.4) is weaker than the continuity of S, which had been disproved in [15]. In fact, the condition (1.4) and the continuity of S are independent of each other, and the following is a modified version of Theorem 1.1:

Theorem 1.3. *Let S and T be commuting self-maps on a complete metric space X satisfying* (1.1) *and the inequality* (1.2)*. If either S is continuous or the condition* (1.4) *holds good, then S and T will have a unique common fixed point.*

The intent of this paper is to prove two generalizations of Theorem 1.2: the first by replacing the continuity of S under a weaker form of (1.2) through the notion of compatibility; and the second as an extension of the first to three self-maps without the compatibility under a stronger form of (1.3) and the restricted completeness of X.

2. Definitions

Sessa [18] introduced the notion of weakly commuting mappings as given below:

Definition 2.1. Self-maps S and T on X are *weakly commuting* if

$$d(STx, TSx) \leq d(Sx, Tx), \quad \text{for all} \quad x \in X. \tag{2.1}$$

This was further generalized by Jungck [8] with the notion of compatibility:

Definition 2.2. Self-maps S and T on X are *compatible* if

$$\lim_{n\to\infty} d(STx_n, TSx_n) = 0 \tag{2.2}$$

whenever $\langle x_n \rangle_{n=1}^{\infty} \subset X$ is such that

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = t, \quad \text{for some} \quad t \in X. \tag{2.3}$$

Obviously, every commuting pair is weakly commuting. The converse need not be true [18]. However, weak commutativity need not imply the existence of sequence of points satisfying the condition (2.3). For instance, consider $S, T : \mathbb{R} \to \mathbb{R}$ given by $Sx = x/2$ and $Tx = Sx + 1$ for all $x \in \mathbb{R}$ with the usual metric d. Then $d(Sx, Tx) = 1$ and $d(STx, TSx) = 1/2$ so that $d(STx, TSx) < d(Sx, Tx)$ for all $x \in \mathbb{R}$. In other words, S and T are weakly commuting. But there is no sequence of numbers satisfying the conditions (2.2) and (2.3). Such maps are *vacuously* compatible.

In what follows, we consider *nonvacuous* compatibility. Now suppose that S and T are weakly commuting and (2.3) holds good for some $\langle x_n \rangle_{n=1}^{\infty} \subset X$. Then writing $x = x_n$ in (2.1), we see that $d(STx_n, TSx_n) \leq d(Sx_n, Tx_n)$ for all n. As $n \to \infty$ this gives (2.2) in view of (2.3). That is, every weakly commuting pair is nonvacuously compatible as well. However, the converse of this is not true [8]. In 2003, Singh and Tomar [19] did a nice comparative study of various weaker forms of commuting maps.

In obtaining fixed points for noncompatible and discontinuous maps, the following notions were introduced:

Definition 2.3. (Pathak et al., [14]) Self-maps S and T on X are *R-weakly commuting of type* (A_g) if there exists an $R > 0$ such that

$$d(TSx, SSx) \leq Rd(Sx, Tx), \quad \text{for all} \quad x \in X,$$

while S and T are *R-weakly commuting of type* (A_f) if there exists an $R > 0$ such that

$$d(TTx, STx) \leq Rd(Sx, Tx), \quad \text{for all} \quad x \in X.$$

Remark 2.1. R-weakly commuting maps of both types (A_g) and (A_f) commute at their coincidence points.

Remark 2.2. The notions of R-weakly commuting and R-weakly commuting of type (A_f) are independent [19].

Remark 2.3. Every nonvacuously compatible pair of maps is R-weakly commuting of type (A_g) or of type (A_f).

Definition 2.4. (Aamri and Moutawakil, [1]) Two self-maps S and T on X satisfy the property E.A. if there is a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in X with the choice (2.3).

Since noncompatibility implies the existence of the sequence $\langle x_n \rangle_{n=1}^{\infty}$ with choice (2.3), the class of all pairs of self-maps with property E.A. is potentially *wider* than that of noncompatible maps.

Definition 2.5. (Pant, [12]) Self-maps S and T on X are *reciprocally continuous* if for any $\langle x_n \rangle_{n=1}^{\infty} \subset X$ with choice (2.3), we have $\lim_{n\to\infty} TSx_n = Tt$ and $\lim_{n\to\infty} STx_n = St$.

Definition 2.6. (Pant et al., [13]) Self-maps S and T on X are *weakly reciprocally continuous* if $\lim_{n\to\infty} TSx_n = Tt$ or $\lim_{n\to\infty} STx_n = St$ for any $\langle x_n \rangle_{n=1}^{\infty} \subset X$ with choice (2.3).

Note that any pair of continuous maps will be reciprocally continuous, and reciprocally continuous maps are obviously weakly reciprocally continuous but neither of the reverse implications is true [12, 13].

3. Main results and discussion

We first prove

Theorem 3.1. *Let S and T be compatible self-maps on a complete metric space X satisfying the inclusion* (1.1) *and the inequality* (1.3). *Suppose that S satisfies the condition*

$$\min\{d(Sx,Sy), d(Tx,Sy), d(y,Sy), d(Sy,Ty)\} \le d(y,Sx) + d(y,Tx), \quad \text{for all } x,y \in X \text{ except for those } x,y \text{ with } Sx = Tx = y. \tag{3.1}$$

Then S and T will have a unique common fixed point.

Proof. Let $x_0 \in X$ be arbitrary. Using the inclusion (1.1), we can choose points $x_1, x_2, x_3, \ldots$ inductively in X such that

$$Tx_{n-1} = Sx_n, \quad \text{for} \quad n = 1, 2, 3, \ldots. \tag{3.2}$$

Then $\langle Sx_n \rangle_{n=1}^{\infty}$ is a Cauchy sequence in X, as shown in [6] and hence converges to some point z in X. That is

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = z, \quad \text{for some} \quad z \in X. \tag{3.3}$$

Now for $n \geq 1$, we note from (1.3) and (3.2) that

$$d(Tz, Tx_n)$$
$$\leq q \max\{d(Sz, Sx_n), d(Sz, Tz), d(Sx_n, Tx_n), d(Sz, Tx_n), d(Sx_n, Tz)\}.$$

Allowing n to approach ∞ in this and using (3.3), we get

$$(3.4) \qquad d(Tz, z) \leq q \max\{d(Sz, z), d(Sz, Tz), 0, d(Sz, z), d(z, Tz)\}$$
$$= \max\{d(Sz, z), d(Sz, Tz)\}.$$

If $Sx_n = z$ for all but finitely many indices, say for $1, 2, \ldots, m$, then

$$Tx_{n-1} = Sx_n = z \quad \text{for} \quad n > m,$$

and by (3.3) and the compatibility, it follows that

$$d(Sz, Tz) = d(STx_n, TSx_n) = 0, \quad \text{for} \quad n > m$$

so that $Sz = Tz$. With this, (3.4) gives $Tz = z$. Thus $Sz = Tz = z$.

Therefore, we assume that $Sx_n \neq z$ for infinitely many n's with the choice (3.2). Then there is a subsequence $\langle Sx_{n_k} \rangle_{k=1}^{\infty}$ such that $Sx_{n_k} \neq z$ for all k which also converges to z.

Without loss of generality, we may assume that $\langle Sx_n \rangle_{n=1}^{\infty}$ itself is such a subsequence. Thus $Sx_n \neq z$ and $Tx_n \neq z$ for all n so that from (3.1), we have

$$\min\{d(Sx_n, Sz), d(Tx_n, Sz), d(z, Sz), d(Sz, Tz)\}$$
$$\leq d(z, Sx_n) + d(z, Tx_n) \text{ for all } n.$$

Now applying the limit as $n \to \infty$ and using (3.3), we get

$$\min\{d(z, Sz), d(Sz, Tz)\} = \text{ so that } d(z, Sz) = 0 \text{ or } d(z, Tz) = 0.$$

Either of these cases together with (3.4) immediately implies that $Tz = z$. Thus z is a common fixed point of S and T.

Uniqueness of the common fixed point follows directly from the inequality (1.3). ∎

Example 3.1. Let $X = [0, 1]$ with the usual metric d.

Define $S, T : X \to X$ by

$$Sx = \begin{cases} \frac{x}{2} & \left(x < \frac{1}{2}\right) \\ \frac{2x}{3} & \left(x \geq \frac{1}{2}\right) \end{cases} \quad \text{and} \quad Tx = \begin{cases} 0 & \left(x < \frac{1}{2}\right) \\ \frac{x}{6} & \left(x \geq \frac{1}{2}\right). \end{cases}$$

Then S and T satisfy the inclusion (1.1), the inequality (1.3) with $q = \frac{1}{2}$ and condition (3.1). Also the maps are compatible. Hence by Theorem 3.1, they have a unique common fixed point. Indeed, 0 is the only common fixed point for them. However, we see that S and T are not commuting, since $STx \neq TSx$ for $x \geq \frac{1}{2}$. Further S is not continuous and the condition (1.4) fails at $x = y = \frac{1}{2}$. Therefore, Theorem 1.2 and Theorem 1.3 do not ensure a common fixed point.

Our second result is

Theorem 3.2. *Let A, S and T be self-maps on X satisfying the inequality*

$$d(Ax, Ty) \le q \max\{d(Sx, Sy), d(Ax, Sx), d(Ty, Sy), \tfrac{1}{2}[d(Ax, Sy) + d(Ty, Sx)]\}, \quad \text{for all } x, y \in X, \tag{3.5}$$

where $0 < q < 1$. Suppose either (A, S) or (T, S) satisfies the property E.A., *and $S(X)$ is a complete subspace of X. Then A, T and S have a common coincidence point. Further if either (A, S) or (T, S) is an R-weakly commuting pair of type (A_g) or of type (A_f), then A, T and S will have a unique common fixed point. In fact, the corresponding point of coincidence common to these maps will be the unique common fixed point.*

Proof. First suppose that (A, S) satisfies the property E.A. Then there is a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = p, \quad \text{for some} \quad p \in X. \tag{3.6}$$

Let $\lim_{n\to\infty} Tx_n = t$. Then we have $t = p$. For, with $x = y = x_n$, (3.5) gives

$$d(Ax_n, Tx_n) \le q \max\{d(Sx_n, Sx_n), d(Ax_n, Sx_n), d(Tx_n, Sx_n), \tfrac{1}{2}[d(Ax_n, Sx_n) + d(Tx_n, Sx_n)]\}.$$

Applying the limit as n tends to ∞ and using (3.6), we get

$$d(p, t) \le q \max\{0, 0, d(t, p), \tfrac{1}{2} d(t, p)\}$$

so that $d(t, p) = 0$ or that $t = p$.

In other words,

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = p. \tag{3.7}$$

Similarly (3.7) can be established if (T, S) satisfies the property E.A.

Now we prove that

$$Ap = Sp = Tp. \tag{3.8}$$

Since $S(X)$ is complete, we get $p \in S(X)$ so that $Sr = p$ for some $r \in X$.

From (3.5), it follows that

$$d(Ax_n, Tr) \le q \max\{d(Sx_n, Sr), d(Ax_n, Sx_n), d(Tr, Sr), \tfrac{1}{2}[d(Ax_n, Sr) + d(Tr, Sx_n)]\}.$$

As $n \to \infty$, this together with (3.7) gives

$$d(p, Tr) \le q \max\{d(p, p), d(p, p), d(Tr, p), \tfrac{1}{2}[d(p, p) + d(Tr, p)]\} = q d(Tr, p)$$

so that $d(Tr,p)=0$ or $Tr=p$. That is

$$Sr = Tr = p. \tag{3.9}$$

Again from (3.5) and (3.9), we get

$$d(Ar,Tr) \le q\max\{d(Sr,Sr), d(Ar,Sr), d(Tr,Sr), \tfrac{1}{2}[d(Ar,Sr)+d(Tr,Sr)]\}$$

or

$$d(Ar,p) \le q\max\{0, d(Ar,p), 0, \tfrac{1}{2}[d(Ar,p)+0]\} = qd(Ar,p)$$

so that $Ar = p$.

Thus r is a common coincidence point of A, S and T, while p is a point of coincidence common to them, that is

$$Ar = Sr = Tr = p. \tag{3.10}$$

Now, let (A,S) be an R-weakly commuting pair of type (A_g) or of type (A_f). Then in view of Remark 2.1, it follows that $Ap = Sp$. Again (3.5) together with $Ap = Sp$ implies that

$$d(Ap,Tp) \le q\max\{d(Sp,Sp), d(Ap,Sp), d(Tp,Sp), \tfrac{1}{2}[d(Ap,Sp)+d(Tp,Sp)]\}$$

or that

$$d(Ap,Tp) \le q\max\{0, 0, d(Tp,Ap), \tfrac{1}{2}[0+d(Tp,Ap)]\} = qd(Tp,Ap).$$

Therefore, $d(Ap,Tp)=0$ or $Ap=Tp$, proving (3.8).

On the other hand, let (T,S) be R-weakly commuting pair of type (A_g) or of type (A_f). Then $Tp = Sp$, again in view of Remark 2.1.
From (3.5) together with $Tp = Sp$, we get that

$$d(Ap,Tp) \le q\max\{d(Tp,Tp), d(Ap,Tp), d(Tp,Tp), \tfrac{1}{2}[d(Ap,Tp)+d(Tp,Tp)]\}$$

or that $d(Ap,Tp) \le q\max\{0, d(Ap,Tp), 0, \frac{1}{2}d(Ap,Tp)]\} = qd(Ap,Tp)$.

Hence $d(Ap,Tp)=0$ or $Ap=Tp$ and (3.8) follows.

Finally p is a fixed point of T. In fact, again from (3.5) we see that

$$d(Ax_n,Tp) \le q\max\{d(Sx_n,Sp), d(Ax_n,Sx_n), d(Tp,Sp), \tfrac{1}{2}[d(Ax_n,Sp)+d(Tp,Sx_n)]\}.$$

Applying the limit as $n\to\infty$, and using (3.7) and (3.8), we get

$$d(p,Tp) \le q\max\{d(p,Tp), 0, 0, \tfrac{1}{2}[d(p,Tp)+d(Tp,p)]\} = qd(p,Tp)$$

showing that $p = Tp$, and hence p is a common fixed point of A, S and T, in view of (3.8).

If u is also a common fixed point of A, S and T, that is $Au = Tu = Su = u$, then from the inequality (3.5), we see that

$$d(p,u) = d(Ap,Tu) \leq q \max\{d(Sp,Su), d(Ap,Sp), d(Tu,Su), \tfrac{1}{2}[d(Ap,Su) + d(Tu,Sp)]\} = qd(p,u)$$

so that $d(p,u) = 0$, since $q < 1$. Thus the common fixed point is unique. ∎

Taking $A = T$ in Theorem 3.2, we get

Corollary 3.1. *Let S and T be compatible self-maps on X satisfying the inequality*

$$d(Tx,Ty) \leq \max\{d(Sx,Sy), d(Tx,Sx), d(Ty,Sy), \tfrac{1}{2}[d(Tx,Sy) + d(Ty,Sx)]\}, \quad \text{for all } x,y \in X, \tag{3.11}$$

with $0 < q < 1$. Suppose that (T,S) satisfies the property E.A., *and $S(X)$ is a complete subspace of X. Then T and S have a common coincidence point. Further if (T,S) is an R-weakly commuting pair of type (A_g) or of type (A_f), then the corresponding point of their coincidence will be the unique common fixed point.*

Corollary 3.2. (Pant et al., [13]) *Let S and T be self-maps on a complete metric space X satisfying the inclusion* (1.1) *and the contraction-type condition*

$$d(Tx,Ty) \leq ad(Sx,Sy) + bd(Tx,Sx) + cd(Ty,Sy) \quad \text{for all } x,y \in X, \tag{3.12}$$

where a, b and c are non negative numbers such that $a + b + c < 1$.
Suppose that any one of the following conditions is true:

(a) *S and T are (nonvacuously) compatible;*
(b) *(S,T) is an R-weakly commuting pair of type (A_g);*
(c) *(S,T) is an R-weakly commuting pair of type (A_f).*

Then S and T have a unique common fixed point, provided they are weakly reciprocally continuous.

Proof. The authors of [13] employed all the three conditions (a)-(c) independently. However, since (a) implies both (b) and (c) respectively, in view of Remark 2.3, it is not necessary to make use of (a). Note that the right hand side of (3.12) is less than or equal to that of (3.11) with the choice $q = a + b + c < 1$. As such, the inequality (3.11) is weaker than (3.12).

Let $x_0 \in X$ be arbitrary. In view of inclusion (1.1), we can choose inductively points $x_1, x_2, \ldots$ in X with the choice (3.2). Also from [13], it follows that $\langle Sx_n \rangle_{n=1}^{\infty}$ is a Cauchy sequence in the complete space X and hence converges to some z in it. This proves that the pair (S,T) satisfies the property E.A. in X. The remainder of the proof of Corollary 3.2 follows from that of Theorem 3.2 with $A = T$, by taking the completeness of the subspace of X in place of the weak reciprocal continuity of (S, T). ■

It follows from Corollary 3.1 that a common fixed point can also be obtained by dropping the inclusion (1.1) and weak reciprocal continuity, and by weakening the inequality (3.12) in Theorem 3.2. Thus, Corollary 3.1 is a significant generalization of Theorem 3.2 under the restricted completeness of the space X.

Corollary 3.1 also suggests that just by replacing the inequality (1.3) with the stronger form (3.11), a common fixed point can be obtained from Theorem 3.1 even by dropping the inclusion (1.1) and the condition (3.1), restricting the completeness of X to its subspace, and by weakening the compatibility through the property E.A.

In other words, Corollary 3.1 is a significant generalization of Theorem 3.1 as well under a sharper form of the inequality (1.3).

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