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Research Article

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Wolfe-type second-order fractional symmetric duality

Abstract: In the present paper, we examine duality results for Wolfe-type second-order fractional symmetric dual programs. These duality results are then used to investigate minimax mixed integer symmetric dual fractional programs. We also discuss self-duality results at the end.

Keywords: Symmetric duality, bonvexity, fractional programming, minimax, duality

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1 Introduction

A pair of primal and dual problems is called symmetric when in the case where the dual is expressed in the form of the primal, then its dual is the primal. Dorn [7] was the first to introduce symmetric duality in the literature. Since then, researchers became engaged in deriving results on different aspects of symmetric duality. Since more parameters are involved, a second-order dual provides tighter bounds for the value of the objective function of the primal problem when approximations are used. Suneja, Lalitha and Khurana [18] studied a pair of Mond–Weir-type second-order dual programs and established the appropriate duality results under η-bonvexity/η-pseudobonvexity assumptions. Gulati and Gupta [8] studied a pair of Wolfe-type second-order symmetric dual programs involving nondifferentiable functions and derived appropriate duality theorems under η_1-bonvexity/η_2-boncavity.

Ahmad [3] considered a pair of Mond–Weir second-order symmetric nondifferentiable multiobjective programs and derived weak, strong and converse duality theorems under η-pseudobonvexity assumptions. Yang, Yang, Teo and Hou [22] derived the appropriate duality results for a pair of second-order symmetric dual programs in multiobjective nonlinear programming under F-convexity. Later on, Gupta and Kailey [9] pointed out some deficiency in strong duality results given in Yang, Yang, Teo and Hou [22] and gave the correct form of the theorem. Mandal and Nahak [11] studied symmetric duality under $(p, r) - \rho - (\eta, \theta)$-invexity assumptions and derived the duality theorems.

Yang, Yang and Teo [21] formulated a pair of Wolfe-type nondifferentiable second-order symmetric primal and dual problems in mathematical programming and established weak and strong duality theorems under second order F-convexity assumptions. They also studied symmetric minimax mixed integer primal and dual problems. Verma and Gulati [20] studied a pair of Wolfe-type nondifferentiable multiobjective second-order symmetric dual programs involving two kernel functions and derived weak, strong and converse duality theorems for this pair under invexity assumptions.

Agarwal, Ahmad, Gupta and Kailey [1] considered a pair of second-order mixed symmetric dual programs involving nondifferentiable functions and proved weak, strong, and converse duality theorems using the notion of second-order F-convexity/pseudoconvexity assumptions. Ahmad [2] focused his attention on mixed symmetric dual programs without nonnegative constraints and derived weak, strong, converse and self-duality theorems. Kim, Lee and Lee [10] derived appropriate duality results for a pair of multiobjective second-order symmetric dual programs, where the objective function contains a support function. Tripathy and Devi [19] extended the results of Ahmad [2], Suneja, Lalitha and Khurana [18] etc. by considering a pair of second-order mixed symmetric duals for a class of nondifferentiable multiobjective programming involving a square root term and established weak duality, strong duality and converse duality theorems under second order (ϕ, ρ)-invexity and (ϕ, ρ)-pseudoinvexity assumptions.

Economic applications often require maximizing the efficiency of an economic system resulting in problems whose objective function is a ratio. Examples of such economic applications include maximization of productivity, maximization of return on investment, minimization of cost/time. Fractional programming deals with all these situations. Apart from these, fractional programming can be used in data envelopment analysis, tax programming, risk and portfolio theory (see, for example, [4, 6, 16, 17]). Keeping this point of view, in the present paper we derive duality results for Wolfe-type second-order fractional symmetric dual programs. We also discuss minimax mixed integer programming problems. The structure of the paper is as follows. In Section 2, we present some preliminaries and definitions. We derive the weak, strong and converse duality theorems in Section 3. In Section 4, we discuss the minimax mixed integer programming problem and self duality in Section 5 followed by conclusion at last.

2 Preliminaries

Let $S_1 \subset \mathbb{R}^n$ and $S_2 \subset \mathbb{R}^m$ be open sets and let $f(x, y)$ be a real-valued twice differentiable function defined on $S_1 \times S_2$. Then, $\nabla_x f$ and $\nabla_y f$ denote gradient vectors of f with respect to x and y respectively and $\nabla_{xy} f$ denotes the $n \times m$ matrix of second-order partial derivatives. All vectors shall be considered as column vectors.

Definition 2.1 ([15]). The function $f(x, y)$ is said to be η_1-bonvex in the first variable at $u \in S_1$ for fixed $v \in S_2$ if there exists a function $\eta_1 : S_1 \times S_1 \to \mathbb{R}^n$, such that for $x \in S_1$ and $q \in \mathbb{R}^n$, we have

$$f(x, v) - f(u, v) + \frac{1}{2} q^\top \nabla_{xx} f(u, v) q \geq \eta_1^\top(x, u)(\nabla_x f(u, v) + \nabla_{xx} f(u, v) q).$$

Definition 2.2 ([15]). The function $f(x, y)$ is said to be η_2-bonvex in the second variable at $v \in S_2$ for fixed $u \in S_1$ if there exists a function $\eta_2 : S_2 \times S_2 \to \mathbb{R}^m$, such that for $y \in S_2$ and $p \in \mathbb{R}^m$ we have

$$f(u, y) - f(u, v) + \frac{1}{2} p^\top \nabla_{yy} f(u, v) p \geq \eta_2^\top(y, v)(\nabla_y f(u, v) + \nabla_{yy} f(u, v) p).$$

3 Second-order fractional symmetric duality

In this paper, we consider the following pair of Wolfe-type second-order fractional symmetric dual programs. Consider the primal problem

$$\min \frac{f(x, y) - y^\top(\nabla_y f(x, y) + \nabla_{yy} f(x, y) p) - \frac{1}{2} p^\top \nabla_{yy} f(x, y) p}{g(x, y) - y^\top(\nabla_y g(x, y) + \nabla_{yy} g(x, y) p) - \frac{1}{2} p^\top \nabla_{yy} g(x, y) p} \tag{WFP}$$

subject to

$$\Big[(g(x, y) - y^\top(\nabla_y g(x, y) + \nabla_{yy} g(x, y) p) - \frac{1}{2} p^\top \nabla_{yy} g(x, y) p)(\nabla_y f(x, y) + \nabla_{yy} f(x, y) p) \\ - (f(x, y) - y^\top(\nabla_y f(x, y) + \nabla_{yy} f(x, y) p) - \frac{1}{2} p^\top \nabla_{yy} f(x, y) p)(\nabla_y g(x, y) + \nabla_{yy} g(x, y) p)\Big] \leq 0,$$

for $x \geq 0$, and the dual problem

$$\max \frac{f(u, v) - u^\top(\nabla_x f(u, v) + \nabla_{xx} f(u, v) q) - \frac{1}{2} q^\top \nabla_{xx} f(u, v) q}{g(u, v) - u^\top(\nabla_x g(u, v) + \nabla_{xx} g(u, v) q) - \frac{1}{2} q^\top \nabla_{xx} g(u, v) q} \tag{WFD}$$

subject to

$$\Big[(g(u,v) - u^{\mathsf T}(\nabla_x g(u,v) + \nabla_{xx} g(u,v)q) - \frac{1}{2}q^{\mathsf T}\nabla_{xx} g(u,v)q)(\nabla_x f(u,v) + \nabla_{xx} f(u,v)q)$$
$$- (f(u,v) - u^{\mathsf T}(\nabla_x f(u,v) + \nabla_{xx} f(u,v)q) - \frac{1}{2}q^{\mathsf T}\nabla_{xx} f(u,v)q)(\nabla_x g(u,v) + \nabla_{xx} g(u,v)q)\Big] \geq 0,$$

for $v \geq 0$, where $f : S_1 \times S_2 \to \mathbb{R}_+$ and $g : S_1 \times S_2 \to \mathbb{R}_+ \setminus \{0\}$ are differentiable functions, and p and q are vectors in $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. It is assumed that in the feasible regions the numerator is nonnegative and the denominator is positive.

For notational convenience, we can express the programs (WFP) and (WFD) in the following equivalent form. For the primal problem we have

$$\min\ l \tag{EWFP}$$

subject to

$$f(x,y) - y^{\mathsf T}(\nabla_y f(x,y) + \nabla_{yy} f(x,y)p) - \frac{1}{2}p^{\mathsf T}\nabla_{yy} f(x,y)p$$
$$- l(g(x,y) - y^{\mathsf T}(\nabla_y g(x,y) + \nabla_{yy} g(x,y)p) - \frac{1}{2}p^{\mathsf T}\nabla_{yy} g(x,y)p) = 0 \tag{3.1}$$

and

$$(\nabla_y f(x,y) + \nabla_{yy} f(x,y)p) - l(\nabla_y g(x,y) + \nabla_{yy} g(x,y)p) \leq 0, \tag{3.2}$$

for $x \geq 0$. For the dual problem we have

$$\max\ m \tag{EWFD}$$

subject to

$$f(u,v) - u^{\mathsf T}(\nabla_x f(u,v) + \nabla_{xx} f(u,v)q) - \frac{1}{2}q^{\mathsf T}\nabla_{xx} f(u,v)q$$
$$- m(g(u,v) - u^{\mathsf T}(\nabla_x g(u,v) + \nabla_{xx} g(u,v)q) - \frac{1}{2}q^{\mathsf T}\nabla_{xx} g(u,v)q) = 0 \tag{3.3}$$

and

$$(\nabla_x f(u,v) + \nabla_{xx} f(u,v)q) - m(\nabla_x g(u,v) + \nabla_{xx} g(u,v)q) \geq 0, \tag{3.4}$$

for $v \geq 0$.

Theorem 3.1 (Weak duality). *Let (x,y,l,p) and (u,v,m,q) be feasible solutions to* (EWFP) *and* (EWFD) *respectively. Assume that*
(i) *$f(\cdot,v) - mg(\cdot,v)$ is η_1-bonvex in the first variable at u for fixed v,*
(ii) *$-f(x,\cdot) + lg(x,\cdot)$ is η_2-bonvex in the second variable at y for fixed x,*
(iii) *$\eta_1(x,u) + u \geq 0$ and $\eta_2(v,y) + y \geq 0$,*
(iv) *$g(x,v) > 0$.*
Then, $l \geq m$.

Proof. By the η_1-bonvexity of $f(\cdot,v) - mg(\cdot,v)$ in the first variable at u for fixed v, we have

$$f(x,v) - mg(x,v) - (f(u,v) - mg(u,v)) + \frac{1}{2}q^{\mathsf T}\nabla_{xx}(f(u,v) - mg(u,v))q$$
$$\geq \eta_1^{\mathsf T}(x,u)(\nabla_x(f(u,v) - mg(u,v)) + \nabla_{xx}(f(u,v) - mg(u,v))q). \tag{3.5}$$

By the η_2-bonvexity of $-f(x,\cdot) + lg(x,\cdot)$ in the second variable at y for fixed x, we have

$$- f(x,v) + lg(x,v) - (-f(x,y) + lg(x,y)) + \frac{1}{2}p^{\mathsf T}\nabla_{yy}(-f(x,y) + lg(x,y))p$$
$$\geq \eta_2^{\mathsf T}(v,y)(\nabla_y(-f(x,y) + lg(x,y)) + \nabla_{yy}(-f(x,y) + lg(x,y))p). \tag{3.6}$$

On adding (3.5) and (3.6), we get

$$\begin{aligned} & f(x,v) - mg(x,v) - (f(u,v) - mg(u,v)) + \frac{1}{2}q^{\mathsf T}\nabla_{xx}(f(u,v) - mg(u,v))q \\ & \qquad - f(x,v) + lg(x,v) - (-f(x,y) + lg(x,y)) + \frac{1}{2}p^{\mathsf T}\nabla_{yy}(-f(x,y) + lg(x,y))p \\ & \geq \eta_1^{\mathsf T}(x,u)(\nabla_x(f(u,v) - mg(u,v)) + \nabla_{xx}(f(u,v) - mg(u,v))q) \\ & \qquad + \eta_2^{\mathsf T}(v,y)(\nabla_y(-f(x,y) + lg(x,y)) + \nabla_{yy}(-f(x,y) + lg(x,y))p). \end{aligned} \tag{3.7}$$

From the dual constraint (3.4) and the condition $\eta_1(x,u) + u \geq 0$ we have

$$(\eta_1(x,u) + u)^{\mathsf T}[\nabla_x f(u,v) + \nabla_{xx} f(u,v)q - m(\nabla_x g(u,v) + \nabla_{xx} g(u,v)q)] \geq 0,$$

or

$$\begin{aligned} & \eta_1^{\mathsf T}(x,u)[\nabla_x f(u,v) + \nabla_{xx} f(u,v)q - m(\nabla_x g(u,v) + \nabla_{xx} g(u,v)q)] \\ & \qquad \geq -u^{\mathsf T}[\nabla_x f(u,v) + \nabla_{xx} f(u,v)q - m(\nabla_x g(u,v) + \nabla_{xx} g(u,v)q)]. \end{aligned} \tag{3.8}$$

Similarly, from the primal constraint (3.2) and the condition $\eta_2(v,y) + y \geq 0$ we have

$$-(\eta_2(v,y) + y)^{\mathsf T}[\nabla_y f(x,y) + \nabla_{yy} f(x,y)p - l(\nabla_y g(x,y) + \nabla_{yy} g(x,y)p)] \geq 0,$$

or

$$\begin{aligned} & -\eta_2^{\mathsf T}(v,y)[\nabla_y f(x,y) + \nabla_{yy} f(x,y)p - l(\nabla_y g(x,y) + \nabla_{yy} g(x,y)p)] \\ & \qquad \geq y^{\mathsf T}[\nabla_y f(x,y) + \nabla_{yy} f(x,y)p - l(\nabla_y g(x,y) + \nabla_{yy} g(x,y)p)]. \end{aligned} \tag{3.9}$$

From (3.7), (3.8) and (3.9), we conclude that

$$\begin{aligned} & f(x,v) - mg(x,v) - (f(u,v) - mg(u,v)) + \frac{1}{2}q^{\mathsf T}\nabla_{xx}(f(u,v) - mg(u,v))q \\ & \qquad - f(x,v) + lg(x,v) - (-f(x,y) + lg(x,y)) + \frac{1}{2}p^{\mathsf T}\nabla_{yy}(-f(x,y) + lg(x,y))p \\ & \geq -u^{\mathsf T}[\nabla_x f(u,v) + \nabla_{xx} f(u,v)q - m(\nabla_x g(u,v) + \nabla_{xx} g(u,v)q)] \\ & \qquad + y^{\mathsf T}[\nabla_y f(x,y) + \nabla_{yy} f(x,y)p - l(\nabla_y g(x,y) + \nabla_{yy} g(x,y)p)], \end{aligned}$$

which on using (3.1) and (3.3) reduces to

$$(l - m)g(x,v) \geq 0.$$

Since $g(x,v) > 0$, it follows from the above inequality that

$$l \geq m.$$

Hence, the theorem follows. □

Theorem 3.2 (Strong duality). *Let f and g be thrice continuously differentiable functions. Let $(\bar{x}, \bar{y}, \bar{l}, \bar{p})$ be an optimal solution of* (EWFP). *Assume that*
(i) $\nabla_{yy} f(\bar{x},\bar{y}) - \bar{l}\nabla_{yy} g(\bar{x},\bar{y})$ *is nonsingular,*
(ii) *the equality* $\bar{p}^{\mathsf T}(\nabla_y(\nabla_{yy} f(\bar{x},\bar{y})\bar{p} - \bar{l}\nabla_{yy} g(\bar{x},\bar{y})\bar{p})) = 0$ *implies that* $\bar{p} = 0$,
(iii) $(\bar{x}^{\mathsf T}\nabla_x g(\bar{x},\bar{y}) - \bar{y}^{\mathsf T}\nabla_y g(\bar{x},\bar{y}))f(\bar{x},\bar{y}) + (\bar{y}^{\mathsf T}\nabla_y f(\bar{x},\bar{y}) - \bar{x}^{\mathsf T}\nabla_x f(\bar{x},\bar{y}))g(\bar{x},\bar{y}) = 0$.
Then, $(\bar{x}, \bar{y}, \bar{l}, \bar{q} = 0)$ *is a feasible solution for* (EWFD) *and the objective values of* (EWFP) *and* (EWFD) *are equal. Furthermore, if the hypotheses of Theorem 3.1 are satisfied, then* $(\bar{x}, \bar{y}, \bar{l}, \bar{q} = 0)$ *is an optimal solution of* (EWFD).

Proof. Since $(\bar{x}, \bar{y}, \bar{l}, \bar{p})$ is an optimal solution of (EWFP), there exist $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$, $\gamma \in \mathbb{R}^m$, $\mu \in \mathbb{R}^n$, such that the following Fritz John conditions are satisfied at $(\bar{x}, \bar{y}, \bar{l}, \bar{p})$:

$$\begin{aligned} & \beta(\nabla_x f(\bar{x},\bar{y}) - \bar{l}\nabla_x g(\bar{x},\bar{y})) + (\gamma - \beta\bar{y})^{\mathsf T}(\nabla_{yx} f(\bar{x},\bar{y}) - \bar{l}\nabla_{yx} g(\bar{x},\bar{y})) \\ & \qquad + \left(\gamma - \beta\bar{y} - \frac{\beta\bar{p}}{2}\right)^{\mathsf T}(\nabla_x(\nabla_{yy} f(\bar{x},\bar{y})\bar{p}) - \bar{l}\nabla_x(\nabla_{yy} g(\bar{x},\bar{y})\bar{p})) = \mu, \end{aligned} \tag{3.10}$$

$$\begin{aligned} & (\gamma - \beta\bar{y} - \beta\bar{p})^{\mathsf T}(\nabla_{yy} f(\bar{x},\bar{y}) - \bar{l}\nabla_{yy} g(\bar{x},\bar{y})) \\ & \qquad + \left(\gamma - \beta\bar{y} - \frac{\beta\bar{p}}{2}\right)^{\mathsf T}(\nabla_y(\nabla_{yy} f(\bar{x},\bar{y})\bar{p}) - \bar{l}\nabla_y(\nabla_{yy} g(\bar{x},\bar{y})\bar{p})) = 0, \end{aligned} \tag{3.11}$$

$$\gamma^{\mathsf{T}}(\nabla_y f(\bar{x},\bar{y}) - \bar{l}\nabla_y g(\bar{x},\bar{y}) + \nabla_{yy} f(\bar{x},\bar{y})\bar{p} - \bar{l}g(\bar{x},\bar{y})\bar{p}) = 0, \tag{3.12}$$

$$\alpha - \beta\Big(g(\bar{x},\bar{y}) - \bar{y}^{\mathsf{T}}(\nabla_y g(\bar{x},\bar{y}) + \nabla_{yy} g(\bar{x},\bar{y})\bar{p}) - \frac{1}{2}\bar{p}^{\mathsf{T}}\nabla_{yy} g(\bar{x},\bar{y})\bar{p}\Big) - \gamma(\nabla_y g(\bar{x},\bar{y}) + \nabla_{yy} g(\bar{x},\bar{y})\bar{p}) = 0, \tag{3.13}$$

$$(\gamma - \beta\bar{y} - \beta\bar{p})^{\mathsf{T}}(\nabla_{yy} f(\bar{x},\bar{y}) - \bar{l}\nabla_{yy} g(\bar{x},\bar{y})) = 0, \tag{3.14}$$

$$\mu^{\mathsf{T}} x = 0, \tag{3.15}$$

$$(\alpha,\beta,\gamma,\mu) \neq 0, \quad (\alpha,\beta,\gamma,\mu) \geq 0. \tag{3.16}$$

Since $\nabla_{yy} f(\bar{x},\bar{y}) - \bar{l}\nabla_{yy} g(\bar{x},\bar{y})$ is nonsingular, it follows from (3.14) that

$$\gamma = \beta(\bar{y} + \bar{p}). \tag{3.17}$$

Now we claim that $\beta \neq 0$. If $\beta = 0$, then from (3.17) we get $\gamma = 0$. This together with (3.10) yields that $\mu = 0$. From (3.13) we have $\alpha = 0$, which contradicts (3.16). Hence, $\beta \neq 0$. From now we assume that $\beta > 0$. Now it follows from (3.11) and (3.17) and assumption (ii) that $\bar{p} = 0$. In particular, by (3.17), $\beta > 0$ and since $\gamma \geq 0$, we have $\bar{y} \geq 0$. Also, it follows from (3.10) that

$$\nabla_x f(\bar{x},\bar{y}) - \bar{l}\nabla_x g(\bar{x},\bar{y}) = \frac{\mu}{\beta} \geq 0. \tag{3.18}$$

Therefore, $(\bar{x},\bar{y},\bar{l},\bar{p} = 0)$ is a feasible solution for the dual problem (EWFD).

It remains to show that the objective values of the two problems have the same value. This is equivalent to the assertion that

$$\frac{f(\bar{x},\bar{y}) - \bar{x}^{\mathsf{T}}\nabla_x f(\bar{x},\bar{y})}{g(\bar{x},\bar{y}) - \bar{x}^{\mathsf{T}}\nabla_x g(\bar{x},\bar{y})} = \frac{f(\bar{x},\bar{y}) - \bar{y}^{\mathsf{T}}\nabla_y f(\bar{x},\bar{y})}{g(\bar{x},\bar{y}) - \bar{y}^{\mathsf{T}}\nabla_y g(\bar{x},\bar{y})}.$$

Now, multiplying (3.18) by $\bar{x}^{\mathsf{T}}$ and using (3.15), we get

$$\frac{\bar{x}^{\mathsf{T}}\nabla_x f(\bar{x},\bar{y})}{\bar{x}^{\mathsf{T}}\nabla_x g(\bar{x},\bar{y})} = \bar{l}. \tag{3.19}$$

Again, since $\bar{p} = 0$, we get from (3.17) that

$$\gamma = \beta\bar{y}. \tag{3.20}$$

Using (3.20) and the conclusion that $\bar{p} = 0$ in (3.12), we have

$$\frac{\bar{y}^{\mathsf{T}}\nabla_y f(\bar{x},\bar{y})}{\bar{y}^{\mathsf{T}}\nabla_y g(\bar{x},\bar{y})} = \bar{l}. \tag{3.21}$$

From (3.19) and (3.21), we get

$$\frac{\bar{x}^{\mathsf{T}}\nabla_x f(\bar{x},\bar{y})}{\bar{x}^{\mathsf{T}}\nabla_x g(\bar{x},\bar{y})} = \frac{\bar{y}^{\mathsf{T}}\nabla_y f(\bar{x},\bar{y})}{\bar{y}^{\mathsf{T}}\nabla_y g(\bar{x},\bar{y})},$$

that is,

$$(\bar{x}^{\mathsf{T}}\nabla_x f(\bar{x},\bar{y}))(\bar{y}^{\mathsf{T}}\nabla_y g(\bar{x},\bar{y})) = (\bar{x}^{\mathsf{T}}\nabla_x g(\bar{x},\bar{y}))(\bar{y}^{\mathsf{T}}\nabla_y f(\bar{x},\bar{y})). \tag{3.22}$$

By assumption (iii), we get

$$\bar{x}^{\mathsf{T}}\nabla_x g(\bar{x},\bar{y}) \cdot f(\bar{x},\bar{y}) + \bar{y}^{\mathsf{T}}\nabla_y f(\bar{x},\bar{y}) \cdot g(\bar{x},\bar{y}) = \bar{y}^{\mathsf{T}}\nabla_y g(\bar{x},\bar{y}) \cdot f(\bar{x},\bar{y}) + \bar{x}^{\mathsf{T}}\nabla_x f(\bar{x},\bar{y}) \cdot g(\bar{x},\bar{y}). \tag{3.23}$$

On subtracting (3.23) from (3.22) and then adding $f(\bar{x},\bar{y}) \cdot g(\bar{x},\bar{y})$ to both sides, we obtain

$$\begin{aligned} &f(\bar{x},\bar{y}) \cdot g(\bar{x},\bar{y}) - f(\bar{x},\bar{y}) \cdot \bar{x}^{\mathsf{T}}\nabla_x g(\bar{x},\bar{y}) - \bar{y}^{\mathsf{T}}\nabla_y f(\bar{x},\bar{y}) \cdot g(\bar{x},\bar{y}) + \bar{y}^{\mathsf{T}}\nabla_y f(\bar{x},\bar{y}) \cdot \bar{x}^{\mathsf{T}}\nabla_x g(\bar{x},\bar{y}) \\ &\quad = f(\bar{x},\bar{y}) \cdot g(\bar{x},\bar{y}) - g(\bar{x},\bar{y}) \cdot \bar{x}^{\mathsf{T}}\nabla_x f(\bar{x},\bar{y}) - \bar{y}^{\mathsf{T}}\nabla_y g(\bar{x},\bar{y}) \cdot f(\bar{x},\bar{y}) + \bar{x}^{\mathsf{T}}\nabla_x f(\bar{x},\bar{y}) \cdot \bar{y}^{\mathsf{T}}\nabla_y g(\bar{x},\bar{y}), \end{aligned}$$

which can be rewritten as

$$\frac{f(\bar{x},\bar{y}) - \bar{x}^{\mathsf{T}}\nabla_x f(\bar{x},\bar{y})}{g(\bar{x},\bar{y}) - \bar{x}^{\mathsf{T}}\nabla_x g(\bar{x},\bar{y})} = \frac{f(\bar{x},\bar{y}) - \bar{y}^{\mathsf{T}}\nabla_y f(\bar{x},\bar{y})}{g(\bar{x},\bar{y}) - \bar{y}^{\mathsf{T}}\nabla_y g(\bar{x},\bar{y})}.$$

Under the assumptions of Theorem 3.1, if $(\bar{x},\bar{y},\bar{l},\bar{q} = 0)$ is not an optimal solution of (EWFD), then there exists another feasible solution (u,v,W,q) of (EWFD) such that $\bar{l} < W$. Since $(\bar{x},\bar{y},\bar{l},\bar{p})$ is a feasible solution of (EWFP) by Theorem 3.1, we have $\bar{l} \geq W$, hence contradiction implies that $(\bar{x},\bar{y},\bar{l},\bar{q} = 0)$ is an optimal solution of (EWFD). □

The converse duality theorem is simply stated as its proof would be analogous to that of Theorem 3.2.

Theorem 3.3 (Converse duality). *Let f and g be thrice continuously differentiable functions. Let $(\bar{u}, \bar{v}, \bar{m}, \bar{q})$ be an optimal solution of* (EWFD). *Assume that*
(i) $\nabla_{xx} f(\bar{u}, \bar{v}) - \bar{m}\nabla_{xx} g(\bar{u}, \bar{v})$ *is nonsingular,*
(ii) *the equality* $\bar{q}^{\top}(\nabla_x(\nabla_{xx} f(\bar{u}, \bar{v})\bar{q} - \bar{m}\nabla_{xx} g(\bar{u}, \bar{v})\bar{q})) = 0$ *implies that* $\bar{q} = 0$,
(iii) $(\bar{u}^{\top}\nabla_x g(\bar{u}, \bar{v}) - \bar{v}^{\top}\nabla_y g(\bar{u}, \bar{v})) f(\bar{u}, \bar{v}) + (\bar{v}^{\top}\nabla_y f(\bar{u}, \bar{v}) - \bar{u}^{\top}\nabla_x f(\bar{u}, \bar{v}))g(\bar{u}, \bar{v}) = 0$.
Then, $(\bar{u}, \bar{v}, \bar{m}, \bar{q} = 0)$ *is a feasible solution for* (EWFP) *and the objective values of* (EWFP) *and* (EWFD) *are equal. Furthermore, if the hypotheses of Theorem 3.1 are satisfied, then* $(\bar{u}, \bar{v}, \bar{m}, \bar{q} = 0)$ *is an optimal solution of* (EWFP).

Remark 3.4. If $g(x, y) = 1$ for all x, y in (WFP) and (WFD), then the problem considered in this paper reduces to the problem considered by Mishra [12] and Mond [14]. If in addition $p = 0$ and $q = 0$, then we get the dual formulated by Chandra, Goyal and Husain [5] and Mond [13].

4 Minimax problems

Let U and V be arbitrary sets of integers in $\mathbb{R}^{n_1}$ and $\mathbb{R}^{m_1}$ respectively. We assume that the first n_1 components of x, $0 \le n_1 \le n$, belong to U and that the first m_1 components of y, $0 \le m_1 \le m$, belong to V. Then, we write

$$(x, y) = (x^1, x^2, y^1, y^2),$$

where $x^1 = (x_1, x_2, \dots, x_{n_1}) \in U$ and $y^1 = (y_1, y_2, \dots, y_{m_1}) \in V$, with x^2 and y^2 being the remaining components of x and y respectively.

Definition 4.1. A vector function $\phi(z^1, z^2, \dots, z^n)$, where $z^1, z^2, \dots, z^n$ are elements of an arbitrary vector space, is called multiplicatively separable with respect to z^1 if there exist vector functions $\psi(z^1)$ independent of $z^2, \dots, z^n$ and $\mu(z^2, \dots, z^n)$ independent of z^1 such that

$$\phi(z^1, z^2, \dots, z^n) = \psi(z^1)\mu(z^2, \dots, z^n).$$

We consider the following pair of Wolfe-type symmetric fractional minimax mixed integer primal and dual programs. For the primal program we have

$$\max_{x^1} \min_{x^2, y} \frac{f(x, y) - (y^2)^{\top}(\nabla_{y^2} f(x, y) + \nabla_{y^2y^2} f(x, y)p) - \frac{1}{2}p^{\top}\nabla_{y^2y^2} f(x, y)p}{g(x, y) - (y^2)^{\top}(\nabla_{y^2} g(x, y) + \nabla_{y^2y^2} g(x, y)p) - \frac{1}{2}p^{\top}\nabla_{y^2y^2} g(x, y)p} \tag{WFP*}$$

subject to

$$\Big[(g(x, y) - (y^2)^{\top}(\nabla_{y^2} g(x, y) + \nabla_{y^2y^2} g(x, y)p) - \frac{1}{2}p^{\top}\nabla_{y^2y^2} g(x, y)p)(\nabla_{y^2} f(x, y) + \nabla_{y^2y^2} f(x, y)p)$$
$$- (f(x, y) - (y^2)^{\top}(\nabla_{y^2} f(x, y) + \nabla_{y^2y^2} f(x, y)p) - \frac{1}{2}p^{\top}\nabla_{y^2y^2} f(x, y)p)(\nabla_{y^2} g(x, y) + \nabla_{y^2y^2} g(x, y)p)\Big] \le 0,$$

for $x^2 \ge 0$ and $x^1 \in U$, $y^1 \in V$. For the dual program we have

$$\min_{v^1} \max_{u, v^2} \frac{f(u, v) - (u^2)^{\top}(\nabla_{x^2} f(u, v) + \nabla_{x^2x^2} f(u, v)q) - \frac{1}{2}q^{\top}\nabla_{x^2x^2} f(u, v)q}{g(u, v) - (u^2)^{\top}(\nabla_{x^2} g(u, v) + \nabla_{x^2x^2} g(u, v)q) - \frac{1}{2}q^{\top}\nabla_{x^2x^2} g(u, v)q} \tag{WFD*}$$

subject to

$$\Big[(g(u, v) - (u^2)^{\top}(\nabla_{x^2} g(u, v) + \nabla_{x^2x^2} g(u, v)q) - \frac{1}{2}q^{\top}\nabla_{x^2x^2} g(u, v)q)(\nabla_{x^2} f(u, v) + \nabla_{x^2x^2} f(u, v)q)$$
$$- (f(u, v) - (u^2)^{\top}(\nabla_{x^2} f(u, v) + \nabla_{x^2x^2} f(u, v)q) - \frac{1}{2}q^{\top}\nabla_{x^2x^2} f(u, v)q)(\nabla_{x^2} g(u, v) + \nabla_{x^2x^2} g(u, v)q)\Big] \ge 0,$$

for $v^2 \ge 0$ and $u^1 \in U$, $v^1 \in V$, where p and q are $(m - m_1)$ and $(n - n_1)$-dimensional vector variables.

To prove the symmetric duality, we shall need the following assumptions.

(i) The numerator is nonnegative and the denominator is positive over their feasible regions H and K respectively.

(ii) The functions $f(x,y)$ and $g(x,y)$ are multiplicatively separable with respect to x^1 or y^1, that is,

$$f(x,y) = f^1(x^1)f^2(x^2,y), \quad g(x,y) = g^1(x^1)g^2(x^2,y).$$

(iii) There hold $f^1(x^1) > 0$ and $g^1(x^1) > 0$ for all $x^1 \in U$.

Let

$$h = \max_{x^1}\min_{x^2,y}\left\{\frac{f(x,y)-(y^2)^\top(\nabla_{y^2}f(x,y)+\nabla_{y^2y^2}f(x,y)p)-\frac{1}{2}p^\top\nabla_{y^2y^2}f(x,y)p}{g(x,y)-(y^2)^\top(\nabla_{y^2}g(x,y)+\nabla_{y^2y^2}g(x,y)p)-\frac{1}{2}p^\top\nabla_{y^2y^2}g(x,y)p}\;\middle|\;(x,y,p)\in H\right\}$$

and

$$k = \min_{v^1}\max_{u,v^2}\left\{\frac{f(u,v)-(u^2)^\top(\nabla_{x^2}f(u,v)+\nabla_{x^2x^2}f(u,v)q)-\frac{1}{2}q^\top\nabla_{x^2x^2}f(u,v)q}{g(u,v)-(u^2)^\top(\nabla_{x^2}g(u,v)+\nabla_{x^2x^2}g(u,v)q)-\frac{1}{2}q^\top\nabla_{x^2x^2}g(u,v)q}\;\middle|\;(u,v,q)\in K\right\}.$$

As $f(x,y)$ and $g(x,y)$ are multiplicatively separable with respect to x^1, it follows that

$$f(x,y) = f^1(x^1)f^2(x^2,y)$$

and

$$g(x,y) = g^1(x^1)g^2(x^2,y).$$

Hence h can be written as

$$h = \max_{x^1}\min_{x^2,y}\left\{\frac{f^1(x^1)}{g^1(x^1)}\cdot\frac{f^2(x^2,y)-(y^2)^\top(\nabla_{y^2}f^2(x^2,y)+\nabla_{y^2y^2}f^2(x^2,y)p)-\frac{1}{2}p^\top\nabla_{y^2y^2}f^2(x^2,y)p}{g^2(x^2,y)-(y^2)^\top(\nabla_{y^2}g^2(x^2,y)+\nabla_{y^2y^2}g^2(x^2,y)p)-\frac{1}{2}p^\top\nabla_{y^2y^2}g^2(x^2,y)p}\;\middle|\;(x,y,p)\in H\right\},$$

where

$$\begin{aligned} H = \Big\{(x,y,p)\;\Big|\; & (g^2(x^2,y)-(y^2)^\top(\nabla_{y^2}g^2(x^2,y)+\nabla_{y^2y^2}g^2(x^2,y)p) \\ & -\frac{1}{2}p^\top\nabla_{y^2y^2}g^2(x^2,y)p)(\nabla_{y^2}f^2(x^2,y)+\nabla_{y^2y^2}f^2(x^2,y)p) \\ & -(f^2(x^2,y)-(y^2)^\top(\nabla_{y^2}f^2(x^2,y)+\nabla_{y^2y^2}f^2(x^2,y)p) \\ & -\frac{1}{2}p^\top\nabla_{y^2y^2}f^2(x^2,y)p)(\nabla_{y^2}g^2(x^2,y)+\nabla_{y^2y^2}g^2(x^2,y)p)\le 0,\ \text{for } x^2\ge 0 \text{ and } x^1\in U,\ y^1\in V\Big\}, \end{aligned}$$

that is,

$$h = \max_{x^1}\min_{y^1}\left\{\frac{f^1(x^1)}{g^1(x^1)}\cdot\phi(y^1)\;\middle|\;x^1\in U,\ y^1\in V\right\},$$

where

$$\begin{aligned} \phi(y^1) = \min_{x^2,y^2}\Big\{ & \frac{f^2(x^2,y)-(y^2)^\top(\nabla_{y^2}f^2(x^2,y)+\nabla_{y^2y^2}f^2(x^2,y)p)-\frac{1}{2}p^\top\nabla_{y^2y^2}f^2(x^2,y)p}{g^2(x^2,y)-(y^2)^\top(\nabla_{y^2}g^2(x^2,y)+\nabla_{y^2y^2}g^2(x^2,y)p)-\frac{1}{2}p^\top\nabla_{y^2y^2}g^2(x^2,y)p}\;\Big| \\ & (g^2(x^2,y)-(y^2)^\top(\nabla_{y^2}g^2(x^2,y)+\nabla_{y^2y^2}g^2(x^2,y)p) \\ & -\frac{1}{2}p^\top\nabla_{y^2y^2}g^2(x^2,y)p)(\nabla_{y^2}f^2(x^2,y)+\nabla_{y^2y^2}f^2(x^2,y)p) \\ & -(f^2(x^2,y)-(y^2)^\top(\nabla_{y^2}f^2(x^2,y)+\nabla_{y^2y^2}f^2(x^2,y)p) \\ & -\frac{1}{2}p^\top\nabla_{y^2y^2}f^2(x^2,y)p)(\nabla_{y^2}g^2(x^2,y)+\nabla_{y^2y^2}g^2(x^2,y)p)\le 0,\ \text{for } x^2\ge 0\Big\}. \end{aligned} \tag{4.1}$$

Similarly, k can be written as

$$k = \min_{v^1}\max_{u^1}\left\{\frac{f^1(u^1)}{g^1(u^1)}\cdot\psi(v^1)\;\middle|\;u^1\in U,\ v^1\in V\right\},$$

where

$$\psi(v^1) = \max_{u^2,v^2} \left\{ \frac{f^2(u^2,v) - (v^2)^\top(\nabla_{x^2} f^2(u^2,v) + \nabla_{x^2x^2} f^2(u^2,v)q) - \frac{1}{2}q^\top \nabla_{x^2x^2} f^2(u^2,v)q}{g^2(u^2,v) - (v^2)^\top(\nabla_{x^2} g^2(u^2,v) + \nabla_{x^2x^2} g^2(u^2,v)q) - \frac{1}{2}q^\top \nabla_{x^2x^2} g^2(u^2,v)q} \right| \\ (g^2(u^2,v) - (v^2)^\top(\nabla_{x^2} g^2(u^2,v) + \nabla_{x^2x^2} g^2(u^2,v)q) \\ - \frac{1}{2}q^\top \nabla_{x^2x^2} g^2(u^2,v)q)(\nabla_{x^2} f^2(u^2,v) + \nabla_{x^2x^2} f^2(u^2,v)q) \\ - (f^2(u^2,v) - (v^2)^\top(\nabla_{x^2} f^2(u^2,v) + \nabla_{x^2x^2} f^2(u^2,v)q) \\ \left. - \frac{1}{2}q^\top \nabla_{x^2x^2} f^2(u^2,v)q)(\nabla_{x^2} g^2(u^2,v) + \nabla_{x^2x^2} g^2(u^2,v)q) \geq 0, \text{ for } v^2 \geq 0 \right\}. \tag{4.2}$$

We set

$$a = \frac{f^2(x^2,y) - (y^2)^\top(\nabla_{y^2} f^2(x^2,y) + \nabla_{y^2y^2} f^2(x^2,y)p) - \frac{1}{2}p^\top \nabla_{y^2y^2} f^2(x^2,y)p}{g^2(x^2,y) - (y^2)^\top(\nabla_{y^2} g^2(x^2,y) + \nabla_{y^2y^2} g^2(x^2,y)p) - \frac{1}{2}p^\top \nabla_{y^2y^2} g^2(x^2,y)p}$$

and

$$b = \frac{f^2(u^2,v) - (v^2)^\top(\nabla_{x^2} f^2(u^2,v) + \nabla_{x^2x^2} f^2(u^2,v)q) - \frac{1}{2}q^\top \nabla_{x^2x^2} f^2(u^2,v)q}{g^2(u^2,v) - (v^2)^\top(\nabla_{x^2} g^2(u^2,v) + \nabla_{x^2x^2} g^2(u^2,v)q) - \frac{1}{2}q^\top \nabla_{x^2x^2} g^2(u^2,v)q}$$

in (4.1) and (4.2) respectively and denote them by the auxiliary programs $(\widehat{\text{WFP}})$ and $(\widehat{\text{WFD}})$ respectively. That is,

$$\min\ a \tag{$\widehat{\text{WFP}}$}$$

subject to

$$f^2(x^2,y) - (y^2)^\top(\nabla_{y^2} f^2(x^2,y) + \nabla_{y^2y^2} f^2(x^2,y)p) - \frac{1}{2}p^\top \nabla_{y^2y^2} f^2(x^2,y)p \\ - a\left\{g^2(x^2,y) - (y^2)^\top(\nabla_{y^2} g^2(x^2,y) + \nabla_{y^2y^2} g^2(x^2,y)p) - \frac{1}{2}p^\top \nabla_{y^2y^2} g^2(x^2,y)p\right\} = 0$$

and

$$(\nabla_{y^2} f^2(x^2,y) + \nabla_{y^2y^2} f^2(x^2,y)p) - a(\nabla_{y^2} g^2(x^2,y) + \nabla_{y^2y^2} g^2(x^2,y)p) \leq 0,$$

for $x^2 \geq 0$, and

$$\max\ a \tag{$\widehat{\text{WFD}}$}$$

subject to

$$f^2(u^2,v) - (v^2)^\top(\nabla_{x^2} f^2(u^2,v) + \nabla_{x^2x^2} f^2(u^2,v)q) - \frac{1}{2}q^\top \nabla_{x^2x^2} f^2(u^2,v)q \\ - b\left\{g^2(u^2,v) - (v^2)^\top(\nabla_{x^2} g^2(u^2,v) + \nabla_{x^2x^2} g^2(u^2,v)q) - \frac{1}{2}q^\top \nabla_{x^2x^2} g^2(u^2,v)q\right\} = 0$$

and

$$(\nabla_{x^2} f^2(u^2,v) + \nabla_{x^2x^2} f^2(u^2,v)q) - b(\nabla_{x^2} g^2(u^2,v) + \nabla_{x^2x^2} g^2(u^2,v)q) \geq 0,$$

for $v^2 \geq 0$.

Theorem 4.2 (Symmetric duality). *Let $(\bar{x}, \bar{y}, \bar{a}, \bar{p})$ be an optimal solution of $(\widehat{\text{WFP}})$. Also, let*

(i) *$f(x,y)$ and $g(x,y)$ be thrice differentiable in x^2 and y^2,*

(ii) *for each feasible solution of $(\widehat{\text{WFP}})$ and $(\widehat{\text{WFD}})$, $f(x,y) - ag(x,y)$ is η_1-bonvex in the first variable at x^2 for each (x^1, y) and $-f(u,v) + bg(u,v)$ is η_2-bonvex in the second variable at v^2 for each (u, v^1),*

(iii) *$\nabla_{y^2y^2} f^2(\bar{x},\bar{y}) - \bar{l}\nabla_{y^2y^2} g^2(\bar{x},\bar{y})$ is nonsingular,*

(iv) *the equality $\bar{p}^\top(\nabla_{y^2}(\nabla_{y^2y^2} f^2(\bar{x},\bar{y})\bar{p} - \bar{l}\nabla_{y^2y^2} g^2(\bar{x},\bar{y})\bar{p})) = 0$ implies that $\bar{p} = 0$,*

(v) *$((\bar{x}^2)^\top \nabla_{x^2} g^2(\bar{x},\bar{y}) - (\bar{y}^2)^\top \nabla_{y^2} g^2(\bar{x},\bar{y})) f^2(\bar{x},\bar{y}) + ((\bar{y}^2)^\top \nabla_{y^2} f^2(\bar{x},\bar{y}) - (\bar{x}^2)^\top \nabla_{x^2} f^2(\bar{x},\bar{y})) g^2(\bar{x},\bar{y}) = 0$.*

Then, $\bar{p} = 0$, the objective values of $(\widehat{\text{WFP}})$ and $(\widehat{\text{WFD}})$ are equal and $(\bar{x}, \bar{y}, \bar{a}, \bar{q} = 0)$ is an optimal solution of problem $(\widehat{\text{WFD}})$.

Proof. The problems $(\widehat{\text{WFP}})$ and $(\widehat{\text{WFD}})$ become a pair of symmetric dual problems discussed in the previous section for given $y^1(=v^1)$ and so Theorem 3.2 becomes applicable in the light of the hypotheses made in this section. Therefore, for $y^1 = \bar{y}^1$,

$$\phi(\bar{y}^1) = \psi(\bar{y}^1).$$

Let us assume that $(\bar{x}, \bar{y}, \bar{a}, \bar{q} = 0)$ is not an optimal solution of problem $(\widehat{\text{WFD}})$. So, there exists $\hat{y}^1 \in V$ such that $\psi(\hat{y}^1) < \psi(\bar{y}^1)$. By the given hypotheses,

$$\phi(\bar{y}^1) = \psi(\bar{y}^1) > \psi(\hat{y}^1) = \phi(\hat{y}^1),$$

which contradicts the optimality of $(\bar{x}, \bar{y}, \bar{a}, \bar{p})$ for $(\widehat{\text{WFP}})$. This confirms the fact that $(\bar{x}, \bar{y}, \bar{a}, \bar{q} = 0)$ is an optimal solution for $(\widehat{\text{WFD}})$ and the optimal values are equal. □

Remark 4.3. If $U = \phi$ and $V = \phi$, then (WFP*) and (WFD*) are reduced to the second-order symmetric dual problems programs of Section 3.

5 Self-duality

A mathematical programming problem is called self-dual if we recast the dual in the form of the primal. The programs (EWFP) and (EWFD) turn out to be self-dual if we take the functions f as skew symmetric and g as symmetric, that is,

$$f(u,v) = -f(v,u), \ \ g(u,v) = g(v,u).$$

By recasting the dual problem (EWFD) as a minimum problem, we have

$$\min \ -m$$

subject to

$$f(u,v) - u^\top(\nabla_x f(u,v) + \nabla_{xx} f(u,v)q) - \frac{1}{2}q^\top \nabla_{xx} f(u,v)q$$
$$- m(g(u,v) - u^\top(\nabla_x g(u,v) + \nabla_{xx} g(u,v)q) - \frac{1}{2}q^\top \nabla_{xx} g(u,v)q) = 0$$

and

$$(\nabla_x f(u,v) + \nabla_{xx} f(u,v)q) - m(\nabla_x g(u,v) + \nabla_{xx} g(u,v)q) \geq 0,$$

for $v \geq 0$, where

$$m = \frac{f(u,v) - u^\top(\nabla_x f(u,v) + \nabla_{xx} f(u,v)q) - \frac{1}{2}q^\top \nabla_{xx} f(u,v)q}{g(u,v) - u^\top(\nabla_x g(u,v) + \nabla_{xx} g(u,v)q) - \frac{1}{2}q^\top \nabla_{xx} g(u,v)q}.$$

By the properties mentioned above, we get

$$\min z$$

subject to

$$f(v,u) - u^\top(\nabla_x f(v,u) + \nabla_{xx} f(v,u)q) - \frac{1}{2}q^\top \nabla_{xx} f(v,u)q$$
$$- z(g(v,u) - u^\top(\nabla_x g(v,u) + \nabla_{xx} g(v,u)q) - \frac{1}{2}q^\top \nabla_{xx} g(v,u)q) = 0$$

and

$$(\nabla_x f(v,u) + \nabla_{xx} f(v,u)q) - z(\nabla_x g(v,u) + \nabla_{xx} g(v,u)q) \leq 0,$$

for $v \geq 0$, where

$$z = \frac{f(v,u) - u^\top(\nabla_x f(v,u) + \nabla_{xx} f(v,u)q) - \frac{1}{2}q^\top \nabla_{xx} f(v,u)q}{g(v,u) - u^\top(\nabla_x g(v,u) + \nabla_{xx} g(v,u)q) - \frac{1}{2}q^\top \nabla_{xx} g(v,u)q}.$$

This shows that the dual problem (EWFD) is identical to (EWFP). Hence, the feasibility of (u, v, m, q) to (EWFD) implies the feasibility of (v, u, m, q) to (EWFP). We now state the following self-duality theorem.

Theorem 5.1. *Let f be skew symmetric and let g be symmetric. Then,* (EWFD) *is self-dual. Furthermore, if* (EWFP) *and* (EWFD) *are dual problems and* $(\bar{x}, \bar{y}, \bar{m}, \bar{p})$ *is a joint optimal solution, then so is* $(\bar{y}, \bar{x}, \bar{m}, \bar{p})$ *and the common optimal value of the objective function is* 0.

6 Conclusion

In this paper, we considered a Wolfe-type second-order fractional dual symmetric program and derived weak, strong and converse duality theorems under η-bonvexity assumptions. We also discussed minimax mixed integer symmetric dual fractional programs and self-duality theorems. The present work can be extended to multiobjective symmetric second-order fractional dual programs and also to nondifferentiable symmetric second-order fractional dual programs. This may be taken as the future task of the authors.

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