SIGMOID FUNCTION IN THE SPACE OF
UNIVALENT λ-PSEUDO STARLIKE FUNCTIONS

G. Murugusundaramoorthy[1] §, T. Janani[2]

[1,2]School of Advanced Sciences
VIT University
Vellore, 632014, INDIA

Abstract: In the present investigation, we obtain initial coefficients of $\lambda-$ pseudo starlike functions related to sigmoid functions and the Fekete-Szegö coefficient functional $|a_3 - \mu a_2^2|$ for certain normalized analytic functions defined on the open unit disk. As an application of the main result, we pointed out the initial coefficients and Fekete-Szegö inequality for a subclasses of starlike functions related to sigmoid functions.

1. Introduction and Preliminaries

The theory of a special function does not have a specific definition but it is of incredibly important to scientist and engineers who are concerned with Mathematical calculations and have a wide application in physics, Computer, engineering etc. Recently, the theory of special function has been outshining by other fields like real analysis, functional analysis, algebra, topology, differential equations.The generalized hypergeometric functions plays a major role in geometric function theory after the proof of Bieberbach conjecture by de-Branges.

§Correspondence author

Activation function is an information process that is inspired by the way biological nervous system such as brain, process information. It composed of large number of highly interconnected processing element (neurons) working together to solve a specific task. This function works in similar way the brain does, it learns by examples and can not be programmed to solve a specific task.The most popular activation function is the sigmoid function because of its gradient descendent learning algorithm. Sigmoid function can be evaluated in different ways, it can be done by truncated series expansion (for details see[4]).

The logistic sigmoid function $h(z) = \frac{1}{1+e^{-z}}$ is differentiable and has the following properties

- It outputs real numbers between 0 and 1.

- It maps a very large input domain to a small range of outputs.

- It never loses information because it is a one-to- one function.

- It increases monotonically.

with all the properties mentioned in[4] sigmoid function is perfectly useful in geometric function theory.

Denote by $\mathcal{A}$ the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \mathbb{U}) \tag{1}$$

which are analytic in the open unit disk $\mathbb{U} := \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ and normalized $f(0) = f'(0) - 1 = 0$. Further, denote by $\mathcal{S}$ the class of analytic,normalized and univalent functions in $\mathbb{U}$.

Let the functions f and g be analytic in the open unit disk $\mathbb{U}$. We say that f is subordinate to g (cf.[6]), written as $f \prec g$ in $\mathbb{U}$ or $f(z) \prec g(z)$ $(z \in \mathbb{U})$, if there exists a Schwarz function $w(z)$, which (by definition) is analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$ $(z \in \mathbb{U})$. It follows from the Schwarz lemma that $f(z) \prec g(z)$ $(z \in \mathbb{U}) \Rightarrow f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$. Furthermore, if the function g is univalent in $\mathbb{U}$, then (see, e.g., [6])

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Longleftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

Lemma 1.1. *(see [9]). If a function $p \in \mathcal{P}$ is given by*

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots \qquad (z \in \mathbb{U}),$$

then $|p_k| \leqq 2$ $(k \in \mathbb{N})$, where $\mathcal{P}$ is the family of all functions p, analytic in $\mathbb{U}$, for which

$$p(0) = 1 \qquad and \qquad \Re\big(p(z)\big) > 0 \qquad (z \in \mathbb{U}).$$

Several authors have discussed various subfamilies of the well-known Bazilevič functions $\mathcal{B}(\alpha)$ satisfy the geometric condition:

$$\Re\left(\frac{f(z)^{\alpha-1} f\ (z)}{z^{\alpha-1}}\right) > 0$$

where α is greater than 1 $(\alpha \in \mathbb{R})$.(see, for details, [3]; see also [5, 10]) from various viewpoints such as the perspective of convexity, inclusion theorems, radii of starlikeness and convexity, boundary rotational problems, subordination relationships, and so on.The class includes the starlike and bounded turning functions as the cases $\alpha = 0$ and $\alpha = 1$ respectively. Further the study has been extended by defining a subclass Bazilevič functions of type α order β if and only if $\Re\left(\frac{f(z)^{\alpha-1} f'(z)}{z^{\alpha-1}}\right) > \beta$ dented by $\mathcal{B}(\alpha, \beta)$.

More Recently Babalola [1]defined a new subclass $\lambda-$pseudo starlike function of order $\beta(0 \leq \beta < 1)$ satisfying the analytic condition

$$\Re\left(\frac{z(f\ (z))^{\lambda}}{f(z)}\right) > \beta \quad (z \in \mathbb{U}, \lambda \geq 1 \in \mathbb{R}) \tag{2}$$

and denoted by $\mathcal{L}_{\lambda}(\beta)$. Further note that

Remark 1.2.   1. Throughout this work, all powers shall mean principal determinations only.

2. If $\lambda = 1$, we have the class of starlike functions of order β, which in this context, are 1$-$pseudo starlike functions of order β satisfying the condition

$$\Re\left(\frac{zf\ (z)}{f(z)}\right) > \beta \quad (z \in \mathbb{U}) \tag{3}$$

denoted by $\mathcal{S}^*(\beta)$.

3. If $\beta = 0$, we simply write $\mathcal{L}$ instead of $\mathcal{L}(0)$.

Babalola [1]remarked that though for $\lambda > 1$, these classes of $\lambda - pseudo-$ starlike functions clone the analytic representation of starlike functions, it is not yet known the possibility of any inclusion relations between them. We recall the following Lemmas which are relevant for our study

Lemma 1.3. *If z is a complex number having positive real part, then for any real number t such that $t \in [0, 1]$, we have $\Re z^t \geq (\Re z)^t$.*

For $\lambda = 2$ we note that functions in $\mathcal{L}_2(\beta) = \mathcal{G}(\beta)$ are defined by

$$\Re\left(f(z)\frac{zf(z)}{f(z)}\right) > \beta, \qquad (z \in \mathbb{U}) \tag{4}$$

which is a product combination of geometric expressions for bounded turning and starlike functions.

In this paper we denote the class of $\lambda-$ Pseudo starlike functions satisfying the condition (2) and related with sigmoid functions by by $\mathcal{L}_\lambda^\beta(\Phi)$ and investigate how the sigmoid function is related to analytic univalent $\lambda-$ Pseudo starlike functions in terms of coefficients bounds and also discuss its Fekete-Szegö problem. Further we pointed out the results for $f \in \mathcal{S}^*(\beta, \Phi)$ and $f \in \mathcal{G}(\beta, \Phi)$ the subclasses related with sigmoid functions satisfying the condition given by (3) and (4)respectively.

2. Initial Coefficients

First we recall the following Lemmas due to Joseph et al [4](also see [8])in order to prove our main result

Lemma 2.1. *[4]Let h be a sigmoid function and*

$$\Phi(z) = 2h(z) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{2^m}\left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}z^n\right)^m, \tag{5}$$

then $\Phi(z) \in \mathcal{P}, |z| < 1$ where $\Phi(z)$ is a modified sigmoid function.

Lemma 2.2. *[4] Let*

$$\Phi_{n,m}(z) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{2^m}\left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}z^n\right)^m \tag{6}$$

then $|\Phi_{n,m}(z)| < 2$.

Lemma 2.3. *[4] If $\Phi(z) \in \mathcal{P}$ and it is starlike, then f is a normalized univalent function of the form(1).*

Taking $m = 1$, Joseph et al [4] remarked the following:

Remark 2.4. Let $\Phi(z) = 1 + \sum_{n=1}^{\infty} C_n z^n$ where $C_n = \frac{-1(-1)^n}{2\ n!}$then

$$|C_n| \leq 2, n = 1, 2, 3, ...$$

this result is sharp for each n.

Theorem 2.5. *If $f \in \mathcal{A}$ and of the form (1) is belonging to $\mathcal{L}_\lambda^\beta(\Phi)$ ($\lambda \geq 1 \in \mathbb{R},$) then*

$$|a_2| \leq \frac{1-\beta}{2(2\lambda-1)} \tag{7}$$

$$|a_3| \leq \frac{(1-\beta)^2(4\lambda-2\lambda^2-1)}{4(2\lambda-1)^2(3\lambda-1)} \tag{8}$$

$$|a_4| \leq \frac{1-\beta}{24(4\lambda-1)} + (1-\beta)^3 \frac{24\lambda^4-80\lambda^3+84\lambda^2-28\lambda+3}{24(2\lambda-1)^3(3\lambda-1)(4\lambda-1)} \tag{9}$$

or

$$|a_4| \leq \begin{cases} \frac{1-\beta}{24(4\lambda-1)} - \frac{(1-\beta)^3}{8(2\lambda-1)^2(4\lambda-1)} \times (F_1 + F_2 + F_3), & \lambda > 1 \\[2mm] \frac{1-\beta}{24(4\lambda-1)} - \frac{(1-\beta)^3}{8(2\lambda-1)^2(4\lambda-1)} F_1, & 1 \leq \lambda < 2 \\[2mm] \frac{1-\beta}{24(4\lambda-1)} - \frac{(1-\beta)^3}{8(2\lambda-1)^2(4\lambda-1)} (F_1 + F_2), & 1 < \lambda \leq 2 \end{cases}$$

where
$F_1 = \frac{4\lambda-2\lambda^2-1}{3\lambda-1}$, $F_2 = \frac{6\lambda(\lambda-1)(4\lambda-2\lambda^2-1)}{8(2\lambda-1)(3\lambda-1)}$ *and* $F_3 = \frac{4\lambda(\lambda-1)(\lambda-2)}{3(2\lambda-1)}$.

Proof. Let $f(z) \in \mathcal{L}_\lambda^\beta(\Phi)$. By Definition there exists $\Phi(z) \in \mathcal{P}$ such that

$$\left(\frac{z(f(z))^\lambda}{f(z)}\right) = \beta + (1-\beta)\Phi(z) \quad (z \in \mathbb{U} \lambda \geq 1 \in \mathbb{R}). \tag{10}$$

where the function $\Phi(z)$ is a modified sigmoid function given by

$$\Phi(z) = 1 + \frac{1}{2}z - \frac{1}{24}z^3 + \frac{1}{240}z^5 - \frac{1}{64}z^6 + \frac{779}{20160}z^7 - \dots \quad . \tag{11}$$

Thus

$$z(f(z))^\lambda = f(z)[\beta + (1-\beta)\Phi(z)] \tag{12}$$

In view of (10), (11) and (12), expanding in series forms we have

$$z + 2\lambda\, a_2 z^2 + [3\lambda a_3 + 2\lambda(\lambda-1)a_2^2]z^3$$
$$+ \left[4\lambda a_4 + 6\lambda(\lambda-1)a_2 a_3 + \frac{4}{3}\lambda(\lambda-1)(\lambda-2)a_2^3\right]z^4 + \dots \quad .$$

$$= z + \left(\frac{1-\beta}{2} + a_2\right) z^2 + \left(a_3 + \frac{1-\beta}{2}a_2\right) z^3$$
$$+ \left[a_4 - \frac{1-\beta}{24} + \frac{1-\beta}{2}a_3\right] z^4 + \left[a_5 - \frac{1-\beta}{24}a_2 + \frac{1-\beta}{2}a_4\right] z^4 ... \quad . \quad (13)$$

Comparing the coefficients of z, z^2 and z^3 and z^4 in(13), we obtain

$$a_2 = \frac{1-\beta}{2(2\lambda-1)} \tag{14}$$

and

$$a_3 = \frac{(1-\beta)^2(4\lambda - 2\lambda^2 - 1)}{4(2\lambda-1)^2(3\lambda-1)}, \tag{15}$$

$$(4\lambda-1)a_4 = \frac{1-\beta}{24} - \frac{1-\beta}{2}a_3 + 6\lambda(\lambda-1)a_2a_3 + \frac{4}{3}\lambda(\lambda-1)(\lambda-2)a_2^3$$
$$= \frac{1-\beta}{24} - \frac{(1-\beta)^3(4\lambda-2\lambda^2-1)}{8(2\lambda-1)^2(3\lambda-1)}$$
$$+ 6\lambda(\lambda-1)\frac{(1-\beta)^3(4\lambda-2\lambda^2-1)}{8(2\lambda-1)^3(3\lambda-1)}$$
$$+ \frac{4}{3}\lambda(\lambda-1)(\lambda-2)\frac{(1-\beta)^3}{8(2\lambda-1)^3}$$

$$a_4 = \frac{1-\beta}{24(4\lambda-1)} - \frac{(1-\beta)^3}{8(2\lambda-1)^2(4\lambda-1)}$$
$$\times \left(\frac{4\lambda-2\lambda^2-1}{3\lambda-1} + \frac{6\lambda(\lambda-1)(4\lambda-2\lambda^2-1)}{8(2\lambda-1)(3\lambda-1)} + \frac{4\lambda(\lambda-1)(\lambda-2)}{3(2\lambda-1)}\right)$$

By simple computation we get

$$a_4 = \frac{1-\beta}{24(4\lambda-1)} - (1-\beta)^3\frac{24\lambda^4 - 80\lambda^3 + 84\lambda^2 - 28\lambda + 3}{24(2\lambda-1)^3(3\lambda-1)(4\lambda-1)} \tag{16}$$

□

.

Corollary 2.6.   *If $f(z) \in \mathcal{A}$ given by (1) belongs to $\mathcal{L}_1^\beta(\Phi) \equiv \mathcal{S}^*(\beta, \Phi)$, then*

$$|a_2| \leq \frac{1-\beta}{2}; \qquad |a_3| \leq \frac{(1-\beta)^2}{8} \quad \text{and} \quad |a_4| \leq \frac{1-\beta}{72} + \frac{(1-\beta)^3}{48}.$$

Corollary 2.7. *If $f(z) \in \mathcal{A}$ given by (1) belongs to $\mathcal{L}_2^\beta(\Phi) \equiv \mathcal{G}(\beta, \Phi)$, then*

$$|a_2| \leq \frac{1-\beta}{6}; \qquad |a_3| \leq \frac{(1-\beta)^2}{180} \quad \text{and} \quad |a_4| \leq \frac{1-\beta}{168} + \frac{27(1-\beta)^3}{22680}.$$

By taking $\beta = 0$ in Corollary 2.6 and 2.7we get

Corollary 2.8. *If $f(z) \in \mathcal{A}$ given by (1) belongs to $\mathcal{L}_1^0(\Phi) \equiv \mathcal{S}^*(\Phi)$, then*

$$|a_2| \leq \frac{1}{2}; \qquad |a_3| \leq \frac{1}{8} \quad \text{and} \quad |a_4| \leq \frac{1}{144}.$$

Corollary 2.9. *If $f(z) \in \mathcal{A}$ given by (1) belongs to $\mathcal{L}_2^0(\Phi(z)) \equiv \mathcal{G}(\Phi)$, then*

$$|a_2| \leq \frac{1}{6}; \qquad |a_3| \leq \frac{1}{180} \quad \text{and} \quad |a_4| \leq \frac{1}{140}.$$

3. The Fekete-Szegö Inequality

Recently there has been interest to obtain the Fekete-Szegö inequality for the subclasses of $\mathcal{S}$ see the works of Ma and Minda [7] and Deniz and Orhan [2](and also see the references cited therein). In this section making use of the values of a_2 and a_3,we prove the following Fekete-Szegö result for the function class $\mathcal{L}_\lambda^\beta(\Phi)$.

Theorem 3.1. *If $f(z) \in \mathcal{A}$ given by (1) be in the class $\mathcal{L}_\lambda^\beta(\Phi)$ and $\mu \in \mathbb{R}$. Then*

$$|a_3 - \mu a_2^2| \leq \frac{(1-\beta)^2}{4(2\lambda - 1)^2} \left| \frac{4\lambda - 2\lambda^2 - 1}{3\lambda - 1} - \mu \right|. \tag{17}$$

Proof. From (14) and (15) we get

$$a_3 - \mu a_2^2 = \frac{(1-\beta)^2(4\lambda - 2\lambda^2 - 1)}{4(2\lambda - 1)^2(3\lambda - 1)} - \mu \left(\frac{1-\beta}{2(2\lambda - 1)} \right)^2 \tag{18}$$

By simple calculation we get

$$a_3 - \mu a_2^2 = \frac{(1-\beta)^2}{4(2\lambda - 1)^2} \left[\frac{4\lambda - 2\lambda^2 - 1}{3\lambda - 1} - \mu \right].$$

Hence, we have

$$|a_3 - \mu a_2^2| \leq \frac{(1-\beta)^2}{4(2\lambda - 1)^2} \left| \frac{4\lambda - 2\lambda^2 - 1}{3\lambda - 1} - \mu \right|. \tag{19}$$

which completes the proof. $\qquad \square$

For taking $\mu = 1$ we get

Remark 3.2. If $f(z) \in \mathcal{A}$ given by (1) be in the class $\mathcal{L}_\lambda^\beta(\Phi)$ then

$$|a_3 - a_2^2| \le \frac{(1-\beta)^2}{4(2\lambda-1)^2} \left| \frac{4\lambda - \lambda^2}{3\lambda - 1} \right|. \tag{20}$$

Theorem 3.3. If $f(z) \in \mathcal{A}$ given by (1) be in the class $\mathcal{L}_\lambda^\beta(\Phi)$, then

$$|a_2 a_4 - a_3^2|$$
$$\le \frac{(1-\beta)^2}{48(2\lambda-1)(4\lambda-1)} \left| 1 - \frac{\lambda(1-\beta)^2[24\lambda^4 - 60\lambda^3 + 44\lambda^2 - 12\lambda + 1]}{(2\lambda-1)^3(3\lambda-1)^2} \right|. \tag{21}$$

Proof. From (14),(15)and (16) we get

$$a_2 a_4 = \frac{(1-\beta)^2}{48(2\lambda-1)(4\lambda-1)}$$
$$\times \left[1 - \frac{(1-\beta)^2[24\lambda^4 - 80\lambda^3 + 84\lambda^2 - 28\lambda + 3]}{(2\lambda-1)^3(3\lambda-1)} \right] \tag{22}$$

$$a_2 a_4 - a_3^2 = \frac{(1-\beta)^2}{48(2\lambda-1)(4\lambda-1)} \left[1 - \frac{(1-\beta)^2[24\lambda^4 - 80\lambda^3 + 84\lambda^2 - 28\lambda + 3]}{(2\lambda-1)^3(3\lambda-1)} \right.$$
$$\left. - \frac{3(1-\beta)^2(4\lambda-1)(4\lambda - 2\lambda^2 - 1)}{(2\lambda-1)^3(3\lambda-1)^3} \right]$$
$$= \frac{(1-\beta)^2}{48(2\lambda-1)(4\lambda-1)} \left[1 - \frac{\lambda(1-\beta)^2[24\lambda^4 - 60\lambda^3 + 44\lambda^2 - 12\lambda + 1]}{(2\lambda-1)^3(3\lambda-1)^2} \right]$$

which gives the desired inequality (21). $\square$

4. Concluding Remarks

In fact, by appropriately selecting the values of λ and β we state the interesting Fekete-Szegö inequality results for the subclasses $\mathcal{S}^*(\beta, \Phi)$ and $\mathcal{G}(\beta, \Phi)$.

References

[1] K. O. Babalola, On λ - pseudostarlike functions, *Journal of Classical Analysis*, **3** (2013), No.2, 137 - 147.

[2] E. Deniz and H. Orhan, The Fekete-Szegö problem for a generalized sub-class of analytic functions, *Kyungpook Math. J.*, **50** (2010), 37-47.

[3] P. L. Duren, *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften, Volume 259, Springer-Verlag, New York (1983).

[4] O. A. Fadipe-Joseph, A. T. Oladipo, U. A. Ezeafulukwe, Modified sigmoid function in univalent function theory, *International Journal of Mathematical Sciences and Engineering Application,* **7**, No. 7 (2013), 313-317.

[5] Y. C. Kim and T. Sugawa, A note on Bazilevič functions, *Taiwanese J. Math.*, **13** (2009), 1489-1495.

[6] S. S. Miller and P. T. Mocanu, *Differential Subordinations: Theory and Applications*, Series on Monographs and Text Books in Pure and Applied Mathematics, Volume 225, Marcel Dekker, New York (2000).

[7] W. Ma and D. Minda, A unified treatment of some special classes of univalent functions, In: *Proceeding of the Conference on Complex Analysis* (Ed-s: Z. Li, F. Ren, L. Yang and S. Zhang), Int. Press (1994), 157-169.

[8] S.O. Olatunji, A.M. Gbolagade, T. Anake and O.A. Fadipe-Joseph, Sigmoid function in the space of univalent function of Bazilevic type, *Scientia Magna,* **97**, No. 3 (2013), 43-51.

[9] C. Pommerenke, *Univalent Functions*, Vandenhoeck and Ruprecht, Göttingen, 1975.

[10] R. Singh, On Bazilevič functions, *Proc. Amer. Math. Soc.*, **28** (1973), 261-271.